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PERIMETER BISECTORS, CUSPS, AND KITES

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Abstract. We identify the specific convex quadrilaterals whose angle bisectors are also perimeter bisectors to be the rhombi and the kites with three congruent acute angles. The proof of this result uses the envelope of the lines that bisect the quadrilateral's perimeter. We also investigate and make some observations regarding these envelopes for more general convex polygons.

1. Introduction

Our main result is

Theorem 1.1. *Let P be a convex quadrilateral. The following are equivalent:*

- *Every angle bisector of P bisects its perimeter.*
- *P is either a kite with three congruent acute angles or a rhombus.*

To prove this result, we needed to consider the envelope of the perimeter bisectors of a convex polygon. For the basic properties of envelopes, we recommend Section 3.5 of [2]. Thus consider a convex polygon P in a plane Π. For each direction measured by an angle θ there is a unique line $L(\theta)$ that bisects the perimeter of P. Consider a perimeter bisector $L(\theta)$ neither of whose endpoints are a vertex of P. Following the construction of an envelope, we define $C(\theta)$ to be the limit of the intersection of $L(\theta)$ with $L(\alpha)$ as α approaches θ. Let $E(P)$ to be the closure of the image of C. In Definition 2.1, we will give a precise definition of $E(P)$ and refer to it as the "bisection envelope" of P.

Unlike the analogous envelope for area bisectors, as described in [3], $E(P)$ is generally not continuous and consists of a union of parabolic segments and isolated points. Indeed, the only triangles with continuous bisection envelopes are the equilateral triangles. However, continuity of $E(P)$ is closely related to when the perimeter bisectors at vertices of P are also angle bisectors. For a formal definition of continuity of $E(P)$, see Definition 2.3.

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When $E(P)$ is continuous, we can imagine a very small car driving along it in such a way that the car does not make any abrupt changes in the direction it is facing. For most of the journey the car will be traveling along a parabolic arc. But when it points in the direction of a perimeter bisector with an endpoint at a vertex, the route will generally involve a cusp (see Definition 4.1), requiring our driver to switch gears from drive to reverse, or vice versa, in order to continue the journey. On the other hand, since all perimeter bisectors are swept through in half a circle, when our driver completes a single lap, the car will be facing in the opposite direction that it started. This is only possible if there are an odd number of cusps in $E(P)$. A quadrilateral has four vertices, and therefore four expected cusps, a contradiction, unless one of the perimeter bisectors joins two vertices. Once we establish the existence of a perimeter bisector that joins two vertices, the remainder of the proof follows using Euclidean methods.

The conditions in Theorem 1.1 do not exactly correspond to continuity of $E(P)$. In addition to the quadrilaterals described in Theorem 1.1, convex quadrilaterals with a continuous bisection envelope also include the parallelograms (see Theorem 5.1).

In order to prove Theorem 5.1, we need an optimization result which is interesting in its own right, so we will briefly explain it here. Fix two points $A = (a, 0)$ and $A' = (-a, 0)$ on the x-axis in the Cartesian plane. For a point X in the upper half plane, define $f(X) = \sin \angle AA'X \sin \angle A'AX$. As X travels along the horizontal half line $(t, 0)$, where $t > 0$, the function $f(X)$ achieves a maximum where this half line intersects the circle with diameter $\overline{AA'}$. However, we will show that as X travels through the first quadrant from apogee to perigee of an ellipse with foci A and A', the upward movement along the ellipse is enough to make f strictly increasing.

2. Definitions

In this section we will outline some terms and definitions used throughout this paper in the order in which they appear.

Let $\Pi = \mathbb{R}^2$. Consider a convex polygon P in Π. For each direction measured by an angle θ there is a unique line in that direction that bisects the perimeter of P. Throughout, we shall denote this line by $L_P(\theta)$ (or just $L(\theta)$ if the context is clear). We define a curve C_+ such that $C_+ : [0, \pi] \to \Pi$ where $C_+(\theta) \equiv \lim_{t \to 0^+} L(\theta) \cap L(\theta + t)$. Similarly, we define C_- using the same formula, except replacing $\theta + t$ with $\theta - t$.

Definition 2.1. *The bisection envelope of P, which we denote by $E(P)$, is the union of the images of C_+ and C_-.*

Examples of bisection envelopes will be given shortly, but we first define terms relating to characteristics of bisection envelopes.

Definition 2.2. *The bisection envelope $E(P)$ is said to be continuous through a perimeter bisector $L(\theta)$ if and only if $C_+(\theta) = C_-(\theta)$.*

Definition 2.3. *The bisection envelope $E(P)$ is said to be continuous if and only if $C_+(\theta) = C_-(\theta)$ for all $\theta \in [0, \pi]$.*

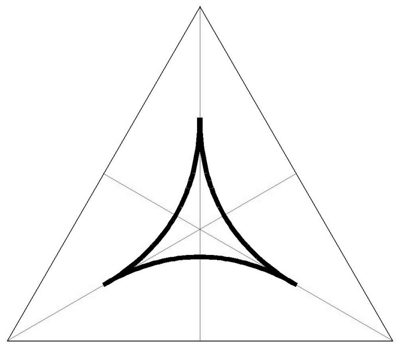

FIGURE 1. The bisection envelope of an equilateral triangle, whose envelope consists of three parabolic segments with endpoints on the perimeter bisectors that pass through the vertices of the triangle, which also are the angle bisectors of the triangle. This envelope is continuous.

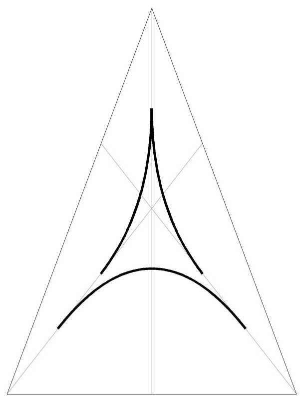

FIGURE 2. The bisection envelope of an isosceles triangle, whose envelope consists of three parabolic segments with endpoints on the perimeter bisectors that pass through the vertices of the triangle. This envelope is continuous through the perimeter bisector that is also the triangle's line of symmetry.

Definition 2.4. *We define the mouth of a parabola to be the open convex region of the plane bounded by the parabola (as opposed to the nonconvex region).*

In order to illustrate these definitions, we provide various examples of bisection envelopes, as depicted in Figures 1, 2, 3, 4, 5, and 6.

Throughout this paper, we found it useful to isolate cases where perimeter bisectors have both endpoints on parallel sides of a polygon. For this reason, we have the following

Definition 2.5. *A polygon is antiparallel if no perimeter bisector of the polygon has endpoints on the interiors of parallel sides.*

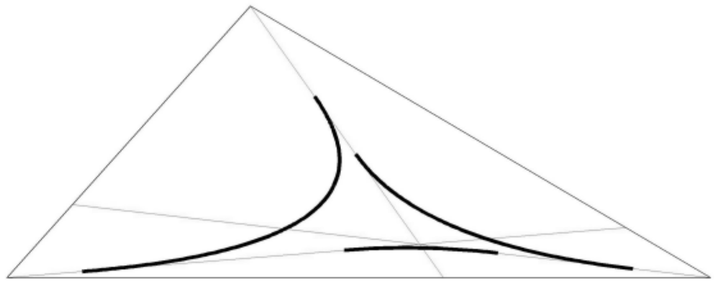

Figure 3. The bisection envelope of a scalene triangle, whose envelope consists of three parabolic segments with endpoints on the perimeter bisectors that pass through the vertices of the triangle. This envelope is not continuous through any perimeter bisector that passes through a vertex.

