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AN ANALYSIS OF MELZAK'S CONJECTURE IN TAXICAB SPACE

DEBRA MIMBS GLADDEN, WILLIAM FULFORD, SAMUEL D. GRUBER, and ALEXANDRA SQUIRES

Abstract. An open optimization problem is minimizing a polyhedron's total edge length with respect to its volume. Melzak conjectured that the polyhedron with the smallest total edge length for a given volume is the equilateral triangular prism. However, Melzak only examined this problem in Euclidean geometry. This paper extends the analysis of total edge length to the Non-Euclidean metric known as Taxicab Geometry. We analyze the total edge length of multiple polyhedra in Taxicab space. Our primary method of calculation is through Lagrange multipliers. We find that the triangular prism is not the edge-length minimizer in Taxicab Geometry and posit that the cube is the total edge length minimizer in this space.

1. Introduction

The shortest polyhedron problem, or Melzak's conjecture, asks a seemingly simple question: what polyhedron in $\mathbb{R}^3$ of unit volume will have the shortest total edge length? The smallest known edge length is that of an equilateral triangular prism, approximately 11.896, but this has not been definitively proven [5]. However, this conjecture has only been thoroughly considered in Euclidean spaces, and has never been considered in taxicab space. This is certainly a topic worth delving into, as taxicab geometry directly affects length and in turn changes Melzak's conjecture. Euclidean geometry is undoubtedly the system that most naturally describes the organic world, but taxicab geometry has relevant and helpful applications in describing the manmade world of buildings and city grids. This paper will make an introductory analysis of the shortest polyhedron problem in taxicab geometry and its fascinating implications.

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After investigating the cube, triangular prism, cylinder, sphere, and tetrahedron, we conclude that the cube will be the polyhedron of unit volume that has the smallest total edge length in taxicab space. There are an endless amount of polyhedra to investigate in order to prove what the minimum is. So it has generally been the practice in furthering research on this conjecture to focus on solids that could reasonably have small edge length. In taxicab geometry, many of the solids that have been deemed relevant to the problem now either no longer exist or will have much larger edge length than they previously did in Euclidean space, resulting in changes to Melzak's conjecture.

2. Taxicab Geometry

Taxicab Geometry[1] garners its name from examining the manner in which a taxicab traverses the blocks of a city. Taxicabs are not permitted to cross through city blocks, as they are blocked by buildings and parks, etc. Thus, we redefine distance, instead of using Euclidean distance, to fit with the idea of a taxicab traversing only in lines that are parallel to coordinate axes.

Definition 1 (Taxicab Distance). *Define the distance, d between two points $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ...q_n)$ by $d(p, q) = \sum_{i=1}^{n} |q_i - p_i|$.*

Graphically, this may be viewed as the sum of the lengths of the projections of the line segments between the points onto coordinate axes.

Proposition 2.1. *Given a line segment, $\overline{pq}$, from point p to point q, the taxicab length of $\overline{pq}$, does not depend upon the reflection of $\overline{pq}$ about coordinate axes nor upon its translation.*

Proposition 2.2. *The taxicab length of $\overline{pq}$ depends upon the rotation of $\overline{pq}$ from the coordinate axes.*

Example 2.1. *Let $p, q \in \mathbb{R}^2$, where $p = (0, 0)$ and $q = (1, 0)$. Then the length of $\overline{pq}$, i.e. $d(p, q) = 1$. Rotating this line segment about the origin $45°$ yields the line segment connecting p to the point $r = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, so that $d(p, r) = \sqrt{2}$.*

Due to the change in distance, a triangle in the plane will necessarily have at least one side with a longer measurement in taxicab geometry as opposed to Euclidean geometry. Additionally, rotating a triangle is likely to change the lengths of its sides within taxicab geometry itself. This can be seen from the previous example. If $\overline{pq}$ were a side of a triangle, and that triangle were rotated where $\overline{pq}$ became $\overline{pr}$, then the length of the side of the triangle will have changed. This behavior we want to eliminate by first rotating a polygon into a standard position before calculating lengths of sides (and any other properties depending on lengths of sides such as perimeter, area, or volume).

