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GENERALIZED ARCHIMEDES ANGLE DIVISION

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Abstract. Trisection is an often maligned topic, but we will show how it leads to a new angle division method using a *neusis*-type geometric construction, motivated by the result on trisection of an angle by Archimedes. This new method generates a new set of integers based on division numbers of the form $2^n + 1$, n a positive integer, which is different from both the set of constructible integers, and the set of integers associated with the q-gons constructible with the *neusis*. Construction methods of the associated regular q-gons of the new angle division numbers are investigated, including examples.

We begin with a review of the Archimedes method for solving the Puzzle of trisecting an angle. Archimedes solved this Puzzle with perhaps the cleverest method of all, using a *neusis* argument with a marked straightedge and a compass [1],[2]. A new method for solving this Puzzle which does not require a marked straightedge is then given. However, it is still similar to the Archimedes method. The advantage of this new method is that it can be generalized to angle division numbers larger than 3.

We will assume angles are acute, unless stated otherwise.

Puzzle. *Given an acute angle $\angle ABC$, a straightedge, and a compass, determine a method to trisect $\angle ABC$.*

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Solution of Archimedes.

Assume we are given an acute angle $\angle ABC$, a straightedge, and a compass.

Construct circle c with center B and radius AB, Figure 1.

Mark the length AB on the end of the straightedge r as DE.

Place the straightedge r on point A with point D on the extension m of side BC back through B, such that we have the order $D - E - A$ on r, Figure 1.

Move point D on line m until point E is on circle c (this is copy r' of r).

Then the triangle $\Delta D'E'B$ is an isosceles triangle, since $D'E' = E'B$, so $\angle E'D'B = \angle E'BD'$.

By the Exterior Angle Theorem, $\angle AE'B = 2\angle E'D'B$.

Also, $\Delta E'AB$ is isosceles, since $E'B = AB$, so $\angle AE'B = \angle E'AB$.

Again, by the Exterior Angle Theorem, $\angle ABC = \angle AD'B + \angle E'AB$.

But $\angle E'AB = 2\angle E'D'B = 2\angle AD'B$, hence $\angle ABC = 3\angle AD'B$.

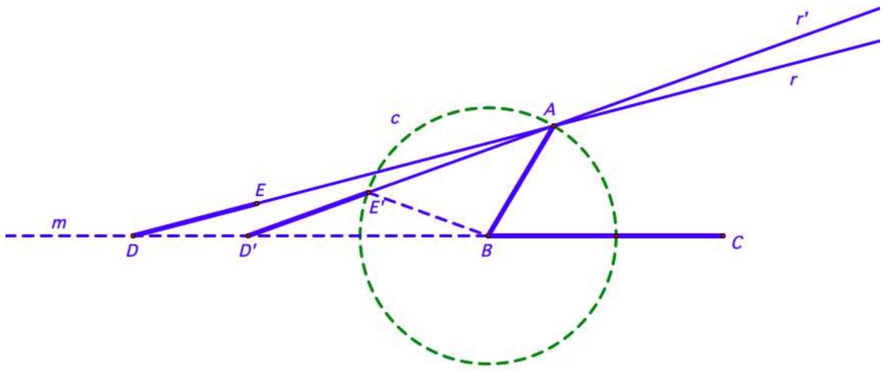


FIGURE 1

Using Dynamic Geometry software the method of Archimedes can be duplicated on a PC. This software made it possible to develop a new variation of Archimedes' solution which uses a compass and straightedge, instead of a compass and a marked straightedge [2],[3]. It is still a *neusis*-type of geometric construction, as the radii of the circles are fixed, but the center of one circle moves with the straightedge, so one of the circles takes the place of the marks on the straightedge.

New Solution.

Given $\angle ABC$, a straightedge, and a compass, choose a point D on AB, and construct the parallel line m to side BC on point D, as shown, Figure 2 Top.

Choose a point E on m within the angle $\angle ABC$, and construct the straightedge EB, so that EB divides $\angle ABC$.

Construct circle c centered on point D with radius DB.

