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Umbilicity of spacelike hypersurfaces immersed in a Lorentzian space form

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Abstract. We establish sufficient conditions to guarantee the umbilicity of complete spacelike hypersurfaces immersed in a Lorentzian space form, under the assumption that the support functions with respect to a fixed nonzero vector are linearly related. Our approach is based on appropriate integrability properties and a suitable parabolicity criterion for complete spacelike hypersurfaces.

1. Introduction

In 2008, Alías, Brasil and Perdomo [2] studied complete hypersurfaces immersed in the unit Euclidean sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, whose support functions with respect to a fixed nonzero vector of the Euclidean space $\mathbb{R}^{n+2}$ are linearly related. In this setting, they showed that such a hypersurface having constant mean curvature must be either totally umbilical or isometric to a Clifford torus.

Later on, working with a different approach of that used in [2], the authors [4, 6] characterized the totally umbilical hypersurfaces and the hyperbolic cylinders of the hyperbolic space $\mathbb{H}^{n+1}$ as the only complete hypersurfaces with constant mean curvature and whose support functions with respect to a fixed nonzero vector of the Lorentz-Minkowski space $\mathbb{L}^{n+2}$ are linearly related. More recently, the authors jointly with dos Santos [7] extended the techniques developed in [4, 6] characterizing constant mean curvature spacelike hypersurfaces in a Lorentzian space form, whose support functions are linearly related.

Proceeding with this picture, our purpose in this paper is to establish sufficient conditions to guarantee the umbilicity of complete spacelike hypersurfaces immersed in a Lorentzian space form, under the assumption that the support functions with respect to a fixed nonzero vector are linearly related. To achieve our goal, we explore appropriate integrability properties due to Yau [13] and a suitable parabolicity criterion for complete spacelike hypersurfaces due to Romero, Rubio and Salamanca [12] (see Sections 3 and 4). Before, in Section 2 we recall some basic facts concerning spacelike hypersurfaces immersed in a Lorentzian space form.

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2. Preliminaries

For an integer ν with $\nu \in \{1, 2\}$, let $\mathbb{R}^{n+2}_{\nu}$ be denoting the $(n+2)$-dimensional semi-Euclidean space endowed with the following metric of index ν

$$\langle u, v\rangle = -\sum_{i=1}^{\nu} u_i v_i + \sum_{i=\nu+1}^{n+2} u_i v_i,$$

for all $u, v \in \mathbb{R}^{n+2}$. In particular, when $\nu = 1$, $\mathbb{R}^{n+2}_1 = \mathbb{L}^{n+2}$ is the so-called Lorentz-Minkowiski space $\mathbb{L}^{n+2}$.

The de Sitter space $\mathbb{S}^{n+1}_1$ is the hyperquadric of $\mathbb{L}^{n+2}$ defined in the following way

$$\mathbb{S}^{n+1}_1 = \{x \in \mathbb{L}^{n+2}\,;\, \langle x, x\rangle = 1\}.$$

Endowed with the induced metric from $\mathbb{L}^{n+2}$, $n \geq 2$, the de Sitter space is complete simply connected $(n+1)$-dimensional Lorentzian manifold with constant sectional curvature one.

On the other hand, when $\nu = 2$, we define the anti-de Sitter space $\mathbb{H}^{n+1}_1$ as being the hyperquadric of $\mathbb{R}^{n+2}_2$ given by

$$\mathbb{H}^{n+1}_1 = \{x \in \mathbb{R}^{n+2}_2\,;\, \langle x, x\rangle = -1\}.$$

Topologically, $\mathbb{H}^{n+1}_1$ is equivalent to the product $\mathbb{S}^1 \times \mathbb{R}^n$ and the semi-Euclidean metric on $\mathbb{R}^{n+2}_2$ induces a Lorentzian metric of constant sectional curvature -1 on $\mathbb{H}^{n+1}_1$. Moreover, the universal covering manifold $\widetilde{\mathbb{H}}^{n+1}_1$ of $\mathbb{H}^{n+1}_1$ is topologically $\mathbb{R}^{n+1}$ (that is, $\widetilde{\mathbb{H}}^{n+1}_1$ is simply connected) and is thus a Lorentzian analogue of the usual Riemannian hyperbolic space $\mathbb{H}^{n+1}$ of negative curvature -1, which is called the *universal anti-de Sitter spacetime* (see, for instance, Section 5.3 of [8] or Section 8.6 of [11]).

