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# 100 CHARACTERIZATIONS OF TANGENTIAL QUADRILATERALS

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**Abstract.** This is the third part in our extensive study of new characterizations of tangential quadrilaterals, preceded by [18, 19]. Here we shall prove 24 more necessary and sufficient conditions for when a convex quadrilateral can have an incircle, making it 100 known such characterizations.

## 1. A basic theorem

We concluded and gave references in our previous paper [19] that we at that time knew of 76 published characterizations of tangential quadrilaterals. These vary from simple ones to quite complex configurations. Since we will use the following theorem six times in the present paper, we start by proving this basic necessary and sufficient condition. (Despite its simplicity, it was not included among the 76 characterizations we referred to.) First, let us define what we mean by a tangential quadrilateral:

*A convex quadrilateral is called* tangential *if and only if it has an interior point that is equidistant from its four sides.*

In case such a point exist, it is called the *incenter* $I$, its distance to the four sides is called the *inradius* $r$, and a circle with center $I$ and radius $r$ is called the *incircle* (the same concepts used in a triangle).

**Theorem 1.1.** *A convex quadrilateral, where two opposite angle bisectors intersect at an interior point, is tangential if and only if this point is equidistant to two opposite sides.*

**Proof.** Suppose the angle bisectors at $A$ and $C$ intersect at an interior point $I$ in a convex quadrilateral $ABCD$. Let $E$, $F$, $G$, $H$ be the projections of $I$ on $AB$, $BC$, $CD$, $DA$ respectively (see the left part of Figure 1). By assumption, we know (for instance) that $IE = IG$. Triangles $AEI$ and $AHI$ are congruent (AAS), as are triangles $CFI$ and $CGI$. Hence $IH = IE = IG = IF$, so $ABCD$ is tangential by definition.

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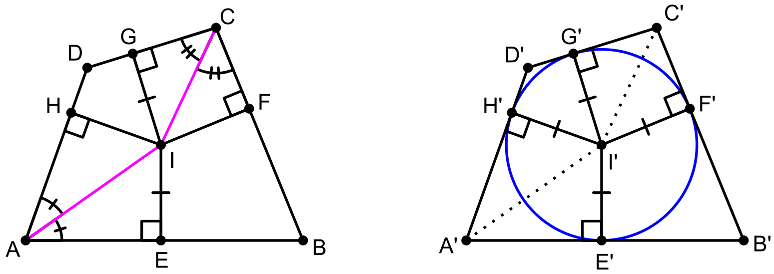


Figure 1. Two pairs of congruent triangles each

Conversely, if $A'B'C'D'$ is a tangential quadrilateral, let $I'$ be an interior point that is equidistant to the sides: $I'E' = I'F' = I'G' = I'H'$ (see the right part of Figure 1). Then triangles $A'E'I'$ and $A'H'I'$ are congruent (RHS), as are triangles $C'F'I'$ and $C'G'I'$, so $\angle E'A'I' = \angle H'A'I'$ and $\angle F'C'I' = \angle G'C'I'$. Hence the angle bisectors at $A'$ and $C'$ intersect at $I'$ (which is a well-known property of tangential quadrilaterals). □

Let's apply this characterization right away. The 'if' part of the following theorem was a problem solved at [6].

**Theorem 1.2.** *In a convex quadrilateral $ABCD$ where the angle bisectors at $A$ and $C$ intersect at the interior point $I$, let $E$, $F$, $G$, $H$ be the projections of $I$ on the sides $AB$, $BC$, $CD$, $DA$ respectively. Suppose that $EG$ and $FH$ intersect at $J$, and that points $K$ and $L$ are on the sides $AD$ and $BC$ respectively such that $KL$ is perpendicular to $IJ$. Then $HK = FL$ if and only if $ABCD$ is a tangential quadrilateral.*

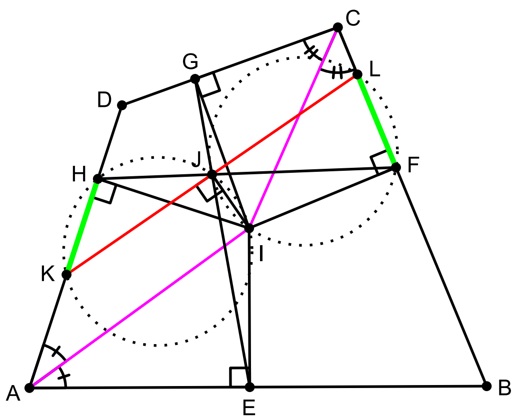


Figure 2. $ABCD$ is tangential $\Leftrightarrow$ $HK = FL$

**Proof.** Using the law of sines, we have (see Figure 2)

$$\frac{HK}{\sin \angle HJK} = \frac{HJ}{\sin \angle HKJ}, \qquad \frac{FL}{\sin \angle LJF} = \frac{JF}{\sin \angle FLJ}.$$