At the intersection F of c with BE construct another circle d centered at F, with radius DF.

Then we have the order $B - F - E$ on line BE.

Note that $DB = DF$, and the circles c and d are congruent throughout the construction, even though the center F of circle d moves on c with the motion of point E on m.

Let G be the intersection, other than D, of circle d with line m.

Move point E on line m until it coincides with point G, Figure 2 Bottom.

Now $\angle DEB = \angle EBC$, since $m//$BC.

Also, $\angle DBF = \angle DFB = 2\angle DEB = 2\angle EBC$, by the Exterior Angle Theorem, since triangles ΔDBF and ΔDFE are isosceles.

Hence $\angle ABC = 3\angle EBC$.

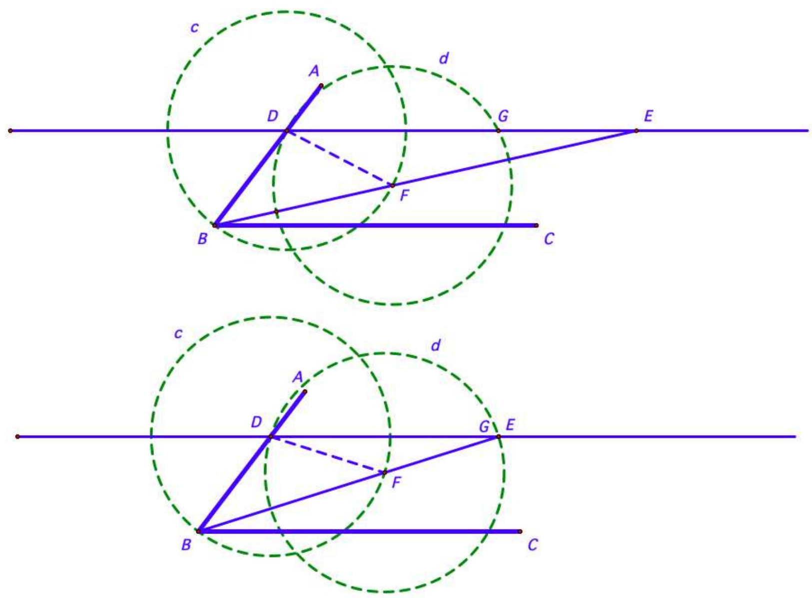


FIGURE 2

Both methods work on angles $\angle ABC$ up to size 135°.

The New Solution construction led to the generalization of the Archimedes solution, the new result, see Theorem below.

The idea behind this construction is this: If we can add one more circle, and one more triangle to ΔDBE in the construction above, it would give a division of $\angle ABC$ by 5, since we're doubling the angles and then adding one more of the angles in each case.

The technique is still *neusis* in character, but uses circles to advance the argument, instead of a marked straightedge.

What we are proposing is a geometric argument which uses the division formula $2^n + 1$, for n a positive integer.

We explore the method for the case $n = 2, 2^n + 1 = 5$, with a demonstration.

Example 1. *Let $\angle ABC$ be a given acute angle. We will show how to divide $\angle ABC$ by 5, which will help illustrate how to generalize the division method for numbers of the form $2^n + 1$, $n \geq 1$.*

As in the New Solution, if $\angle ABC$ is a given acute angle, choose a point D on AB, and construct a parallel line m to BC on D.

Let E be a point on m such that EB divides $\angle ABC$, Figure 3 Top.

Construct a circle c centered on D with radius DB, and let F be the point of intersection of c with EB.

Construct a second circle d on F with radius DF.

Move E off to the right if needed, and let G be the intersection of circle d with BE, so that we have the order $B - F - G - E$ on BE.

Construct a third circle e centered on G with radius DG.

Notice how the base side of the previous isosceles triangle becomes the short side of the next isosceles triangle.

Let H is the intersection of circle e with line m, so we have the order $D - H - E$ on line m.

Now move point E to coincide with point H, Figure 3 Bottom.