From now on, we consider $L^{n+1}_1(c)$, where $c \in \{-1, 0, 1\}$, denoting the Lorentz-Minkowiski space if $c = 0$, the de Sitter space if $c = 1$ and the anti-de Sitter space if $c = -1$. A smooth immersion $x : \Sigma^n \to L^{n+1}_1(c)$ of an n-dimensional connected manifold Σ^n is said a *spacelike hypersurface* if the induced metric via x is a Riemannian metric on Σ^n, which is also denoted by $\langle\ ,\rangle$. Since $L^{n+1}_1(c)$ is time-oriented, we can choose an unique unit normal vector field N on Σ^n which is a future-directed timelike vector field in $L^{n+1}_1(c)$ and, hence, we may assume that Σ^n is oriented by N. In this setting, we will denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of $L^{n+1}_1(c)$ and Σ^n, respectively.

Fixed a nonzero vector $a \in \mathbb{R}^{n+2}_{\nu}$, let us consider the support functions $l_a : \Sigma^n \to \mathbb{R}$, given by $l_a(p) = \langle x(p), a\rangle$, and $f_a : \Sigma^n \to \mathbb{R}$, given by $f_a(p) = \langle N(p), a\rangle$. Then, we can write

$$a = a^{\top} - f_a N + c l_a x, \tag{1}$$

and a direct calculation allows us to conclude that

$$\nabla l_a = a^{\top} \quad\text{and}\quad \nabla f_a = -A(a^{\top}).$$

So, for all $X \in \mathfrak{X}(\Sigma)$,

$$\nabla_X \nabla l_a = \nabla_X a^{\top} = -f_a AX - c l_a X. \tag{2}$$

Taking the trace in (2), we get

$$\Delta l_a = nHf_a - cnl_a. \tag{3}$$

3. Umbilicity via integrability properties

We start this section recalling a suitable description of totally umbilical hypersurfaces of $L_1^{n+1}(c)$. Let $a \in \mathbb{R}_\nu^{n+2}$ be a fixed nonzero vector with $\langle a, a\rangle \in \{-1, 0, 1\}$, and take the smooth function $g : L_1^{n+1}(c) \to \mathbb{R}$ defined by $g(x) = \langle a, x\rangle$. It is not difficult to see that, for every $\tau \in \mathbb{R}$ with $\langle a, a\rangle - c\tau^2 \neq 0$, the set

$$L_\tau = g^{-1}(\tau) = \{x \in L_1^{n+1}(c)\,;\, \langle a, x\rangle = \tau\} \tag{4}$$

is a totally umbilical hypersurface in $L_1^{n+1}(c)$, with Gauss map given by

$$N_\tau(p) = \frac{1}{\sqrt{|\langle a, a\rangle - c\tau^2|}}(a - c\tau x). \tag{5}$$

With respect to this orientation, the shape operator and mean curvature are, respectively,

$$Av = \frac{c\tau}{\sqrt{|\langle a, a\rangle - c\tau^2|}}v, \quad \text{and} \quad H^2 = \frac{\tau^2}{|\langle a, a\rangle - c\tau^2|}.$$

Hence, with a straightforward computation we get

$$l_a = \frac{|\tau|}{\sqrt{|\langle a, a\rangle - c\tau^2|}}f_a = |H|f_a. \tag{6}$$

Motivated by (6), our aim is establish sufficient conditions to guarantee the umbilicity of complete spacelike hypersurfaces immersed in $L_1^{n+1}(c)$ whose support functions l_a and f_a are linearly related, that is,

$$l_a = \lambda f_a, \tag{7}$$

for some smooth function $\lambda : \Sigma^n \to \mathbb{R}$.