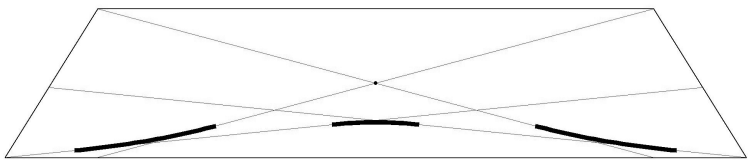

Figure 4. The bisection envelope of a trapezoid, whose envelope consists of three parabolic segments with endpoints on the perimeter bisectors that pass through the bottom two vertices of the trapezoid and an isolated point which is on the intersection of the perimeter bisectors coming from the top two vertices.

Consider a convex polygon P with parallel sides AB and CD such that there exists more than one perimeter bisector with endpoints on those parallel sides. Without loss of generality, let $AB \leq CD$. Within side AB, let XY be the subsegment consisting of the points on AB such that they are endpoints of perimeter bisectors with their other endpoint on segment CD. Similarly define VW on CD. Note that $XY = VW$. We shorten each of the lengths of AB and CD by collapsing XY and VW to points. We repeat this process until we obtain a polygon Q, which is antiparallel.

Definition 2.6. *We call Q the collapsed polygon of P and the process by which Q is formed the act of collapsing P.*

Note that the collapsed polygon could be a degenerate point, such as in the case in which P is a parallelogram.

3. Bisection Envelopes

In this section, we discuss some general properties of bisection envelopes.

Theorem 3.1. *Let P be a convex polygon. Then $E(P)$ is a union of parabolic sections and isolated points. Let $B = \{L(\theta) \mid \theta \in [0, \pi]\}$. Let B_v be the subset of lines in B that go through a vertex of P, and let L_1 and L_2*

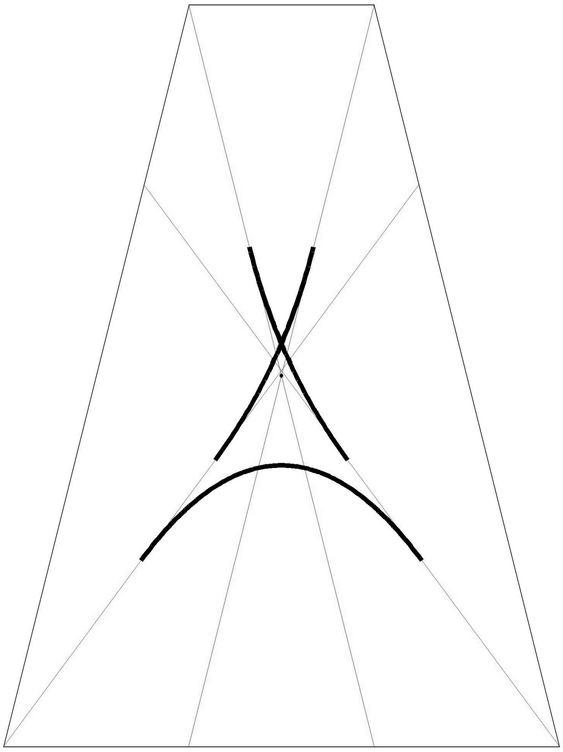

FIGURE 5. The bisection envelope of a trapezoid, whose envelope consists of three parabolic arcs and an isolated point with two parabolic arcs that cross over each other. The endpoints of these arcs rest on perimeter bisectors of the trapezoid, and the isolated point lies on the intersection of two perimeter bisectors.

be consecutive perimeter bisectors in B_v. The endpoints of perimeter bisectors between L_1 and L_2 slide along two specific sides of P. If these sides are parallel, the corresponding section of $E(P)$ will be a point located at the midpoint of any of these perimeter bisectors. Otherwise, the corresponding section of $E(P)$ is an arc of a parabola.

In order to prove this result, we use

Lemma 3.1. *Let A be an angle with vertex V. Fix a positive real number l. Let S_l be the set of line segments with endpoints on opposite sides of A which cut off a section of A of length l. Let E be the envelope of the line segments in S_l as parameterized by their direction. Then E is an arc of a parabola with axis the angle bisector of A. Furthermore, each segment in S_l is tangent to E and separates the mouth of E from the vertex V. If $\overline{P_1P_2} \in S_l$ and $Q = E \cap \overline{P_1P_2}$, then $P_1Q/QP_2 = VP_2/VP_1$. Finally, clockwise rotation through the line segments in S_l corresponds to motion along E from right to left according to a person standing on E and facing into the mouth of the parabola.*

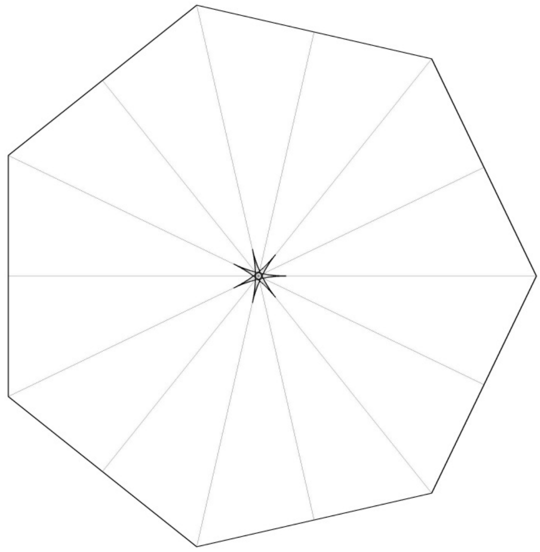

FIGURE 6. The bisection envelope of a regular heptagon, whose envelope consists of seven parabolic segments with endpoints on the perimeter bisectors that pass through the vertices of the heptagon, which also are the angle bisectors of the heptagon. This envelope is continuous.

Proof. Let $L = \overline{P_1P_2} \in S_l$. Let x be a real number. Let $P_1(x)$ be the point on $\overleftrightarrow{VP_1}$ whose signed distance from P_1 is x (oriented so that V is on the positive side of P_1). Define $P_2(x)$ so that $\overline{P_1(x)P_2(x)} \in S_l$. Furthermore, define $L(x)$ to be $\overline{P_1(x)P_2(x)}$. Let $Q(x)$ be the intersection of L and $L(x)$ and let Q be the limit of $Q(x)$ as $x \to 0$. Finally, denote by $d(\cdot,\cdot)$ the Euclidean distance between two points.

Let $\theta = m\angle P_1Q(x)P_1(x) = m\angle P_2Q(x)P_2(x)$. By the law of sines,

$$\frac{d(P_1, Q(x))}{\sin\angle P_1P_1(x)Q(x)} = \frac{|x|}{\sin\theta} = \frac{d(P_2, Q(x))}{\sin\angle P_2P_2(x)Q(x)}.$$

Taking the limit as $x \to 0$, we determine

$$\frac{d(P_1, Q)}{\sin\angle VP_1P_2} = \frac{d(P_2, Q)}{\sin\angle VP_2P_1}.$$

Again applying the law of sines, we conclude that

$$\frac{d(P_1, Q)}{d(P_2, Q)} = \frac{\sin\angle VP_1P_2}{\sin\angle VP_2P_1} = \frac{d(V, P_2)}{d(V, P_1)}. \tag{1}$$

Introduce an xy-coordinate plane in which $V = (0, -m)$, where $m > 0$ and the sides of A intersect the x-axis at $(n, 0)$ and $(-n, 0)$, where $n > 0$. Furthermore, do this in such a way that $l = 2\sqrt{n^2 + m^2}$, so the segment connecting $(-n, 0)$ and $(n, 0)$ is in S_l.

The setup of Lemma 3.1 is illustrated in Figure 7. The two sides of the angle correspond to the equations $y = -(m/n)x - m$ and $y = (m/n)x - m$.