Definition 2. *A taxicab standard position for a polyhedron is one entirely in the first octant which minimizes the volume of the polyhedron based on all possible rotations of that polyhedron.*

[1] A nice discussion of the properties of taxicab geometry is given by Krause [9], and an exposition of regular polygons is given in [13].

Note that it is possible for polyhedra to have multiple standard positions. For instance rotating a rectangle already in a standard position by $\frac{\pi}{2}$ results in another standard position due to the sides being again parallel to the coordinate axes.

While length does not remain constant from Euclidean geometry to taxicab geometry, the formula for area of a triangle (and, consequently, the formula for volume of its associated prism) will translate directly [11], assuming the figure is in a standard position. (This is not necessarily true for all solids.) This is because the triangle's area and volume are solely dependent on vertical and horizontal measurements that do not change when moved to taxicab space.

Additionally, the definition of a triangle is not dependent on length in any way. As such, a nonspecific triangle undergoes no changes that impact the calculation of length when moved to taxicab space. This is not the case with other figures, as the rotation of a figure can affect its area (or volume).

Theorem 2.1. *Rotations do not preserve area in taxicab geometry.*

Example 2.2. *Let S be the unit square in Euclidean geometry as seen in Figure 1. (Notice S is in a standard position.) As the sides of the S are parallel to the coordinate axes, the square in taxicab geometry has the same area as in Euclidean geometry, i.e. $A(S) = 1$. Let S' be the square found by rotating S about its center by $\frac{\pi}{4}$. The length of each side of this square is now $\sqrt{2}$, and $A(S') = 2$.*

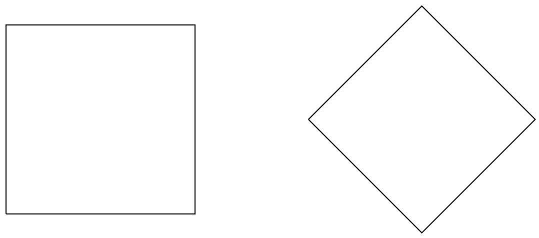

FIGURE 1. Two squares. The second is a rotation of the first by $\frac{\pi}{4}$. Consequently, these figures are of equal Euclidean area (in fact are congruent) but of different Taxicab area. [11]

It is not merely rotations that cause a difficulty in taxicab geometry. Some properties that are considered straightforward in Euclidean geometry (such a regularity) are troublesome in taxicab geometry.

Definition 3. *A polygon of n sides is said to be regular in taxicab geometry if it has n equal angles and n sides of equal taxicab length.*

Hanson showed that in taxicab geometry equilateral triangles (triangles with equal angles and equal taxicab length sides) no longer exist. If two sides of a triangle are the same length, this forces the third side to be of different length. As a consequence, equilateral triangular prisms likewise do not exist [7]. It is immediately clear then that Melzak's conjecture will not hold in taxicab geometry.

Unlike a nonspecific triangle, taxicab geometry changes the shape of a circle, as a circle is defined in terms of distance.

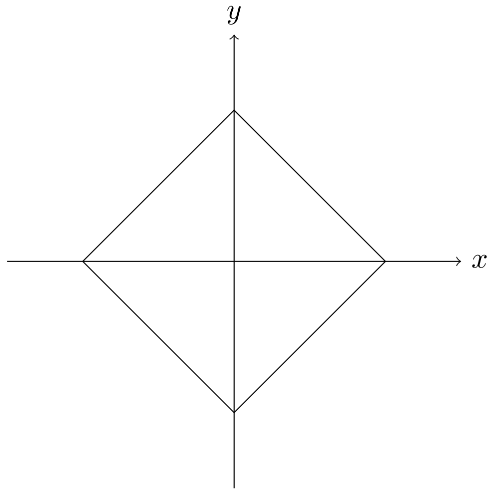

Figure 2. The unit circle in taxicab geometry

Definition 4. *(Taxicab Unit Circle) The unit circle in taxicab geometry is represented by* $|x| + |y| = 1$.

This is what some would call a diamond (i.e. a rotated square). (See Figure 2.) Any other figure whose definition relies directly on a certain distance will be translated in a distinct manner when moved to taxicab geometry.