Thus

$$\begin{aligned}\frac{HK}{FL} &= \frac{HJ}{\sin\angle HKJ}\cdot\frac{\sin\angle HJK}{\sin\angle LJF}\cdot\frac{\sin\angle FLJ}{JF}\\ &= \frac{HJ}{\sin\angle HIJ}\cdot\frac{\sin\angle JIF}{JF}\\ &= \frac{HI}{\sin\angle HJI}\cdot\frac{\sin\angle FJI}{FI} = \frac{HI}{FI}\end{aligned}$$

where we used that $HKIJ$ and $FLJI$ are always cyclic quadrilaterals. Hence

$$HK = FL \quad \Leftrightarrow \quad HI = FI$$

and the last equality is valid if and only if $ABCD$ is a tangential quadrilateral according to Theorem 1.1. □

Next we have a configuration with three exterior triangle incircles.

**Theorem 1.3.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides $AB$, $CD$ and $BC$, $DA$ intersect at $J$ and $K$ respectively. Assume the vertices are labeled such that $JK$ lies outside of $D$. Suppose the angle bisectors at $A$ and $C$ intersect at the interior point $I$, and let $G$ and $H$ be the projections of $I$ on the sides $CD$ and $DA$ respectively. If the incircles in triangles $CDK$, $ADJ$, $DJK$ are tangent to $CJ$ and $AK$ at $L$, $N$, $O$ and $Q$, $M$, $P$ respectively, then*

$$MH + PQ = NO + GL$$

*if and only if $ABCD$ is a tangential quadrilateral.*

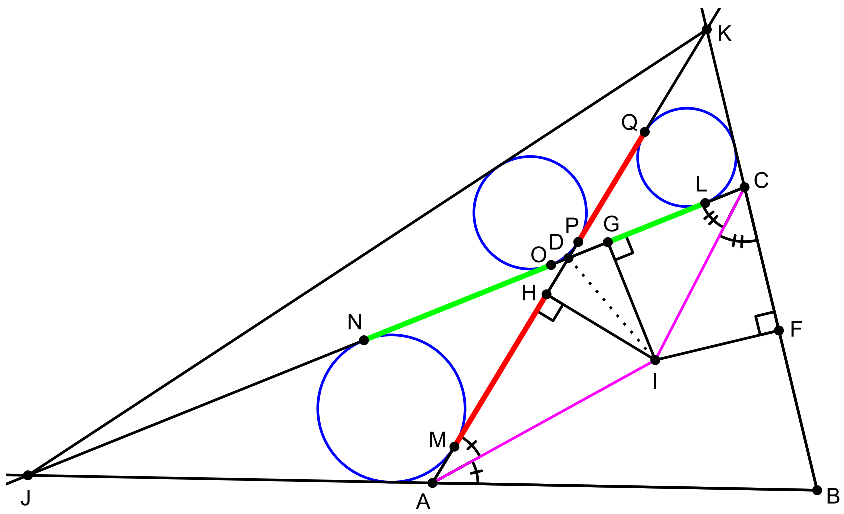


FIGURE 3. $ABCD$ is tangential $\Leftrightarrow$ $MH + PQ = NO + GL$

**Proof.** A consequence of the two tangent theorem is that $MQ = NL$ and $DO = DP$ in all convex quadrilaterals (see Figure 3). When $ABCD$ is tangential, then the incircle is tangent to $CD$ and $DA$ at $G$ and $H$ respectively, so $DG = DH$. By subtraction, we get $MH + PQ = NO + GL$.

Conversely, if $MH + PQ = NO + GL$ holds in a convex quadrilateral, then we get $DG = DH$ by applying the same equalities from the two tangent theorem as before. Thus triangles $DHI$ and $DGI$ are congruent (RHS), so $HI = GI$. Triangles $CIG$ and $CIF$ are also congruent (AAS), so $GI = FI$. Hence $HI = FI$, and $ABCD$ is tangential according to Theorem 1.1. □

There is the following excircle version of the previous theorem with an identical proof, so this proof is omitted (see Figure 4).