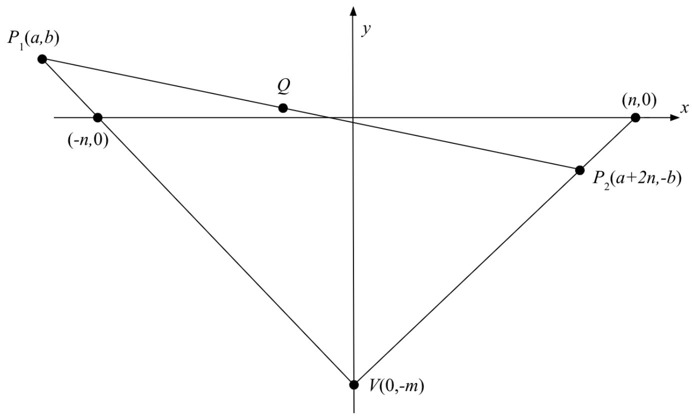

FIGURE 7. The setup used in the proof of Lemma 3.1. Segment $\overline{P_1P_2}$ and the segment connecting $(-n,0)$ and $(n,0)$ both cut off the same length l of the angle.

Without loss of generality, assume $P_1 = (a,b)$ is on the line $y = -(m/n)x - m$ with $-2n < a < 0$.

Because the distance between P_1 and $(-n,0)$ is equal to the distance between P_2 and $(n,0)$, and the sides of A have opposite slopes, we know that $P_2 = (n,0) - (-n-a,b) = (a+2n,-b)$. By (1), we know $P_1V/P_2V = P_2Q/P_1Q$. By similarity, $P_1V/P_2V = -a/(a+2n)$. Thus,

$$\begin{aligned} Q &= \frac{a+2n}{2n}(a+2n, (\frac{m}{n})a+m) - \frac{a}{2n}(a, (\frac{-m}{n})a - m) \\ &= (2a+2n, \frac{m}{4n^2}(2a+2n)^2), \end{aligned}$$

which shows that Q is on the parabola $y = (m/4n^2)x^2$ with axis the angle bisector of A. Therefore, E is an arc of $y = (m/4n^2)x^2$ (which extends from $(-2n,m)$ to $(2n,m)$). Its derivative is $(m/2n^2)x$. Note that the slope of $\overleftrightarrow{P_1P_2}$ is

$$\frac{(2m/n)a+2m}{2n} = \frac{ma+mn}{n^2} = (2a+2n)(\frac{m}{2n^2}),$$

which agrees with the slope of the tangent at Q. Therefore $\overleftrightarrow{P_1P_2}$ is the tangent to E at Q.[1]

Next, we show that $\overleftrightarrow{P_1P_2}$ separates the mouth of the parabola from V. Note that $\overleftrightarrow{P_1P_2}$ intersects the y-axis at $(0, -\frac{m(a+n)^2}{n^2})$, so we must show that

$$-m < -\frac{m(a+n)^2}{n^2} < 0,$$

which follows since $m > 0$ and $-2n < a < 0$, so $-n < a+n < n$.

Because the derivative of the parabola is $\frac{m}{2n^2}x$, which is linear and has a positive slope, traveling from right to left along this parabola as defined by

[1]This also follows from general properties of envelopes. See, for example, Section 3.5b, page 295 of [2].

looking into its mouth corresponds to clockwise rotation of tangent lines.

Proof. [Proof of Theorem 3.1] Consider two consecutive lines L_1 and L_2 in B_v. The set of lines in B between L_1 and L_2 have endpoints on a fixed pair of sides of P. Call these sides s_1 and s_2.

There are two cases to consider: either s_1 and s_2 are parallel or they are not.

If they are not parallel, then the perimeter bisectors between L_1 and L_2 also cut off the same length of the angle formed by s_1 and s_2. Hence, Lemma 3.1 applies and we see that the section of $E(P)$ generated by perimeter bisectors between L_1 and L_2 is a parabolic arc.

If s_1 and s_2 are parallel, then the envelope of the lines between L_1 and L_2 will simply be a point because any pair of bisecting lines in this set will intersect each other halfway between s_1 and s_2.

3.1. **Collapsed Polygons.** Note that, in the process of collapsing one pair of parallel sides, the sides of P do not change direction, but all perimeter bisectors, except for the one parallel to the pair of parallel sides being collapsed, will change direction. In fact, one can imagine all the perimeter bisectors having their endpoints glued in place along the perimeter of P, because the collapsing process does not change the relative location of the endpoints of perimeter bisectors along the boundary. Let X, Y, V, and W be as in the setup of Definition 2.6. Any bisecting line with endpoints on XY or VW will, after collapsing the polygon, become a bisecting line whose endpoints are the points that XY and VW become after the collapse.

Because of the way the perimeter bisecting lines and the sides of a polygon behave under collapsing, after collapsing a pair of parallel sides in this polygon, the number of parabolic sections in its bisection envelope will remain unchanged.

Proposition 3.1. *The bisection envelope of a convex polygon is a point if and only if its collapsed polygon is a point.*

Proof. By the proof of Theorem 3.1, the bisection envelope of a polygon will be a point if and only if every perimeter bisector has endpoints on a pair of parallel sides. From our description of the collapsed polygons, we know that, if the polygon does not collapse into a point, then the collapsed polygon has perimeter bisectors that do not have endpoints on a pair of parallel sides.

4. Continuity of Bisection Envelopes

By Lemma 3.1, a bisection envelope is continuous through any perimeter bisector whose endpoints are not vertices. Recall (Definition 2.3) that a bisection envelope is continuous if and only if it is continuous through every perimeter bisector. Note that bisection envelopes are not continuous in general. For example, among triangles, only the equilateral triangle has a continuous bisection envelope. In this section, we state our general results on the continuity of bisection envelopes.

In the case that the bisection envelope is continuous, we can define the following functions, which will be used in the proofs of Lemma 4.1 and

Theorem 4.1. Let $\mathbb{P}^1$ be the real projective line, which we will parameterize by real numbers θ modulo π. Given $\theta \in \mathbb{P}^1$, there is a unique perimeter bisector L in that direction, which we have called $L(\theta)$. Because the bisection envelope is continuous, we can parameterize the envelope by the function $e : \mathbb{P}^1 \to \Pi$ by defining $e(\theta) = C_+(\theta) = C_-(\theta)$. Let S^1 be the unit circle parameterized by a real number t modulo 2π. Let $\pi : S^1 \to \mathbb{P}^1$ be the canonical projection. Define $f : S^1 \to \Pi$ by $f = e \circ \pi$. Finally, let $v(t) = \lim_{h\to 0^+} \frac{f(t+h)-f(t)}{h}$. Note that in this definition, h approaches 0 only from above.

Imagine traveling around the bisection envelope in a very small car, with $e(t)$ denoting the car's position. If we interpret t as time, as t increases, the perimeter bisectors $L(t)$ rotate counterclockwise. Let the vector $(\cos t, \sin t)$ represent the direction the car is facing, and note that the vector $v(t)$ points in the direction the car moves.

Definition 4.1. *We define a cusp to be a point occurring at the intersection of two parabolic arcs such that moving through the point from one parabola to the other reverses the sign of* $v(t) \cdot (\cos t, \sin t)$.

Note that the sign of $v(t) \cdot (\cos t, \sin t)$ is positive exactly when the mouth of the parabola that the car is traveling along is on the left side (with respect to the direction the car is facing) of the perimeter bisector tangent to this parabola at the location of the car.

Lemma 4.1. *Let P be an antiparallel polygon. Consider a perimeter bisector of P with exactly one endpoint at a vertex of P. The bisection envelope of this polygon is continuous through this perimeter bisector if and only if the perimeter bisector is an angle bisector. Furthermore, the sign of* $v(t) \cdot (\cos t, \sin t)$ *is reversed after traveling through this perimeter bisector, meaning that the bisection envelope has a cusp at this perimeter bisector.*

Proof. Let AB be a perimeter bisector of P such that A is a vertex of P and B is on the interior of some side $\overline{CD}$ of P. In order to apply Lemma 3.1, we extend CD as well as the sides of the polygon which have A as an endpoint. Since A is not an endpoint of a side that is parallel to CD, a triangle is formed. Let the other vertices of this triangle be E and F. Note that, by Theorem 3.1, if the bisection envelope is to be continuous through AB, there must exist a point G on AB such that the parabolas resulting from $\angle AEB$ and $\angle AFB$ are both tangent to $\overline{AB}$ at G. See Figure 8.