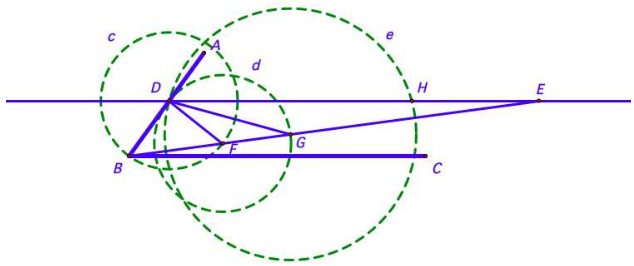
The centers F and G, of the circles d and e, will move with E.

Then $\angle EBC = \angle DEB$, since $m//$BC.

Also, $\angle DGB = 2\angle DEB = 2\angle EBC$, by the Exterior Angle Theorem, since ΔEDG is isosceles.

Similarly, since ΔDBF and ΔDFG are isosceles, $\angle DBE = \angle DFB = 2\angle DGF = 4\angle EBC$.

Hence $\angle ABC = 5\angle EBC$.



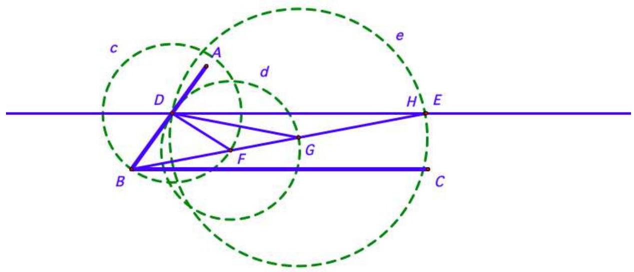


FIGURE 3

The new angle division method is now clear. By induction it can be shown that, by this new angle division method, the number $2^n + 1$ divides a given acute angle $\angle ABC$, when n is a positive integer.

Theorem. *By the new angle division method,* $2^n + 1$ *divides any acute angle* $\angle ABC$*, if* n *is a positive integer.*

Proof.
By the Puzzle, New Solution, we know the method works for $n = 1$. The number of circles needed for this result is 2.

Assume $2^{n-1} + 1$ divides $\angle ABC$, for $n > 1$, where line m is on point D, D on AB, and $m//BC$, as above.

Then there are n circles used to show that $\angle DBK = 2^{n-1}\angle KBC$, where K is the point on m corresponding to point E in Example 1.

Thus $\angle ABC = (2^{n-1} + 1)\angle KBC$.

Move point K off to the right on m if needed, and add another circle to the n circles, using the intersection point L of the nth circle with KB as the center point, and DL as the radius.

Move the intersection point K on m to coincide with the point M, the intersection point of this new $(n + 1)$th circle z with m.

The previous set of n nested triangles will be unchanged in their relationship with each other, and the formula $\angle ABC = (2^{n-1} + 1)\angle KBC$ still holds when the point K is moved on m.

This is because each of the isosceles triangles has its equal sides equal to the radius of one of the constructed circles, so when the circles center is moved the triangle remains isosceles.

The base side of the previous isosceles triangle becomes the short side of the next isosceles triangle.

We will denote this new point K by K', Figure 4.