**Theorem 1.4.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides $AB$, $CD$ and $BC$, $DA$ intersect at $J$ and $K$ respectively. Assume the vertices are labeled such that $JK$ lies outside of $D$. Suppose the angle bisectors at $A$ and $C$ intersect at the interior point $I$, and let $G$ and $H$ be the projections of $I$ on the sides $CD$ and $DA$ respectively. If the excircles to triangles $ADJ$ and $CDK$ that are tangent to $AJ$ and $CK$ respectively are tangent to the extensions of $CJ$ and $AK$ at $N'$, $L'$ and $M'$, $Q'$ respectively, and the incircle in $DJK$ is tangent to $CJ$ and $AK$ at $O$ and $P$ respectively, then*

$$M'H + PQ' = N'O + GL'$$

*if and only if $ABCD$ is a tangential quadrilateral.*

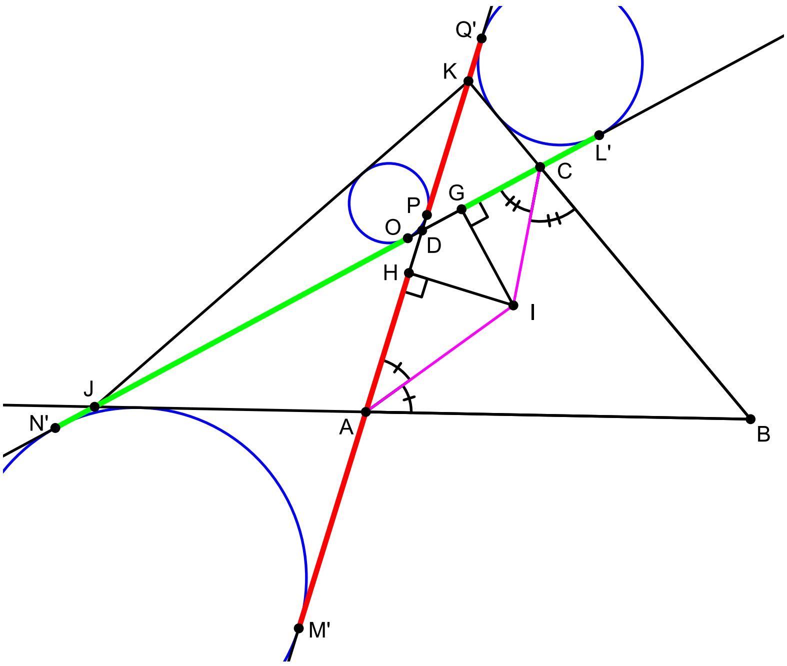


FIGURE 4. $ABCD$ is tangential $\Leftrightarrow$ $M'H + PQ' = N'O + GL'$

## 2. Variations of Pitot's theorem

The most important necessary and sufficient condition for when a convex quadrilateral $ABCD$ can have an incircle is the famous *Pitot's theorem*

$$AB + CD = BC + DA.$$

One not so well-known variant of it is

(1) $$BJ + BK = DJ + DK$$

where $J$ is the intersection of opposite sides $AB$ and $CD$, and $K$ is the intersection of $BC$ and $DA$. This characterization of tangential quadrilaterals was stated in [8, p. 644] and a proof using contradiction was given in [25, p. 147]. These sources however neglected to point out that for this theorem to be valid, the line segment $JK$ most be outside one of the vertices $A$ or $C$. In the other case, when $JK$ is outside one of the vertices $B$ or $D$, then the correct characterization is

$$AJ + AK = CJ + CK. \tag{2}$$

We let the reader confirm this by drawing figures for the four different possible cases and compare them with these equalities.

Based on (1) we shall prove another similar variant of Pitot's theorem.

**Theorem 2.1.** *In a convex quadrilateral $ABCD$ that is not a trapezoid and where the angle bisectors at $B$ and $D$ intersect at the interior point $I$, let $E$, $F$, $G$, $H$ be the projections of $I$ on the sides $AB$, $BC$, $CD$, $DA$ respectively. If $J$ is the intersection of opposite sides $AB$ and $CD$, $K$ is the intersection of $BC$ and $DA$, and the segment $JK$ is outside one of the vertices $A$ or $C$, then*