For this, we will extend a nomenclature of Section 4 in [10] and consider the following open sets

$$\mathcal{O} = \{p \in \mathbb{S}_1^{n+1}\,;\, \langle p, a\rangle \neq 0\} \subset \mathbb{S}_1^{n+1},$$

and

$$\mathcal{A} = \{p \in \mathbb{H}_1^{n+1}\,;\, \langle p, a\rangle \neq 0\} \subset \mathbb{H}_1^{n+1}.$$

In the de Sitter space, the connected spacetime $\mathcal{O}^+$ is given by $\langle p, a\rangle > 0$ and the connected spacetime $\mathcal{O}^-$ is given by $\langle p, a\rangle < 0$. According to Aledo, Alías and Romero [1], when the vector a is timelike, the connected spacetime $\mathcal{O}^+$ is denoted by past chronological and the connected spacetime $\mathcal{O}^-$ is denoted by future chronological of a equator of $\mathbb{S}_1^{n+1}$ determined by vector a.

For our purpose, we need to quote an extension of Hopf's theorem on a complete Riemannian manifold due to Yau in [13]. In what follows, we will consider the following set

$$\mathcal{L}^p(\Sigma) := \{h : \Sigma^n \to \mathbb{R} : \int_\Sigma |h|^p d\Sigma < +\infty\}.$$

Lemma 3.1. *Let Σ^n be an n-dimensional, complete Riemannian manifold and let $h : \Sigma^n \to \mathbb{R}$ be a subharmonic (or superharmonic) function on Σ^n. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then h must be harmonic.*

Now, we are in position to present our first result.

Theorem 3.1. *Let $x : \Sigma^n \to \mathcal{A} \subset \mathbb{H}^{n+1}_1$ be a complete spacelike hypersurface satisfying (7) for some a nonzero timelike vector $a \in \mathbb{R}^{n+2}_2$. Suppose that Σ^n lies under a totally umbilical hypersurface of $\mathbb{H}^{n+1}_1$ determined by a and that its mean curvature H satisfies $H^2 \leq \lambda^2$. If $|a^\top| \in \mathcal{L}^1(\Sigma)$, then Σ^n is a totally umbilical hypersurface.*

Proof. Since $x(\Sigma) \subset \mathcal{A}$ we have that the function l_a has strict sign and, consequently, from relation (7) we get that λ also has strict sign on Σ^n. Moreover, as we are supposing that Σ^n lies below a totally umbilical spacelike hypersurface determined by a, there exists $\rho \in \mathbb{R}$ such that $|l_a| \leq \rho$.

Taking $c = -1$ in (3), we obtain

$$\Delta l_a = nHf_a + nl_a. \tag{8}$$

Computing the Laplacian of function l_a^2, using (8) and (7), we have

$$\begin{aligned} \frac{1}{2}\Delta l_a^2 = l_a \Delta l_a + |\nabla l_a|^2 &= nHf_a l_a + nl_a^2 + |\nabla l_a|^2 \\ &= n\left(\frac{H}{\lambda} + 1\right) l_a^2 + |\nabla l_a|^2 \geq n\left(\frac{H}{\lambda} + 1\right) l_a^2. \end{aligned} \tag{9}$$

Thus, since $H^2 \leq \lambda^2$, from (9) we conclude that l_a^2 is a bounded subharmonic function on Σ^n.

On the other hand, since we are assuming $|a^\top| \in \mathcal{L}^1(\Sigma)$, we get

$$|\nabla l_a^2| = 2|l_a||\nabla l_a| \leq 2\rho|a^\top| \in \mathcal{L}^1(\Sigma).$$

Hence, applying Lemma 3.1 we have that l_a^2 is a harmonic function on Σ^n. So, returning to (9) we obtain that $|\nabla l_a|^2 = 0$. Consequently, we conclude that l_a is a constant function. Thus, $\Sigma^n \subset L_\tau$, for some $\tau \in \mathbb{R}$. Since Σ^n is complete and L_τ is connected, we conclude that, in fact, $\Sigma^n = L_\tau$.