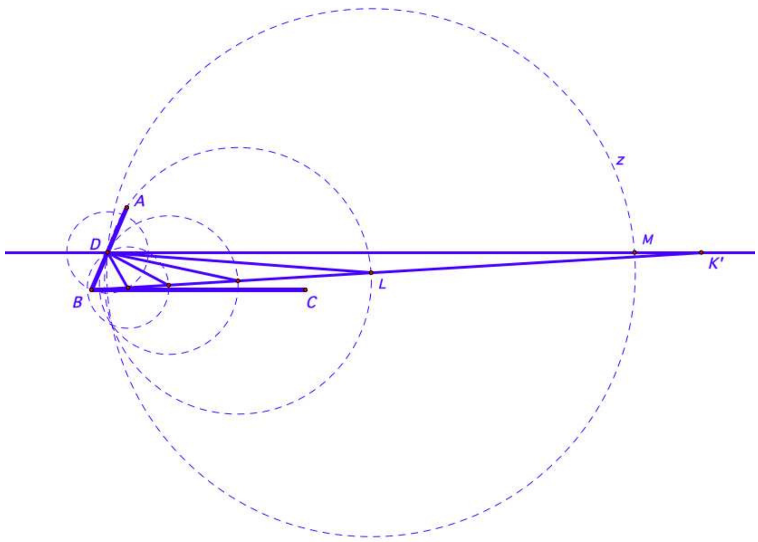


Figure 4

Then, since this $(n+1)$th circle z adds one more multiple of angles, and $\angle DK'L.= \angle LDK'$, by the Exterior Angle Theorem on the related isosceles triangle, we have $\angle DBK' = 2(2^{n-1})\angle DK'L = 2^n\angle DK'L = 2^n\angle DK'B$ since $\angle DK'B = \angle DK'L$.

But $\angle DK'B = \angle K'BC$, thus $\angle ABC = \angle DBK' + \angle K'BC = 2^n\angle K'BC + \angle K'BC = (2^n+1)\angle K'BC$.

We will indicate how this new angle division is related to the construction of regular polygons. Since these polygons are not constructible, we use Transformation Geometry.

Given a $90°$ angle $\angle ABC$, where AB and CB are unit length sides, let c be the unit circle about B. If we make a division of $\angle ABC$ by 2^n+1, $n \geq 1$, we get an angle $\angle EBC$, where the point E is on c, Figure 5.

Let $\angle FBC$ equal 4 copies of $\angle EBC$ formed by reflection, or rotation, then F is on c.

Thus $m\angle FBC = 4m\angle EBC$, and $(2^n+1)m\angle EBC = 90°$.

We then have a division of c by 2^n+1, since $(2^n+1)m\angle FBC = (2^n+1)4m\angle EBC = 4(2^n+1)m\angle EBC = 360°$.

These $\angle FBC$ sections are then the *division units* which determine the sides of the regular (2^n+1)-gon by reflection or rotation.

Connect the intersection points C and F of the $\angle FBC$ division units to obtain the sides of the (2^n+1)-gon.

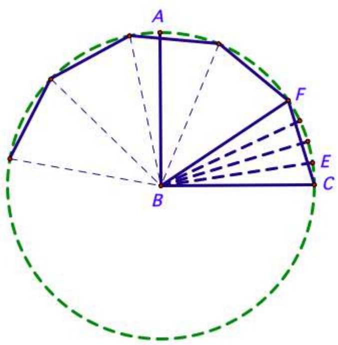


FIGURE 5

For example, we can use trisection to determine a 3-gon (and also a 4-gon, 6-gon, and 12-gon).

Let $\angle ABC$ be a $90°$ angle, with AB, and BC unit segments, and construct the unit circle c centered at B, as shown, Figure 6 Left. Of course we could have constructed a $30°$ angle, but we are demonstrating the general division method here.

Divide $\angle ABC$ by 3, using the method in Puzzle, New Solution, to obtain angle $\angle EBC$, E on c.

Then reflect or rotate $\angle EBC$ to obtain the angle $\angle A'BC$, where $\angle A'BA = \angle EBC$.

Since m$\angle ABC = 3m\angle EBC = 90°$, and we know that $\angle A'BC = 4\angle EBC$, we have $3m\angle A'BC = 3(4m\angle EBC) = 4(3m\angle EBC) = 360°$.

Thus 3 of the $\angle A'BC$ division units cover the entire circular region.

Connect the points C, A', and D', and hide the radii line segments and the circle, to obtain the 3-gon $CA'D'$, Figure 6 Right.

Connect the other points as shown to get the other regular polygons in Figure 6 Right.