By applying Lemma 3.1 to $\angle AEB$ and $\angle AFB$, we see that $BE/AE = AG/GB = BF/AF$. By the converse of the angle bisector theorem (see [4], in particular, Proposition 3 of Book 6), $\overline{AB}$ bisects the polygon's angle at A.

Next, we show that the sign of $v(t) \cdot (\cos t, \sin t)$ reverses when traveling through AB, resulting in a cusp. In order for this sign to reverse, the mouths of the parabolas generated from $\angle AEB$ and $\angle AFB$ must be on opposite sides of AB, that is, E and F are on opposite sides of AB. We note that, since AEF is a triangle and AB goes through its interior, E and F must be on opposite sides of AB, thus the sign of $v(t) \cdot (\cos t, \sin t)$ must reverse through AB, and a cusp is generated.

Lemma 4.1 immediately yields

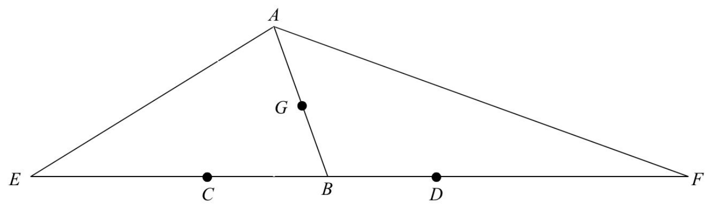

Figure 8. This triangle arises in the proof of Lemma 4.1. Its sides are the extensions of the sides of polygon P that contain vertex A and the side CD of P. Segment $\overline{AB}$ is a perimeter bisector of P.

Corollary 4.1. *In an antiparallel polygon, if every perimeter bisector having a vertex as an endpoint has exactly one endpoint at a vertex and is an angle bisector of the polygon, then the bisection envelope of the polygon is continuous.*

Theorem 4.1. *Consider an antiparallel polygon with the properties that no perimeter bisector connects two of its vertices and its bisection envelope is continuous. Then its bisection envelope consists of an odd number of cusps.*

Proof. By Lemma 4.1, at each cusp of the bisection envelope, the sign of $v(t) \cdot (\cos t, \sin t)$ will change. Since $f(t)$ factors through $\mathbb{P}^1$, the period of $f(t)$ is π; $v(0) = v(\pi)$ and $(\cos 0, \sin 0) = -(\cos \pi, \sin \pi)$, therefore $v(0) \cdot (\cos 0, \sin 0) = -v(\pi) \cdot (\cos \pi, \sin \pi)$. (Note that $v(t)$ is never 0 in an antiparallel polygon.) This equality can only hold when there are an odd number of cusps in the envelope, since the sign of $v(t) \cdot (\cos t, \sin t)$ changes after each cusp.

Corollary 4.2. *If an antiparallel polygon has an even number of vertices and a continuous bisection envelope, then there must be a perimeter bisector which connects two of its vertices.*

Proof. Consider a polygon P that satisfies the conditions of the hypothesis; however, assume for the sake of contradiction that there are no perimeter bisectors connecting two vertices of P. By Theorem 3.1, cusps between parabolas occur on perimeter bisectors with an endpoint at a vertex of P. Thus, in this case, the number of cusps in the bisection envelope equals the number of vertices of P, which is even. However, since P has a continuous bisection envelope, by Theorem 4.1, there should be an odd number of cusps, a contradiction. Therefore, there must be a perimeter bisector that connects two vertices of P.

4.1. **Non-antiparallel Polygons.** The results described thus far in this section are not necessarily true when a polygon is not antiparallel. We will now consider the non-antiparallel case.

We begin with the setup and all the hypotheses of Lemma 4.1 except allow the polygon P to be non-antiparallel.

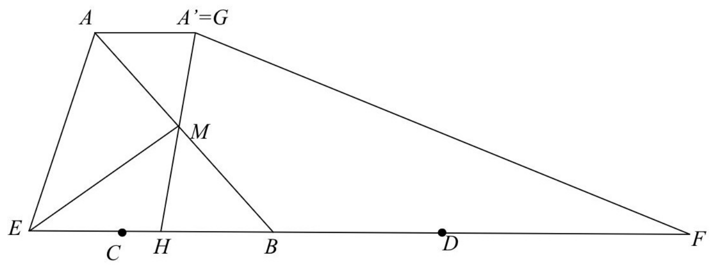

FIGURE 9. Trapezoid $AA'FE$, where $\overline{AA'}$ is parallel to $\overline{EF}$. Points A, $A' = G$, C, and D are vertices of a polygon, $\overline{AB}$ and $\overline{A'H}$ are perimeter bisectors of this polygon which intersect at their midpoint M.

So let P be a non-antiparallel polygon. Let A be a vertex of P and define B so that $\overline{AB}$ is a perimeter bisector of P. Furthermore, assume that A is the endpoint of a side that is parallel to side $\overline{CD}$. Let $\overline{AA'}$ be the side of the polygon parallel to $\overline{CD}$. Let us assume that there exists a perimeter bisector with both endpoints on $\overline{AA'}$ and $\overline{CD}$. Without loss of generality, we can assume that rotating $\overline{AB}$ clockwise results in the perimeter bisectors with endpoints on $\overline{AA'}$ and $\overline{CD}$. We will show that if the bisection envelope is continuous through $\overline{AB}$, then $\overline{AB}$ must still be an angle bisector. However, at $\overline{AB}$, $E(P)$ may or may not have a cusp.

First we show that $\overline{AB}$ is still an angle bisector. By Theorem 3.1, the bisection envelope resulting from perimeter bisectors with endpoints on $\overline{AA'}$ and $\overline{CD}$ will be the midpoint of $\overline{AB}$, which we label M. Let us extend $\overline{CD}$ and the side with an endpoint at A that is not $\overline{AA'}$ and let their intersection point be E. By Lemma 3.1, $AE/BE = BM/AM$. Since M is the midpoint of AB, $AE/BE = 1$. Thus, ΔAEB is isosceles. Because $\angle EAB = \angle ABE$ and $\angle A'AB$ and $\angle ABE$ are alternate interior angles, $\overline{AB}$ is an angle bisector.

We will now show that the sign change described in Lemma 4.1 occurs for some non-antiparallel cases, but not for others.

Let $\overline{GH}$ be the first perimeter bisector that has a vertex as an endpoint when rotating the perimeter bisectors clockwise starting at $\overline{AB}$, labeled so that G is on AA' and H is on CD. Note that GH intersects M.

For now, assume that G and H are not both vertices of P as we will address that situation later in this subsection. Then there are two cases to consider: Either G is a vertex or H is a vertex.

Suppose G is a vertex of the polygon, meaning that $G = A'$. See Figure 9. In this case, we claim that the sign of $v(t) \cdot (\cos t, \sin t)$ still changes as we pass through M. Note that a parabola is generated both before and after the endpoints of the perimeter bisectors travel along $\overline{AA'}$ and $\overline{BH}$: the parabola generated beforehand results from $\angle AEB$, and the parabola generated afterwards results from $\angle A'FB$. Since E and F are on opposite sides of both $\overline{AB}$ and $\overline{A'H}$, when moving towards M and when emerging from M, the mouths of the parabolas generated will change sides of the

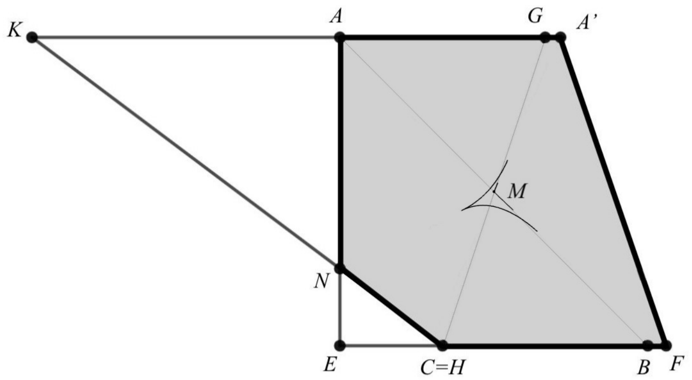

Figure 10. The example described in 4.1.1.

perimeter bisectors with respect to the direction $(\cos t, \sin t)$. As a result, the sign of $v(t) \cdot (\cos t, \sin t)$ reverses.