For the proof of our next result, we need the following analytical tool which is a consequence of Theorem 3 in [13].

Lemma 3.2. *Let Σ^n be a complete Riemannian manifold and let $h : \Sigma^n \to \mathbb{R}$ be a nonnegative subharmonic function on Σ^n. If $h \in \mathcal{L}^p(\Sigma)$ with $p > 1$, then h must be constant on Σ^n.*

Now, we state and prove our second result.

Theorem 3.2. *Let $x : \Sigma^n \to \mathcal{O} \subset \mathbb{S}^{n+1}_1$ be a complete spacelike hypersurface satisfying (7) for some nonzero vector $a \in \mathbb{L}^{n+2}$. Suppose that Σ^n lies below a totally umbilical determined by a and that its mean curvature H satisfies $H^2 \geq \lambda^2$. If one of the following situations occurs,*

(i) *$l_a \in \mathcal{L}^{p/2}(\Sigma)$, for some $p > 1$;*
(ii) *$|a^\top| \in \mathcal{L}^1(\Sigma)$,*

then Σ^n is a totally umbilical hypersurface.

Proof. As in the proof of Theorem 3.1, the condition $x(\Sigma) \subset \mathcal{O}$ guarantees us that the function l_a has strict sign and, hence, from (7) we get once more that λ also has strict sign on Σ^n.

On the other hand, taking $c = 1$ in (3) and using (7), we have

$$\frac{1}{2}\Delta l_a^2 = n\left(\frac{H}{\lambda} - 1\right) l_a^2 + |\nabla l_a|^2 \geq n\left(\frac{H}{\lambda} - 1\right) l_a^2. \tag{10}$$

So, taking into account that $H^2 \geq \lambda^2$, from (10) we obtain that l_a^2 is a subharmonic function on Σ^n.

Let us assume that the condition (i) occurs. So, we have that $l_a^2 \in \mathcal{L}^p(\Sigma)$ for $p > 1$. Since l_a^2 is a positive function, Lemma 3.2 guarantees us that l_a^2 is constant and, consequently, l_a is constant on Σ^n. Hence, there exists $\tau \in \mathbb{R}$ such that $\Sigma^n \subset L_\tau$. Since Σ^n is complete and connected, we conclude that $\Sigma^n = L_\tau$.

Finally, if the condition (ii) holds, the proof follows as in last part of the proof of Theorem 3.1.

According to Section 4 of Montiel [10], when $a \in \mathbb{L}^{n+2}$ is a timelike vector, the Lorentz-Minkowiski space $\mathbb{L}^{n+2}$ can be foliated by means of parallel spacelike hyperplanes orthogonal to a. In this setting, we have following result

Theorem 3.3. *Let $x : \Sigma^n \to \mathbb{R}_1^{n+1}$ be a complete spacelike hypersurface satisfying* (7) *for some nonzero timelike vector $a \in \mathbb{R}_1^{n+2}$. Suppose that Σ^n lies below a spacelike hyperplane orthogonal to a and passing through the origin, and that its mean curvature H satisfies $\frac{H}{\lambda} \geq 0$. If $|a^\top| \in \mathcal{L}^1(\Sigma)$, then Σ^n is a totally umbilical hypersurface.*

Proof. Since Σ^n lies below a spacelike hyperplane orthogonal to a and passing through the origin, the function l_a has strict sign. Consequently, λ does not vanish on Σ^n and the ratio $\frac{H}{\lambda}$ is, indeed, well defined.

Taking $c = 0$ in equation (3) and using (7), we get

$$\frac{1}{2}\Delta l_a^2 = \frac{nH}{\lambda} l_a^2 + |\nabla l_a|^2 \geq \frac{nH}{\lambda} l_a^2. \tag{11}$$

Hence, since we are assuming that $\frac{H}{\lambda} \geq 0$, we have that l_a^2 is a subharmonic function on Σ^n. At this point, we can reason once more as in the last part of the proof of Theorem 3.1.