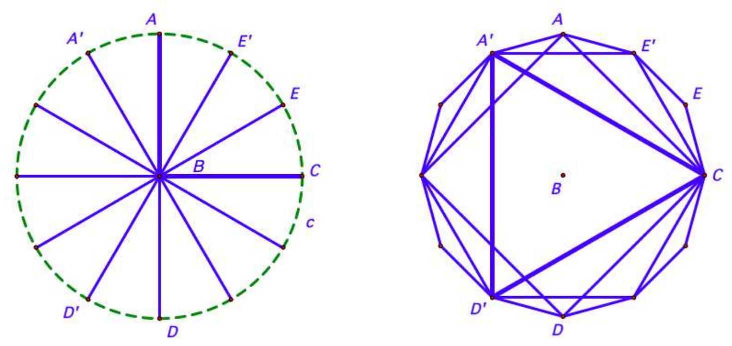


FIGURE 6

By the technique in the above example, we cannot only divide angles by numbers of the form $2^n + 1$, for $n \geq 1$, but we can also divide angles by products of numbers of this type, $(2^m + 1)(2^n + 1), m, n \geq 1$. This also includes powers of these numbers, $(2^m + 1)^r, m, r \geq 1$.

The n-bisections by numbers which are powers of 2 can be combined with these products: $2^n(2^m + 1)(2^q + 1)$, for $m, q \geq 1, n \geq 0$.

The general form of the divisors of this type is therefore: $2^n p[1]p[2]...p[m]$, $m \geq 1, n \geq 0$, where each of the distinct factors $p[k]$ has the form $(2^s + 1)^t$, where $s, t \geq 1$.

We call this new method, which uses these new numbers as divisors, the *Generalized Archimedes Angle Division Method* (GAADM for short).

The standard division by the product $(2^m + 1)(2^n + 1)$ is accomplished by dividing the given angle $\angle ABC$, by $2^m + 1, m \geq 1$, to obtain $\angle EBC$, and then divide this angle by $2^n + 1, n \geq 1$, to obtain $\angle FBC$.

Thus $\angle ABC = (2^m + 1)\angle EBC = (2^m + 1)(2^n + 1)\angle FBC$.

Clearly, the product of the division operations is commutative.

For example we can divide an angle $\angle ABC$ by 25, by first dividing the angle $\angle ABC$ by 5 to get $\angle EBC$, as in Example 1 above.

Then divide $\angle EBC$ by 5 to get the desired result, say $\angle FBC$, such that $\angle ABC = 25\angle FBC$.

We can also divide an angle $\angle ABC$ by 11, even though 11 does not have the form $2^n + 1$, for some n. First divide $\angle ABC$ by $2^5 + 1 = 33$, to get $\angle EBC$, then we can join 3 of these small angles $\angle EBC$ together to form a new division unit angle $\angle FBC$ which determines the division of $\angle ABC$ by 11 of these $\angle FBC$ units.

Other divisors, such as 13 (use 65 = 5 x 13) and 19 (use 513 = 27 x 19), are possible to construct by similar arguments. Division by other primes, such as 23, 29, . . . , is theoretically possible.

Division by 7 is not possible, since $(2^n{+}1)\text{mod}7$ is equal to either 2, 3, or 5, for all $n \geq 1$.

For a general example, consider the divisor $(2^1)(2^1+1)(2^2+1) = 30$. The division of $\angle ABC$ is accomplished by bisecting the given angle to obtain the angle $\angle DBC$, Figure 7. Then divide angle $\angle DBC$ by 3 to obtain the angle $\angle EBC$, and last divide angle $\angle EBC$ by 5, to get $\angle FBC$.

This last angle $\angle FBC$ then satisfies $\angle ABC = 30\angle FBC$.

Similar multiplication techniques allow us to divide angles by some multiples of 25.

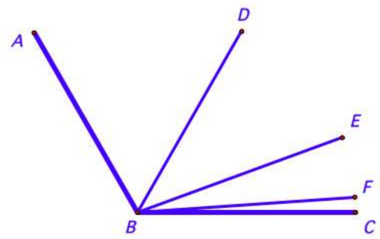


FIGURE 7

Example 2. *To demonstrate the Generalized Archimedes Angle Division Method and associated $q-gon$ construction for a nontrivial case we construct a heptadecagon (17-gon), in honor of Carl Friedrich Guass, who was the first to construct this polygon in 1796, using a Euclidean construction.*

First we divide a right angle $\angle ABC$ by 17, Figure 8.

We will need to construct 5 circles.