Now suppose H is a vertex of the polygon (and not G). When H is a vertex of the polygon, it must coincide with C. Let us denote the point generated by the intersection of $\overline{AA'}$ and the side with an endpoint at C that is not parallel to $\overline{AA'}$ as K.

Example 4.1.1. We present a pentagon in which the sign of $v(t) \cdot (\cos t, \sin t)$ does not change from one parabolic arc to another because the vertices of the angles defining parabolic sections before and after the endpoints of the perimeter bisectors travel along $\overline{AG}$ and $\overline{BC}$ are on the same side of $\overline{AB}$ and $\overline{GC}$, by assigning actual coordinates to the labeled points in Figure 10 as follows.

$$\begin{array}{llll} A=(-1,1) & A'=(\frac{13}{30},1) & B=(1,-1) & C=H=(-\frac{1}{3},-1) \\ F=(\frac{47}{42},-1) & E=(-1,-1) & G=(\frac{1}{3},1) & K=(-3,1) \\ & M=(0,0) & N=(-1,-\frac{1}{2}) & \end{array}$$

Here, our pentagon is $ANCFA'$, as illustrated in the above diagram. We leave it to the reader to verify that $\overline{AB}$ and $\overline{GC}$ are perimeter bisectors of pentagon $ANCFA'$ and that the bisection envelope is continuous through M (which means checking that $KG = KC$ and $AE = EB$).

Example 4.1.2. We will now give an example of a non-antiparallel pentagon P which has a bisection envelope that contains two parabolic sections that meet along a perimeter bisector that connects two of its vertices, but not at a cusp.

Let $P = NXMYZ$ be the pentagon where:

$$\begin{array}{lll} N=(1,0) & X=(-1,-1) & M=(-1,0) \\ \multicolumn{2}{l}{Y=(\frac{\sqrt{5}-1}{3+2\sqrt{3}+\sqrt{5}}-1, \frac{2\sqrt{5}-2}{3+2\sqrt{3}+\sqrt{5}})} & Z=(\frac{2\sqrt{15}-2\sqrt{3}}{3+2\sqrt{3}+\sqrt{5}}+1, \frac{2\sqrt{5}-2}{3+2\sqrt{3}+\sqrt{5}}) \end{array}$$

The pentagon and its bisection envelope are illustrated in Figure 11: We claim that $\overline{MN}$ is a perimeter bisector of P and $E(P)$ is continuous through $\overline{MN}$. Furthermore, $v(t) \cdot (\cos t, \sin t)$ does not change sign through $\overline{MN}$.

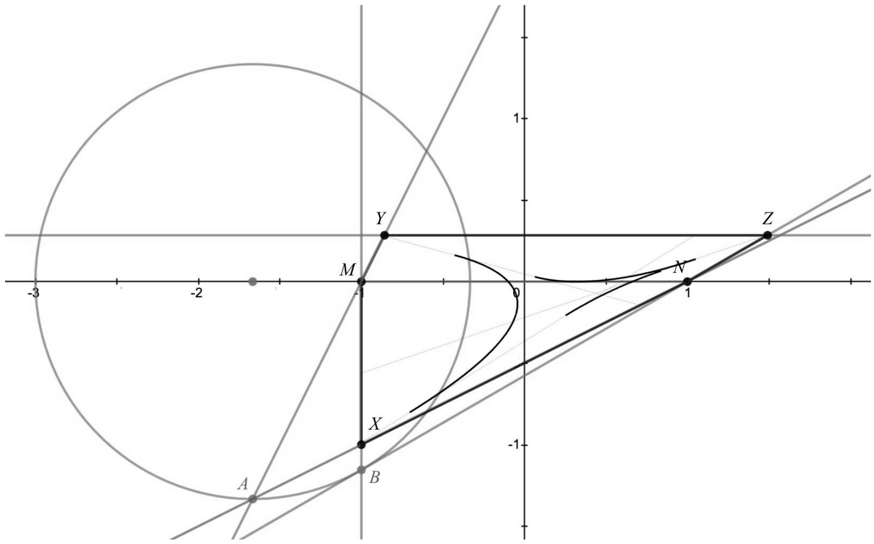

Figure 11. The example described in 4.1.2.

Let $A \equiv (x_a, y_a)$ be the intersection of $\overleftrightarrow{MY}$ and $\overleftrightarrow{NX}$. Similarly, let $B \equiv (x_b, y_b)$ be the intersection $\overleftrightarrow{MX}$ and $\overleftrightarrow{NZ}$.

For $E(P)$ to be continuous through $\overline{MN}$, we require that $AN/AM = BN/BM$, by Lemma 4.1. The locus of points Q for which $QN/QM = AN/AM = BN/BM$ is one of the Apollonian circles associated with M and N, that is, a circle with equation $(x - \frac{1+f^2}{1-f^2})^2 + y^2 = \frac{4f^2}{(f^2-1)^2}$, where $f = AN/AM = BN/BM$. We constructed our pentagon by choosing $f = 2$, then picked two nice points on this circle, namely $A = (-5/3, -4/3)$ and $B = (-1, -\frac{2\sqrt{3}}{3})$. We then set X to be the intersection of $\overline{NA}$ and $\overline{MB}$. Finally, we found Y and Z on $\overleftrightarrow{MA}$ and $\overleftrightarrow{NB}$, respectively, so that Y and Z share their y-coordinates and so that $\overline{MN}$ would be a perimeter bisector of the resulting pentagon.

We note that, by Lemma 3.1, the parabolas generated by the angles at A and B do not form a cusp because A and B are on the same side of $\overleftrightarrow{MN}$.

5. Continuity for Triangles and Quadrilaterals

5.1. **Triangle Case.**

Proposition 5.1. *If the bisection envelope of a triangle is continuous through a perimeter bisector that has an endpoint on a vertex, then that vertex is the apex of an isosceles triangle.*

Proof. Consider triangle ABC. Suppose that $\overline{AP}$ is a perimeter bisector and the bisection envelope is continuous through $\overline{AP}$. By Lemma 4.1, $\overline{AP}$ is an angle bisector. By the angle bisector theorem, there exists a constant k such that $BP = k \cdot AB$ and $CP = k \cdot AC$. Since $\overline{AP}$ is also a perimeter bisector, we must have $AB + BP = AC + CP$, that is $AB + k \cdot AB = AC + k \cdot AC$, hence $AB = AC$.

Corollary 5.1. *The bisection envelope of a triangle is continuous if and only if the triangle is equilateral.*

Proof. By Proposition 5.1, every vertex is the apex of an isosceles triangle, therefore, that triangle must be equilateral.

5.2. **Quadrilateral Case.** Here, we apply our results about the continuity of bisection envelopes to prove Theorem 1.1. We also determine exactly which convex quadrilaterals have continuous bisection envelopes. But first, we state and prove a technical fact about isosceles trapezoids which we will invoke twice in the sequel.