Example 2.3. *The unit circle has four sides each with length* 2*, so that the circumference of the circle is* 8*. In fact, for any circle,* $|x| + |y| = r^2$*, centered at the origin, the length of each side is* $2r$*, where* r *is the radius of the circle. Thus, the circumference is* $8r$*, showing that* $\pi = 4$ *in taxicab geometry.*

3. Melzak's Conjecture

Melzak's Conjecture seeks the polyhedron with minimal total edge length for a given volume, or the polyhedron which will minimize the ratio $R = \frac{P}{\sqrt[3]{V}}$, where P is the perimeter and V is the volume. Besicovitch and Eggleston have shown that, in Euclidean space, the cube has the smallest edge length of the regular polyhedra [1], but the equilateral triangular prism has a smaller edge length than any of the platonic solids [5]. It has further been shown that in Euclidean space, the equilateral triangular prism is the edge length minimizer for all triangular prisms [2]. Melzack conjectures that the equilateral triangular prism is indeed the edge length minimizer for all polyhedra in Euclidean space [10], but this has not been definitely proven. Coxeter and Toth analyzed edge length in spherical and cylindrical spaces, but their analysis was not on total minimizers or polyhedra of unit volume [4].

Indeed, it seems as though every solid must be analyzed individually to discover how to translate it into taxicab in a way that meets its definition as well as how to find an accurate way to represent its area and volume. Janssen discusses several types of taxicab areas and volumes [8], and Çolakoğlu and Kaya investigate the volume of a tetrahedron [3]. From their research it is clear that there are multiple ways to interpret every unique solid and derive its volume. Some shapes cannot exist in taxicab space, such as an equilateral triangle [7]. However, other more complicated solids are possible in taxicab that are not in Euclidean, and many of them fall into two classes of solids that are mutually exclusive in Euclidean. For example, the taxicab cylinder, sphere, and paraboloids are classified as polyhedra. Thus, it can be seen that the switch between metrics introduces entirely new classes of polyhedra to Melzack's problem.

In considering a certain type of polyhedron and finding the minimum edge length it achieves, we use the method of Lagrange multipliers. The condition that must be satisfied to use Lagrange multipliers is merely that the partial derivatives are continuous. This is true for each of our volume and edge length formulas. Each value found using this method must either be a maximum or minimum value. For each solid we only find one value, and to confirm that it is indeed a minimum we must only find one possible greater value, which in this case clearly exists.

3.1. **Rectangular Prisms.** The cube is the edge length minimizer of the regular polygons in Euclidean space [1] and is unchanged in its transition to taxicab space. In Euclidean geometry the triangle is considered a basic shape; it is the shape with the least number of sides, and every regular shape can be comprised of triangles. So in Euclidean space it is reasonable that the triangle has the least total edge length given a unit area. In taxicab geometry, however, the triangle is now measured solely in terms of its legs, so the triangle is dependent on the corresponding rectangle. In fact, we can now think of every shape being made up of the rectangles that determine the side lengths, as triangles do in Euclidean space. Thus it seems likely that rectangular prisms could be the solid to minimize total edge length for a given volume in taxicab geometry.

Proposition 3.1. *The cube achieves the minimum total edge length of rectangular prisms with a given unit volume.*

Proof. Let R be a rectangular prism with unit volume. Then $P = 4l + 4w + 4h$, where l is the length, w is the width, and h is the height of the rectangular prism. Also, $V = lwh = 1$. Using the method of Lagrange multipliers, we consider $P - \lambda V = 4l + 4w + 4h - \lambda lwh = 0$. Finding critical points yields a system of equations,

$$\begin{cases} 4 = \lambda wh \\ 4 = \lambda lh \\ 4 = \lambda lw \end{cases} = \begin{cases} \lambda = \frac{4}{wh} \\ \lambda = \frac{4}{lh} \\ \lambda = \frac{4}{lw} \end{cases} \tag{1}$$

Solving this system of equations yields $l = w = h = 1$. Therefore, the optimization for the rectangular prisms of unit volume must be a cube of side length 1, giving an edge length of $R = 12.000$. □

Interestingly, the cube has the same edge length value in taxicab space as it does in Euclidean space. This is expected for the cube because the lengths x, y, and z are unchanged lengths.

3.2. **Triangular Prisms.** Although regular triangular prisms do not exit in taxicab space, as they are conjectured to be the total edge length minimizer in Euclidean geometry, a reasonable place to search for the minimizer in taxicab space would be with other triangular prisms.