Choose a point D on AB, and construct a line m parallel to BC through D. Then Choose a point E on line m in the interior of $\angle ABC$, and connect E to B.

Move E off to the right if necessary, and construct a circle c_1 with center D and radius DB.

At the intersection F of the circle c_1 with BE construct a 2nd circle c_2 with center F and radius DF.

At the intersection G of this circle c_2 with BE, where $B{-}F{-}G$, construct a 3rd circle c_3 with center G and radius DG.

At the intersection H of this circle c_3 with BE, where $B - F - G - H$, construct a 4th circle c_4 with center H and radius DH.

At the intersection J of this circle c_4 with BE, where $B - F - G - H - J$, construct a 5th circle c_5 with center J and radius DJ.

Move E to the intersection K of circle c_5 with the parallel line m so that we have $B - F - G - H - J - K$, and connect D to F, G, H, and J, respectively.

We know $\angle EBC = \angle DEB$, because $m//BC$.

Also, $\angle DJB = 2\angle DEB = 2\angle EBC$ by the Exterior Angle Theorem, since ΔDJE is isosceles.

Similarly, since ΔDHJ, ΔDGH, ΔDFG and ΔDBF are isosceles, we have $\angle DBF = \angle DFB = 2\angle DGB = 4\angle DHB = 8\angle DJB = 16\angle DEB = 16\angle EBC$, all by the Exterior Angle Theorem, plus the equality above.

Thus $\angle ABC = \angle DBE + \angle EBC = \angle DBF + \angle EBC = 16\angle EBC + \angle EBC = 17\angle EBC$.

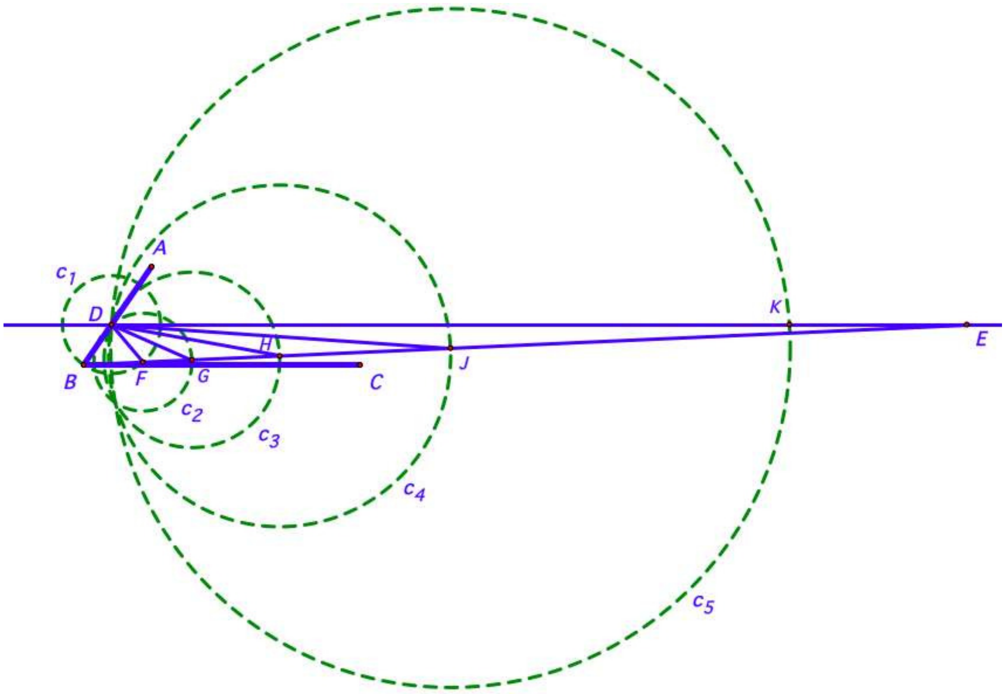


Figure 8

Construction of the associated regular 17-gon then follows as in the examples above.

Let $\angle ABC$ be a right angle, with AB and CB unit segments, and construct a unit circle c about the center point B.