Lemma 5.1. *If all angle bisectors of an isosceles trapezoid are perimeter bisectors, then the trapezoid must be a square.*

Proof. Consider an isosceles trapezoid $ABDC$, labeled so that $\overline{CD}$ is parallel to $\overline{AB}$ and $CD \leq AB$. Let $\overline{CE}$ be a perimeter bisector. Note that E is on $\overline{AB}$ since $AB \geq CD$. Because each direction corresponds to a unique perimeter bisector, $\overline{CE}$ must be the angle bisector at C. Similarly, if $\overline{DG}$ is a perimeter bisector, then it is the angle bisector at D and G is on $\overline{AB}$. Because $\overline{CE}$ and $\overline{DG}$ are perimeter bisectors, $CD = EG$, and, by symmetry, $CGED$ is a rectangle, and $\overline{CE}$ and $\overline{DG}$ intersect at their midpoint, which we label H. Observe that $m\angle ECA = m\angle DCE = m\angle CEA$. Hence ΔACE is isosceles. Similarly, ΔBDG is isosceles. Thus, $AE = AC = BD = BG$.

Let $a = AE$ and $c = CD$. Let $\theta = m\angle CAB = m\angle DBA$. Then $AB = 2a - c$ and $c = a(1 - \cos\theta)$. Because ΔACE is isosceles, the angle bisector at A passes through H and reintersects the boundary of the trapezoid at a point we label J. Note that J must be on $\overline{BD}$ since $AB \geq CD$. By hypothesis, $\overline{AJ}$ is a perimeter bisector; therefore, $AB + BJ = 2a$, so $BJ = c$. Applying the law of sines to ΔABJ, we obtain

$$\frac{1 - \cos(\theta)}{\sin(\theta/2)} = \frac{1 + \cos(\theta)}{\sin(\pi - 3\theta/2)},$$

whose only solutions, up to multiples of 2π, are $\theta = \pi/2, 3\pi/2$. The solution $\theta = 3\pi/2$ is extraneous and the solution $\theta = \pi/2$ shows that our trapezoid is a square.

Proof. [Proof of Theorem 1.1] We split this proof into three cases based on the number of pairs of parallel sides in the quadrilateral.

Case 1: No Pairs of Parallel Sides
Consider a convex quadrilateral which has no pairs of parallel sides and angle bisectors that bisect its perimeter. We will show that the only such quadrilaterals are the kites with three equal acute angles. If none of the perimeter bisectors connect vertices of our quadrilateral, then by Lemma 4.1, its bisection envelope must be continuous. However, by Corollary 4.2 there must be an odd number of cusps. Since we are in an antiparallel case, all four angle bisectors must be cusps, a contradiction. Therefore, there must exist a perimeter bisector connecting two vertices of this quadrilateral.

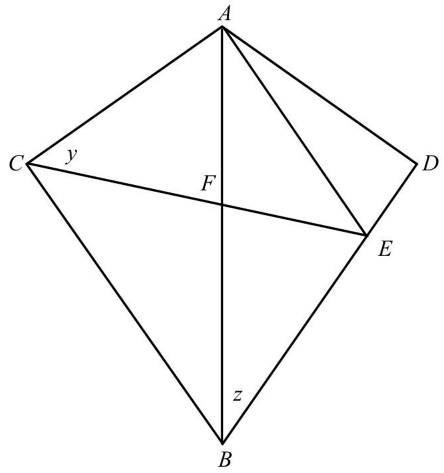

FIGURE 12. A kite used in the proof of Case 1 of Theorem 1.1.

Remark 5.1. *In searching for an elementary proof of Theorem 1.1, our stumbling block was deducing the existence of a perimeter bisector that connected two vertices. So it is in showing that such a perimeter bisector exists that we seem to need our general results on bisection envelopes.*

So we may assume that our quadrilateral has a perimeter bisector that connects two of its vertices. Since this perimeter bisector must also be an angle bisector, the quadrilateral must be either a kite or a parallelogram. In the current case, we are considering quadrilaterals with no pairs of parallel sides, so we determine that our quadrilateral must be a kite.

We will now show that this kite must have three equal acute angles. Consider a kite $ACBD$ labeled so that $\overline{AB}$ is an angle bisector. Let $\overline{CE}$ be the angle bisector at C, and let F be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$. By swapping the labels of A and B if necessary, we may assume that E is on $\overline{BD}$. See Figure 12. Since $\overline{CE}$ and $\overline{AB}$ are angle bisectors, we have $\frac{AC}{BC} = \frac{AF}{BF} = \frac{EF}{CF}$; since $\overline{CE}$ and $\overline{AB}$ are also perimeter bisectors, we must have $AC = BE$. Therefore, $\Delta FBC \sim \Delta FAE$, so $\frac{AC}{BC} = \frac{AE}{BC}$, which shows $AE = AC$. So, $m\angle CEB = \pi - m\angle CED = \pi - (m\angle CEA + m\angle AED) = \pi - (y + 2y)$. Since $m\angle ECB = y$ and the angles of a triangle sum to π, we conclude that $m\angle EBC = 2y$ and our quadrilateral has three equal angles, namely B, C, and D. Furthermore, note that these three angles must be acute, since otherwise E would be on $\overline{AD}$ ($\overline{CE}$ is a perimeter bisector and if the angles are obtuse, then $CB + BD$ is less than half the perimeter of $ACBD$).

We now show that any kite with 3 equal acute angles has angle bisectors which are also perimeter bisectors. So consider the same setup as above, except assume that the angles at C, B, and D all have the same acute measure, and that $\overline{CE}$ is an angle bisector. Then $BC/AC = BF/AF$, and since $\overline{AB}$ is an angle bisector, $BC/BE = CF/EF$. Furthermore, since $\Delta ACB \sim \Delta BCE$, $BC/BE = BC/AC$. As a result, $CF/EF = BF/AF$, thus, $\Delta ACF \sim \Delta BFE$, and since ΔBCF is isosceles, $\Delta ACF \cong \Delta EBF$, so $AC = BE = AD$. We note that, since AB is a perimeter bisector, $AC + BC$ is half of the perimeter of $ACBD$. Thus, $BC + BE$ is also half of the perimeter of $ACBD$, making $\overline{CE}$ a perimeter bisector.

We have therefore determined that for quadrilaterals having no pairs of parallel sides, having angle bisectors which are also perimeter bisectors is equivalent to being a kite with three equal acute angles.

Case 2: One Pair of Parallel Sides
Consider a trapezoid $ABDC$ labeled so that $\overline{AB}$ is parallel to $\overline{CD}$ and such that $CD < AB$. Assume that all of its angle bisectors are also perimeter bisectors. Let E be the point where the angle bisector at C intersects the boundary of the trapezoid. By assumption $\overline{CE}$ is also a perimeter bisector, and since $CD < AB$, E must be on AB. By similar reasoning, if F is the intersection of the angle bisector at D with the boundary of the trapezoid, then F is on $\overline{AB}$. Observe that $m\angle FDB = m\angle CDF = m\angle DFB$ and $m\angle ECA = m\angle DCE = m\angle CEA$; therefore, ΔACE and ΔFDB are isosceles triangles. Since CE and DF are perimeter bisectors of $ABDC$, both $AC + AE$ and $BD + BF$ are half the perimeter; therefore $AC = BD$ and $ABDC$ is an isosceles trapezoid. By Lemma 5.1, we see that our trapezoid must be a square, which is a contradiction (to having only one pair of parallel sides).

Case 3: Two Pairs of Parallel Sides
In all parallelograms, the diagonals are perimeter bisectors, but are angle bisectors only when the parallelogram is a rhombus. Conversely, the diagonals of a rhombus are always perimeter and angle bisectors.

It is not the case that the equivalent statements in Theorem 1.1 are equivalent to the quadrilateral having a continuous bisection envelope. Instead, we have

Theorem 5.1. *Let Q be a convex quadrilateral. The following are equivalent:*

- *The bisection envelope of Q is continuous.*
- *Q is either a kite with three congruent acute angles or a parallelogram.*

Proof. We shall split into three cases depending on the number of pairs of parallel sides in the quadrilateral.