4. Parabolic spacelike hypersurfaces

Let $(M^n, \langle\,,\,\rangle_M)$ be a connected, n-dimensional, oriented Riemannian manifold, $I \subset \mathbb{R}$ an open interval and $f : I \to \mathbb{R}$ a positive smooth function. Also, in the product manifold $\overline{M}^{n+1} = I \times M^n$ let π_I and π_M denote the canonical projections onto the factors I and M^n, respectively.

In this section, we will consider a class of Lorentzian manifolds obtained by furnishing $\overline{M}^{n+1}$ with the following Lorentzian metric

$$\langle u, v\rangle_p = -((\pi_I)_* u)((\pi_I)_* v) + (f \circ (\pi_I))^2(p)\langle((\pi_M)_* u, (\pi_M)_* v)\rangle_{\pi_M(p)},$$

for all $p \in \overline{M}^{n+1}$ and $u,\ v \in T_p\overline{M}$.

According to the nomenclature established in [3], $(\overline{M}^{n+1}, \langle\,,\rangle)$ is called a *generalized Robertson-Walker* (GRW) spacetime with warping function f and Riemannian fiber M^n. For simplicity, we will just write $\overline{M}^{n+1} = -I \times_f M^n$.

Given a spacelike hypersurface immersed in a GRW spacetime $x : \Sigma^n \to -I \times_f M^n$, since

$$\partial_t = (\partial/\partial t)_{(t,p)},\ t \in I,\ p \in M^n,$$

is a unitary timelike vector field globally defined on $-I \times_f M^n$ which determines a time-orientation on it, the future-pointing Gauss map of Σ^n is the unique unitary timelike normal vector field N globally defined on Σ^n in the same time-orientation as ∂_t. In this setting, the *hyperbolic angle* of Σ^n is the smooth function $\Theta : \Sigma^n \to (-\infty, -1]$, which is defined by $\Theta := \langle N, \partial_t\rangle$.

Considering the previous digression, we have the following parabolicity criterion due to Romero, Rubio and Salamanca (cf. Theorem 3 of [12]).

Lemma 4.1. *Let $x : \Sigma^n \to -I \times_f M^n$ be a complete spacelike hypersurface in a GRW spacetime $-I \times_f M^n$, whose Riemannian fiber M^n has parabolic universal Riemannian covering. If the hyperbolic angle Θ of Σ^n is bounded and the warping function f on Σ^n satisfies*

(i) $\sup_\Sigma f(h) < +\infty$ *and*

(ii) $\inf_\Sigma f(h) > 0$,

where $h : \Sigma^n \to \mathbb{R}$ is defined by $h = (\pi_I)|_\Sigma$, then Σ^n is parabolic.

Exploring a suitable GRW model of the de Sitter space, we obtain

Theorem 4.1. *Let $x : \Sigma^n \to \mathcal{O} \subset \mathbb{S}^{n+1}_1$ be a complete spacelike hypersurface satisfying* (7) *for some bounded function λ and nonzero timelike vector $a \in \mathbb{L}^{n+2}$. If the image of the Gauss map of Σ^n is contained into a geodesic ball of $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ and its mean curvature H satisfies $H^2 \geq \lambda^2$, then Σ^n is isometric to $\mathbb{S}^n(r)$ form some radius $r \geq 1$.*

Proof. According to Example 4.2 of [10], the de Sitter space $\mathbb{S}^{n+1}_1$ is isometric to the following GRW spacetime

$$-\mathbb{R} \times_{\cosh t} \mathbb{S}^n,$$

where $\mathbb{S}^n$ is the unit Euclidean sphere endowed with its canonical metric. We observe that the timelike unit vectors ∂_t and N_τ are identified via the isometry between the warped and quadric models of $\mathbb{S}^{n+1}_1$. In this setting, from (5) we have

$$|\Theta| = |\langle N, \partial_t\rangle| = \frac{1}{\sqrt{1+\tau^2}}|f_a|. \tag{12}$$

Since we are supposing that the Gauss map of Σ^n is contained into a geodesic ball of $\mathbb{H}^{n+1}$, the function f_a is bounded and, consequently, from (12) we get that the hyperbolic angle of Σ^n is bounded. Moreover, since λ is a bounded function on Σ^n, it follows from relation (7) that the function l_a is also bounded on Σ^n.