Proposition 3.2. *Consider the generic triangular prism shown in Figure 3, where e is the portion of the base to the left of the height, a is the portion of the base to the right, b is the height, and h is the z-axis extension. The total edge length minimizer of the triangular prisms with unit volume achieves a total edge length of greater than that of the cube.*

Proof. Let T be a triangular prism with unit volume, as seen in Figure 3. Then $P = 2(a+e) + 3h + 2(e+b) + 2(a+b) = 4e + 4a + 4b + 3h$, and $V = \frac{1}{2}(a+e)bh = \frac{1}{2}abh + \frac{1}{2}ebh = 1$. Using the method of Lagrange multipliers, we consider $P - \lambda V = 4e + 4a + 4b + 3h - \lambda\frac{1}{2}(a+e)bh = 0$. Finding critical points yields a system of equations,

$$\begin{cases} 4 = \lambda\frac{1}{2}bh \\ 4 = \lambda(\frac{1}{2}ah + \frac{1}{2}eh) \\ 3 = \lambda(\frac{1}{2}ab + \frac{1}{2}eb) \end{cases} \tag{2}$$

Solving this system of equations yields $\lambda = 2\sqrt[3]{12} \to a+e = \frac{2\sqrt[3]{12}}{4}, b = \frac{2\sqrt[3]{12}}{4}, h = \frac{2\sqrt[3]{12}}{3}$. Thus, $P = 6\sqrt[3]{12} \approx 13.737$, which is larger than that of the cube. $\square$

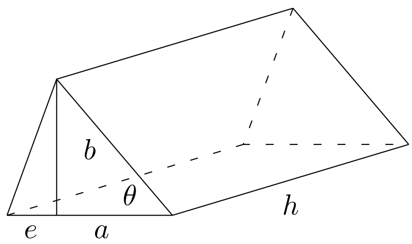

FIGURE 3. A triangular prism - the class containing the conjectured total edge length minimizer in Euclidean space

As a result of Proposition 3.1 and Proposition 3.4, we are able to state the following Theorem.

Theorem 3.1. *In taxicab space, the triangular prism does not minimize edge length as Melzack conjectured for Euclidean space.*

Coxeter and Toth have previously shown that the rankings of polyhedra by Euclidean edge length minimization does not necessarily hold in non-Euclidean space, given a sufficiently large inradius ψ [4]. Our conclusion furthers that argument, and shows that the ranking does not necessarily hold between polyhedra, regardless of the length of their ψ.

In Euclidean space, there exist triangular prisms with both greater and lesser edge length than a cube. For example, a right triangular prism has edge length $R \approx 12.383$.

Consider isosceles triangular prisms with a projection into z equal to the length of its legs l. We find that there exists two prisms with an edge length equal to a cube at $\theta \approx 47.991$ and $\theta \approx 73.375$, where θ is the interior angle between the legs. These triangular prisms have a minimum at $\theta = 60$ [2]. Thus, the triangular prisms with $47.991 < \theta < 73.375$ have an edge length less than that of a cube. This can be seen in Figure 4.[2]

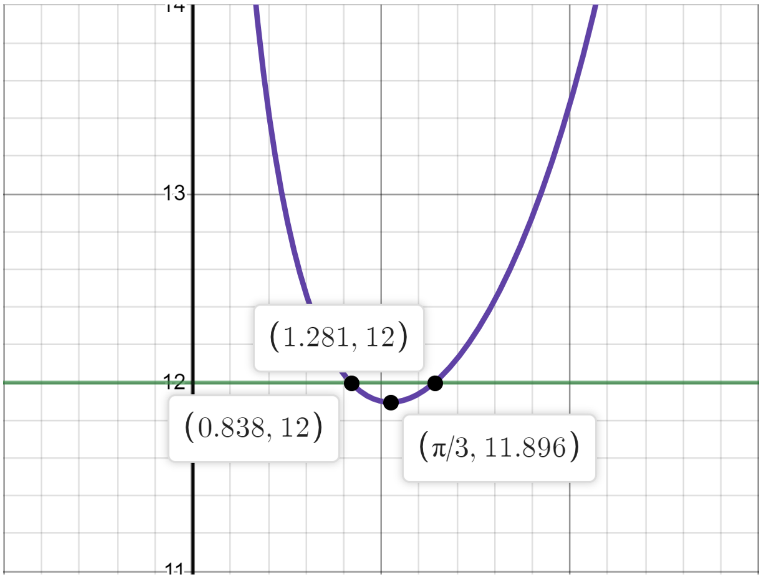

FIGURE 4. The edge length of an isosceles triangular prism with a projection into z equal to its legs, based on the interior angle between the legs.