Case 1: No Pairs of Parallel Sides.
Consider a convex quadrilateral which has no pairs of parallel sides and a continuous bisection envelope. By Corollary 4.2 this quadrilateral must have a perimeter bisector connecting two of its vertices.

Let us therefore consider a quadrilateral $ACBD$ that has a perimeter bisector connecting two vertices A and B. Because $\overline{AB}$ is a perimeter bisector, $AC + CB = AD + DB$, that is, C and D are on an ellipse with foci A and B. We shall show that continuity of the bisection envelope through $\overline{AB}$ implies that $ACBD$ is a kite (or a parallelogram, although here we are considering

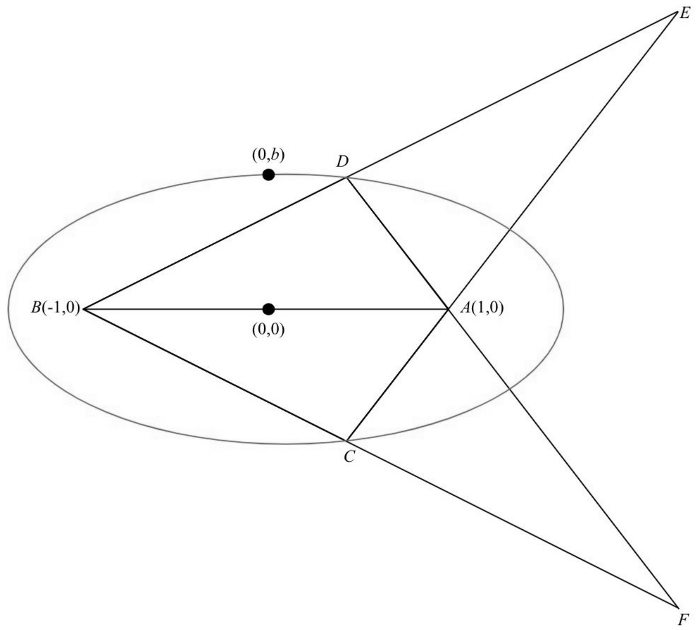

FIGURE 13. An illustration of Case 1 in the proof of Theorem 5.1.

the case with no parallel sides). Let $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$ intersect at E, and let $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect at F. By Lemma 3.1, $AE/AF = BE/BF$. By the law of sines, this is equivalent to $\frac{\sin\angle DBA}{\sin\angle ABC} = \frac{\sin\angle CAB}{\sin\angle BAD}$, or $\sin\angle DBA \sin\angle BAD = \sin\angle ABC \sin\angle CAB$.

We claim that the only points P on the ellipse that satisfy

$$\sin\angle DBA \sin\angle BAD = \sin\angle ABP \sin\angle PAB$$

are D and its images under symmetries of the ellipse. To show this, by the symmetry of the ellipse, it suffices to show that $\sin\angle ABP \sin\angle PAB$ is strictly monotone as P travels over a quadrant of the ellipse. In particular, place $ACBD$ in an xy-coordinate plane so that the ellipse defined by the points X such that $AX + XB = AD + DB$ has major axis the x-axis and minor axis the y-axis and so that $A = (-1, 0)$ and $B = (1, 0)$. Let the highest point on the ellipse be $(0, b)$. See Figure 13.

Let $s(x, y) = \sin\angle ABP \sin\angle PAB$, where $P = (x, y)$, so

$$s(x, y) = \frac{y^2}{\sqrt{((x+1)^2 + y^2)((x-1)^2 + y^2)}}.$$

Define $x(t) = \sqrt{b^2+1}\cos t$ and $y(t) = b\sin t$ so that $(x(t), y(t))$ is a parameterization of the ellipse. We will show that $s(x(t), y(t))$ is strictly increasing for $t \in [0, \pi/2]$. We simplify our computations by showing, instead, that $g(t) \equiv \frac{1}{s(x(t),y(t))^2}$ is strictly decreasing for $t \in [0, \pi/2]$.

We compute that

$$\begin{aligned}\frac{dg}{dt} &= \frac{4x(x^2+y^2-1)}{y^4}(-\sqrt{b^2+1}\sin t) + \frac{8x^2-4x^4-4x^2y^2-4y^2-4}{y^5}(b\cos t)\\ &= -\frac{4b^3\cos t}{y^5}(b^2+\sin^2 t).\end{aligned}$$

Because $b, y > 0$ and $\cos t > 0$ for $t \in (0, \pi/2)$, we see that $dg/dt < 0$ for $t \in (0, \pi/2)$, thus $ds/dt > 0$ for $t \in (0, \pi/2)$, as desired.

This establishes that there are at maximum four solutions to the equation

$$\sin\angle DBA \sin\angle BAD = \sin\angle ABP \sin\angle PAB,$$

namely D and its images under the symmetries of the ellipse. If D is not on the minor axis of the ellipse, then two of these solutions are degenerate, one, namely the point symmetric to D in the center of the ellipse, corresponds to a parallelogram, and one, namely the point mirror symmetric to D in the major-axis, corresponds to a kite. If $D = (0, b)$, then there are only two solutions, one degenerate and one corresponding to a rhombus. In this case, we are assuming that our quadrilateral has no parallel sides, so we conclude that this quadrilateral is a kite.

Note that the perimeter bisector with one endpoint at C must have its other endpoint on the interior of a side (since C is not on the minor axis). Since the bisection envelope is continuous, by Lemma 4.1, this perimeter bisector must be an angle bisector. Hence all angle bisectors of our kite are also perimeter bisectors, and so by Theorem 1.1, we conclude that our kite must have three equal acute angles.

Case 2: One Pair of Parallel Sides.
We will now show that the bisection envelope of a quadrilateral containing one pair of parallel sides (i.e. a trapezoid) is never continuous.

Consider a trapezoid $ABDC$ labeled so that $\overline{AB}$ is parallel to $\overline{CD}$ and such that $CD < AB$; we disregard the case where $AB = CD$ since we consider parallelograms in a later section.

We claim that an endpoint of one of the perimeter bisectors from vertex C or D must be on $\overline{AB}$. Let E be the other endpoint of the perimeter bisector from C. If our claim is false, we must have E on $\overline{BD}$. If $BE > CD$, then the perimeter bisector at D would have its other endpoint on $\overline{BD}$ as well, which is impossible. But if $BE \le CD$, since $AB > CD$, the other endpoint of the perimeter bisector at D must be on $\overline{AB}$, as desired. By relabeling if necessary, we may assume that E is on $\overline{AB}$. Let F be the point on $\overline{CD}$ which is furthest from C such that the perimeter bisector with endpoint F has its other endpoint on $\overline{AB}$. Note that $G = A$ or $F = D$.

These two possibilities are illustrated in Figure 14 and Figure 15. Let H be the intersection of $\overline{CE}$ and $\overline{FG}$. Note that H is the midpoint of both $\overline{CE}$ and $\overline{FG}$ since $\overline{CE}$ and $\overline{FG}$ are diagonals of parallelogram $CFEG$. By the proof of Theorem 3.1, the bisection envelope generated by perimeter bisectors between $\overline{CE}$ and $\overline{FG}$ will be H. By Lemma 3.1, if the bisection envelope of this trapezoid is to be continuous, $AE/AC = CH/EH = 1$, hence ΔCAE is isosceles and $\overline{AH}$ is an angle bisector. Since $\overline{CE}$ is a perimeter bisector, $2AC$ is half of the perimeter of trapezoid $ABDC$.