Then let $\angle EBC$ be the division of $\angle ABC$ by 17, E on c, Figure 9 Left.

Form the division unit $\angle PBC$ from 4 copies of $\angle EBC$ by reflections or rotations.

Thus $\angle PBC = 4\angle EBC$, $\angle ABC = 4\angle PBC + \angle EBC = 17\angle EBC$, and $17m\angle PBC = 360°$.

Connect C and P to obtain a side of the 17-gon.

Reflect or rotate ΔPBC by iteration around the circle to get the sides of the 17-gon, Figure 9 Right.

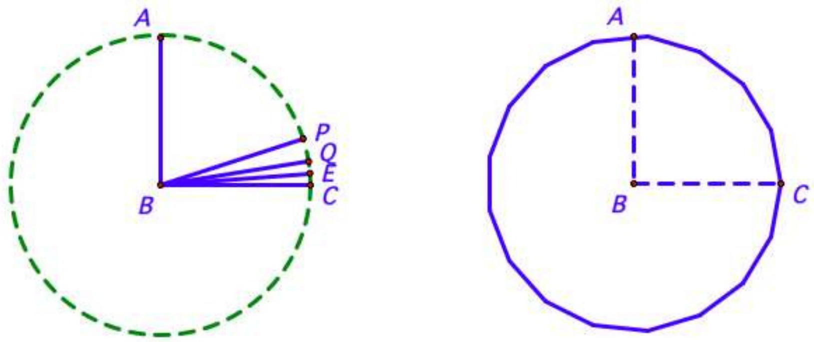


FIGURE 9

Conclusions.

This is a partial list of the division numbers which create q-gons using the numbers $2^n + 1, n \geq 1$, and their multiples, but excluding bisections:
3, 5, 9, 15, 17, 25, 27, 33, 45, 51, 65, 75, 81, 85, 99, 114, ...

If we include bisections $2^n, n \geq 0$, and multiples in the above set of numbers, we have numbers of the general form $2^n p[1]p[2]...p[m]$, for $m \geq 1, n \geq 0$, where each of the distinct $p[k]$ terms has the form $(2^s + 1)^t, s, t \geq 1$.

Here is the general list of all the numbers above, with the allowed multiples, plus bisections. We denote this set by GAADM: 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 17, 18, 20, 24, 25, 27, 30, 32, 33, 34, 36, 40, 45, 48, 50, 51, 54, 60, 64, 65, 66, 68, 72, 75, 80, 81, 84, 85, 90, 96, 99, 100, 102, ...

Notice that this set of numbers contains the number 25, and some multiples of 25. These numbers are on the list of open question integers for the *neusis* constructions [4], [5].

The numbers in the following list are the *constructible integers* such that the q-gons are derived from divisors constructed using a compass and straight edge. They are of the type: $2^n p[1]p[2]...p[m], m \geq 1, n \geq 0$, where each of the $p[k]$ terms are distinct Fermat Primes [4], OEIS A003401: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, ...

The following is the set of numbers which result from the *neusis* constructions of polygons [5]. OEIS A122254: 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 42, 44, 45, 48, 51, 52, 54, 55, 56, 57, 60, 63, 64, 65, 66, 68, 70, 72, 73, 74, 76, 77, 78, 80, 81, 84, 85, 88, 90, 91, 95, 96, 97, 99, 102, ...

The following is the list of those numbers which correspond to polygons which are not constructible with *neusis* [5], OEIS A048136: 23, 29, 43, 46, 47, 49, 53, 58, 59, 67, 69, 71, 79, 83, 86, 87, 89, 92, 94, 98, 103, 106, 107, ...

The set of numbers denoted by GAADM contains the set of constructible integers, except for the number 1, since these Fermat Primes are numbers of the form $2^n + 1$, for some n.

Even if we add the primes 11, 13, 19, 23, . . . , and their multiples to the set of numbers denoted by GAADM, based on our discussion above, this set appears to be contained in the set of numbers which result from the *neusis* constructions of polygons listed above [5], OEIS A122254, except for the number 25 and the multiples of 25. The validity of this observation is not known at this time.

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