While the equilateral triangle does not exist in taxicab space, isosceles triangular prisms with a projection into z equal to the leg length and interior angle $47.991 < \theta < 73.375$ exist in both taxicab and Euclidean geometry, but we have already shown that the cube has a smaller edge length than the minimum edge length of a triangular prism in taxicab. Therefore, these prisms have a smaller edge length than a cube in Euclidean geometry, but a greater edge length than a cube in taxicab geometry.

We have shown that Melzak's conjecture does not hold in taxicab space. In order to amend his conjecture to be more appropriate for taxicab geometry, we first explore two more classes of figures - cylinders and spheres.

3.3. **Cylinders.** [3] We define a cylinder as a circle in the $x-y$ plane extended into z, as in Figure 5 As shown by Thompson, the volume of a taxicab cylinder is $V = \frac{1}{2}\pi r^2 l = 2r^2 l$.

[2]The equation for this graph is $y = \frac{7+2\sqrt{2(1-\cos x)}}{\sqrt[3]{\frac{\sin x}{2}}}$, where y is the prism's edge length to volume ratio, and x is the interior angle θ. For ease of calculation, assume the leg length $= 1$. The actual length of the legs is irrelevant, as the ratio $R = \frac{P}{\sqrt[3]{V}}$ is the same for all similar triangular prisms.

[3]For a further exposition of Cylinders see [6].

Interestingly, this volume only differs from the Euclidean volume by a constant multiplier [11].

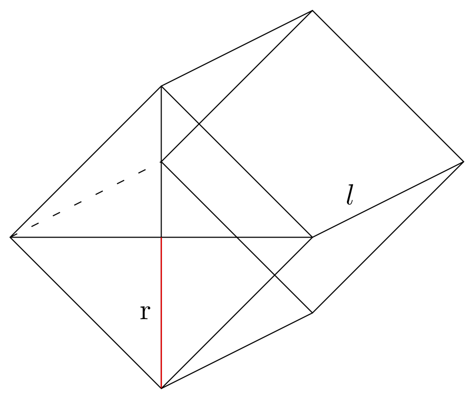

FIGURE 5. A taxicab cylinder

Proposition 3.3. *Consider the generic cylinder shown in Figure 5. The total edge length minimizer of the cylinders with unit volume achieves a total edge length greater than that of the cube.*

Proof. Let C be a taxicab cylinder with unit volume, as seen in Figure 5. Then $P = 8r + 8r + 4l = 16r + 4l$, and $V = 2r^2 l = \frac{1}{2}\pi r^2 l = 1$. Using the method of Lagrange multipliers, we consider $P - \lambda V = -\lambda lwh = 0$. Finding critical points yields a system of equations,

$$\begin{cases} 16 = 4rl\lambda \\ 4 = 2r^2\lambda \end{cases} = \begin{cases} \lambda = \frac{4}{rl} \\ \lambda = \frac{2}{r^2} \end{cases} \tag{3}$$

Solving this system of equations yields $\lambda = 4\sqrt[3]{2}, r = \frac{\sqrt[3]{2}}{2}, andl = \sqrt[3]{2}$. Thus, $P = 12\sqrt[3]{2} \approx 15.119$ which is larger than that of the cube. □

Consequently, the cylinder can be eliminated from consideration of the edge length minimizer.

3.4. **Spheres.** As stated by Çolakoğlu and Kaya, a taxicab sphere[4] has the volume $V = \frac{4}{3}r^3$ [3].

Proposition 3.4. *Consider the generic sphere shown in Figure 6. The total edge length minimizer of the spheres with unit volume achieves a total edge length greater than that of the cube.*

Proof. Let S be a taxicab sphere with unit volume, as seen in Figure 6. Then $P = 24r$, and $V = \frac{4}{3}r^3 = 1$. So that $r = \frac{\sqrt[3]{6}}{2}$, and $P = 24(\frac{\sqrt[3]{6}}{2}) \approx 21.8055$ which is larger than that of the cube. □

[4]This is the most common interpretation of a taxicab sphere but it is not the only one [11].