Hence, we are in position to apply Lemma 4.1 obtaining that Σ^n is parabolic. Therefore, taking into account once more (10), we conclude that the function l_a is constant on Σ^n, implying that it must be a totally umbilical round sphere $\mathbb{S}^n(r)$ for some radius $r = \cosh t \geq 1$.

Proceeding, we establish the following result.

Theorem 4.2. *Let $x : \Sigma^2 \to \mathcal{O} \subset \mathbb{S}^3_1$ be a complete spacelike surface satisfying* (7) *for some nonzero null vector $a \in \mathbb{L}^3$. Suppose that Σ^2 lies between two totally umbilical spacelike surfaces determined by a, and that its mean curvature H satisfies $H^2 \geq \lambda^2$. If one of the following situations occurs*

(i) *the image of the hyperbolic Gauss map of Σ^2 is contained in the closure of the interior domain enclosed by a horosphere of $\mathbb{H}^2$ determined by the nonzero null vector a;*

(ii) *Σ^2 has nonnegative Gaussian curvature,*

then Σ^2 is isometric to $\mathbb{R}^2$.

Proof. Since $a \in \mathbb{L}^3$ is a nonzero null vector, from Example 4.2 of [10] we have that each connect component of $\mathcal{O}$ can be modeled by the following GRW spacetime

$$-\mathbb{R} \times_{e^t} \mathbb{R}^2.$$

Let us consider the situation of item (i). Since the image of the hyperbolic Gauss map of Σ^2 is contained in the closure of the interior domain enclosed by a horosphere of $\mathbb{H}^2$ determined by the nonzero null vector a, we get the function f_a is bounded. Moreover, from (5) we have that the hyperbolic angle Θ of Σ^2 is given by

$$|\Theta| = |\langle N, \partial_t \rangle| = \frac{1}{|\tau|}|f_a|. \tag{13}$$

Hence, from (13) we obtain that Θ is bounded on Σ^2.

Our assumption that Σ^2 is between two totally totally umbilical determined by a guarantees the existence of $\tau_*, \tau^* > 0$ such that $\tau_* \leq l_a \leq \tau^*$. Thus, we have that l_a is bounded on Σ^2. Consequently, taking into account the isometry between the quadric and GRW models of $\mathcal{O}$, it is not difficult to verify that we can apply Lemma 4.1 obtaining that Σ^2 is parabolic.

Finally, supposing that item (ii) holds, that is, Σ^2 has nonnegative Gaussian curvature, by Huber's Theorem [9] we have that Σ^2 is parabolic. Therefore, in both cases, using again (10), we get that l_a is constant and, hence, Σ^2 must be isometric to $\mathbb{R}^2$.

Since $\mathbb{L}^3 = -\mathbb{R} \times \mathbb{R}^2$, we can reason as in the proof of Theorem 4.2 obtaining our last result.

Theorem 4.3. *Let $x : \Sigma^2 \to \mathbb{L}^3$ be a complete spacelike surface satisfying* (7) *for some nonzero timelike vector $a \in \mathbb{L}^3$. Suppose that Σ^2 lies below a spacelike plane orthogonal to a and passing through the origin, and that its mean curvature H satisfies $\frac{H}{\lambda} \geq 0$. If one of the following situations occurs*

(i) *the image of the Gauss map of Σ^2 is contained into a geodesic ball of $\mathbb{H}^2$;*

(ii) *Σ^2 has nonnegative Gaussian curvature,*

then Σ^2 is isometric to $\mathbb{R}^2$.

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