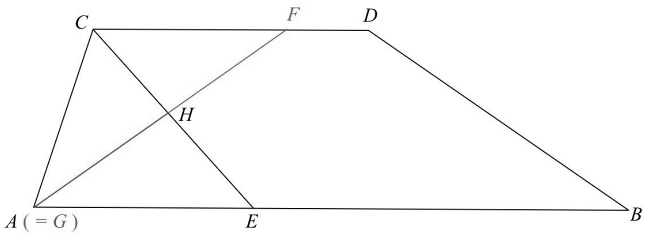

Figure 14. Trapezoid $ACBD$ with perimeter bisectors $\overline{CE}$ and $\overline{FG}$ where $A = G$, as appears in the proof of Theorem 5.1.

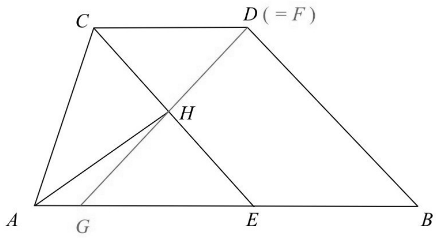

Figure 15. Trapezoid $ACBD$ with perimeter bisectors $\overline{CE}$ and $\overline{FG}$ where $D = F$, as appears in the proof of Theorem 5.1.

Suppose $G = A$ (see Figure 14). Then $2AC = FD+DB+BA \geq FE+EA$. However, $FE = AC = AE$, so $F = D$ and $E = B$, implying that $ABDC$ is a parallelogram, which we are not allowing here.

So assume $F = D$ (see Figure 15). Then $BD + BG = 2AC$. By Lemma 3.1, $BD = BG$, thus, $BD = BG = AC = AE$, and the trapezoid is isosceles. By Lemma 4.1, all of the angle bisectors of this isosceles trapezoid must also be perimeter bisectors. We therefore apply Lemma 5.1 to determine that this isosceles trapezoid must be a square, which we address in the next section. Thus, there are no quadrilaterals with one pair of parallel sides that have continuous bisection envelopes.

Case 3: Two Pairs of Parallel Sides.
The bisection envelope of a quadrilateral containing two pairs of parallel sides is always continuous, since the bisection envelope will be a point: by Theorem 3.1, since in a parallelogram, all perimeter bisectors pass through the intersection of the parallelogram's diagonals, its bisection envelope is a point.

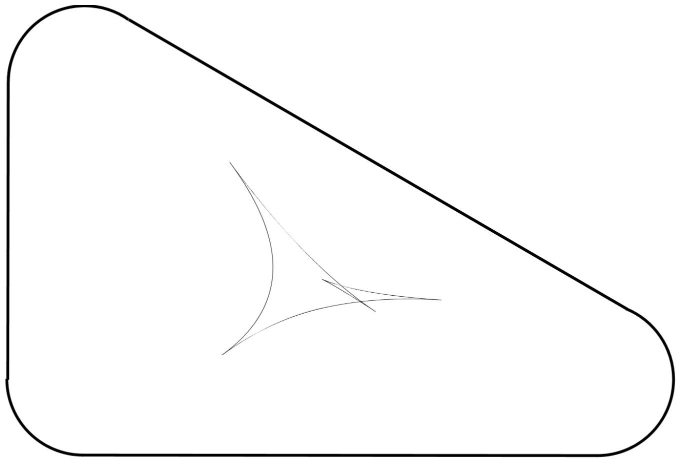

Figure 16. A triangle with rounded corners with its bisection envelope to illustrate an example of a region with a continuously differentiable boundary.

6. Conclusion

It may seem that Theorem 1.1 should have a proof that uses purely Euclidean methods. Note that the quadrilaterals in our theorem all have a perimeter bisector that connects two vertices, and, once we establish this fact, the remainder of our proof is Euclidean. So if one could provide a Euclidean proof of the existence of a perimeter bisector that connects two vertices (or, if one could find a proof that avoids determining the existence of such a perimeter bisector as an intermediary step), a purely Euclidean proof would be possible. However, we deduced the existence of such a perimeter bisector by studying properties of the envelope generated by the family of perimeter bisectors. Indeed, we think the argument involving envelopes is not only a crucial feature of the present work, but also rather satisfying.

Perimeter bisectors for triangles have been studied and it is known that the three perimeter bisectors from each vertex of a triangle are concurrent. In [5], Todd considered the envelope of the perimeter bisectors of a triangle, but his motivation was to find lines that simultaneously bisect the area and the perimeter of the triangle. In [1], Berele and Catoiu recognize the envelope of the perimeter bisecting lines of a triangle to be a union of three parabolic arcs and reveal many of its properties.

Of further interest might be a study of the cusps of a bisection envelope of a finite convex region with continuously differentiable boundary, more specifically in knowing what the conditions are that produce a cusp in the continuously differentiable case. In this case, the bisection envelope is always continuous. See Figure 16 for an example. We conclude by providing formulas for the bisection envelope in the continuously differentiable case. Let the boundary of the region be given by $C : [0, L] \to \Pi$, where C is continuously differentiable, $C(0) = C(L)$, and the parameter corresponds to arc length (so that L is the perimeter of the region). The perimeter bisectors are the line segments joining $C(s)$ with $C(\frac{L}{2} + s)$, so we define $P(s) \equiv C(\frac{L}{2} + s) - C(s)$. Then the perimeter bisector through

$C(s)$ is parameterized by $C(s)+xP(s)$ and the perimeter bisector through $C(s+t)$ is parameterized by $C(s+t)+yP(s+t)$. The intersection of these two perimeter bisectors is found by solving the system of linear equations $C(s)+xP(s)=C(s+t)+yP(s+t)$, then substituting the solution for x or y into the appropriate parameterization. Note that because our region is convex, both x and y will be positive (in fact, between 0 and 1). Let $E:[0,L]\to\Pi$ parameterize the bisection envelope. Using these considerations, we compute

$$\begin{aligned}
E(s) &= \lim_{t\to 0}(C(s)+\frac{|(C(s+t)-C(s))\times P(s+t)|}{|P(s)\times P(s+t)|}P(s))\\
&= C(s)+\lim_{t\to 0}\frac{\frac{d}{dt}|(C(s+t)-C(s))\times P(s+t)|}{\frac{d}{dt}|P(s)\times P(s+t)|}P(s) \quad \text{(by L'Hôpital's rule)}\\
&= C(s)+\lim_{t\to 0}\frac{|(C(s+t)-C(s))\times P'(s+t)+C'(s+t)\times P(s+t)|}{|P(s)\times P'(s+t)|}P(s)\\
&= C(s)+\frac{|P(s)\times C'(s)|}{|P(s)\times P'(s)|}P(s)
\end{aligned}$$

Note that $P(s)\times C'(s)$ can never be 0 because neither $P(s)$ nor $C'(s)$ is zero and the perimeter bisector at $C(s)$ cannot be tangent to the boundary (for, otherwise, our shape would have to be the line segment that connect $C(s)$ with $C(s+L/2)$. Therefore, $P(s)\times C'(s+L/2)=-P(s+L/2)\times C'(s+L/2)$ points in the opposite direction to $P(s)\times C'(s)$. Thus,

$$|P(s)\times(C'(s+\frac{L}{2})-C'(s))|=|P(s)\times C'(s)|+|P(s)\times C'(\frac{L}{2}+s)|,$$

and therefore

$$\begin{aligned}
E(s) &= C(s)+\frac{|P(s)\times C'(s)|}{|P(s)\times P'(s)|}P(s)\\
&= C(s)+\frac{|P(s)\times C'(s)|}{|P(s)\times(C'(s+\frac{L}{2})-C'(s))|}P(s)\\
&= C(s)+\frac{|P(s)\times C'(s)|}{|P(s)\times C'(s)|+|P(s)\times C'(\frac{L}{2}+s)|}P(s).
\end{aligned}$$

Note that this last expression shows that we may think of the bisection envelope in the continuously differentiable case as given by Lemma 3.1 applied to the angle whose sides are the tangent lines at the endpoints of each perimeter bisector.

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