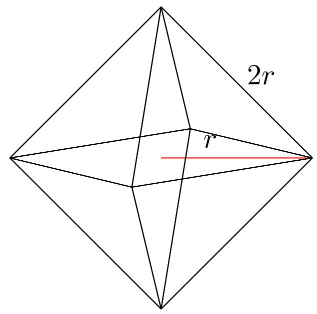

FIGURE 6. A taxicab sphere

Similarly, a square-based tetrahedron will have volume half that of the taxicab sphere. Thus, it is obvious to calculate the edge length for a taxicab hemisphere. Neither of these solids achieve an edge length lower than that of the cube, despite their unique properties.

Conjecture 3.1. *The cube is the total edge length minimizer for given volume in taxicab geometry.*

We will be unable to provide a definitive proof of this at this time, for the same reasons that no proof has been found in Euclidean geometry. Nevertheless, we have proven that the triangular prism is not the edge length minimizer in taxicab as it is in Euclidean, and that the ranking of polyhedra by edge length does not necessarily translate to the same ranking in other, non-Euclidean spaces, such as taxicab.

3.5. **Table of Results.** In Figure 7 we compile the known minimum edge lengths for some relevant polyhedra, adding to the table set forth for Euclidean by Dorf and Hall [5]. Some of the taxicab polyhedra are not considered in Euclidean, and vice versa, denoted by DNE.

4. Areas for Future Research

The edge length of other taxicab polyhedra could also be calculated and compared against the cube. Interested inquirers could also examine minimizing edge length in other non-Euclidean spaces. Coxeter and Toth examine minimization in hyperbolic space and spherical space for polyhedra of a given inradius ψ. This could be generalized beyond a given ψ, or minimization could be examined in other non-Euclidean spaces, such as elliptic space.

Since the ranking of polyhedra by edge length minimization does not necessarily remain between geometric spaces, an analysis ranking polyhedron in non-Euclidean spaces may also yield interesting results. For example, in Euclidean space the platonic solids can be ranked, from least to greatest, as cube, tetrahedron, dodecahedron, octahedron, icosahedron. This does not necessarily hold in non-Euclidean spaces, so the ranking would need to be reevaluated for these spaces.

Polyhedron	**Euclidean**	**approx value**	**taxicab**	**approx tc value**
Platonic Icosahedron	$30\sqrt[3]{\frac{12}{5(3+\sqrt{5}}}$	23.131	DNE	DNE
Octahedron	$12\sqrt[3]{\frac{3}{\sqrt{2}}}$	15.419	$12\sqrt[3]{6}$	21.805
Platonic Dodecahedron	$30\sqrt[3]{\frac{4}{15+7\sqrt{2}}}$	15.217	DNE	DNE
Triangular Pyramid	$9\sqrt[3]{\frac{6}{\sqrt{2}}}$	14.342	DNE	DNE
SquareBase Tetrahedron	$6\sqrt[3]{6\sqrt{2}}$	12.238	$8\sqrt[3]{12}$	18.315
Cube	$12 \cdot 1$	12.000	$12 \cdot 1$	12.000
Triangular Prism	$9\sqrt[3]{\frac{4}{\sqrt{3}}}$	11.896	$6\sqrt[3]{12}$	13.737
Cylinder	DNE	DNE	$12\sqrt[3]{2}$	15.119

Figure 7. A Comparison of Euclidean and Taxicab Edge Lengths: Note that in taxicab space, the platonic solid forms of the Platonic icosahedron, dodecahedron, and triangular pyramid (which the Euclidean space calculation is based on) do not exist since there are no regular taxicab triangles or regular taxicab pentagons.[7] Additionally, A taxicab sphere is an Octahedron.[11]

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DEPARTMENT OF MATHEMATICAL SCIENCES
LEE UNIVERSITY
CLEVELAND, 37320 TN, US

E-mail address: dgladden@leeuniversity.edu

E-mail address: wfulfo00@leeu.edu

E-mail address: sgrube00@leeu.edu

E-mail address: asquir00@leeu.edu