$$EJ + FK = GJ + HK$$

*if and only if $ABCD$ is a tangential quadrilateral.*

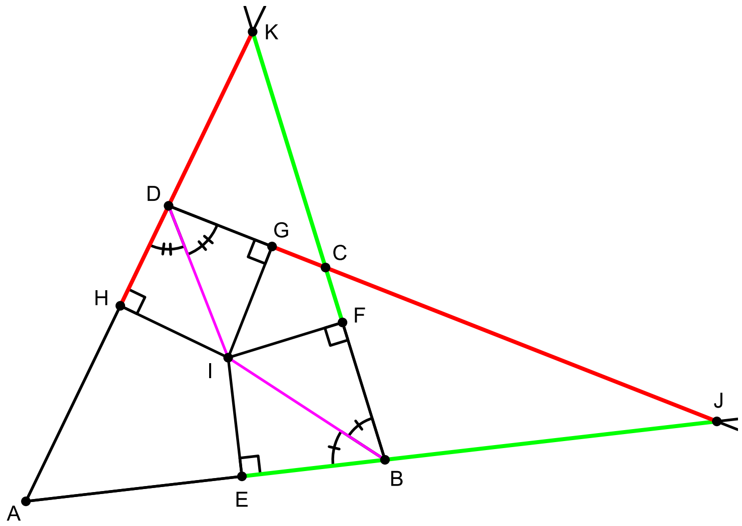


FIGURE 5. $ABCD$ is tangential $\Leftrightarrow$ $EJ + FK = GJ + HK$

**Proof.** We have that $ABCD$ is a tangential quadrilateral if and only if

$$\begin{aligned} & BJ + BK = DJ + DK \\ \Leftrightarrow\quad & BJ + KF + FB = GJ + DG + DK \\ \Leftrightarrow\quad & BJ + KF + EB = GJ + DH + DK \\ \Leftrightarrow\quad & EJ + FK = GJ + HK \end{aligned}$$

since $FB = EB$ and $DG = DH$ due to congruent triangles $BEI$ and $BFI$ (AAS), and $DGI$ and $DHI$ respectively (see Figure 5). □

In the next characterization, as well as several times in the next two sections, we will need the following well-known triangle formulas. Proving them is left as an easy exercise for the reader. The first equality is the *two tangent theorem* (two tangents from an external point have equal length).

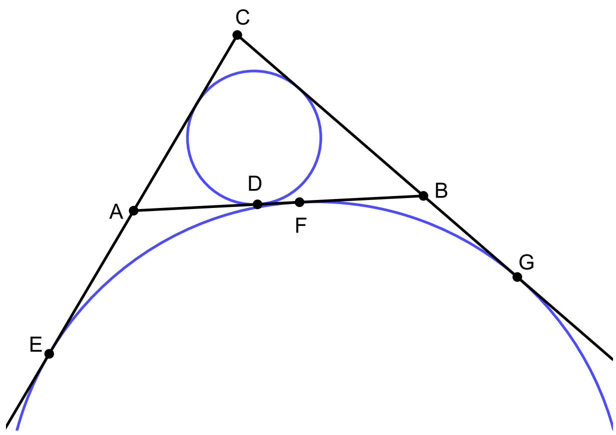


FIGURE 6. Tangent points for the incircle and an excircle to a triangle

**Lemma 2.1.** *In triangle $ABC$, suppose the incircle is tangent to $AB$ at $D$, and that one of the excircles is tangent to $AB$ at $F$ and the extensions of $CA$ and $CB$ at $E$ and $G$ respectively. Then*

$$AE = AF = BD = \tfrac{1}{2}(AB + BC - CA),$$
$$AD = BF = BG = \tfrac{1}{2}(AB - BC + CA).$$

The following theorem include distances between the tangency points of the incircles and the excircles to the two triangles created by a diagonal.

**Theorem 2.2.** *In a convex quadrilateral $ABCD$, suppose the incircles in triangles $ABC$ and $CDA$ are tangent to $AB$, $BC$, $CD$, $DA$ at $E_1$, $F_1$, $G_1$, $H_1$ respectively, and the excircles to those triangles are tangent to the same sides at $E_2$, $F_2$, $G_2$, $H_2$ respectively. The relation*

$$\pm E_1E_2 \pm F_1F_2 \pm G_1G_2 \pm H_1H_2 = 0$$

*holds if and only if $ABCD$ is a tangential quadrilateral, where the minus sign for each term is chosen if on any side of the quadrilateral the tangency point of the incircle comes before the tangency point of the excircle when tracing around the quadrilateral in the same direction the vertices are labeled.*

**Proof.** We prove the theorem in the case that is shown in Figure 7. There we have

$$\begin{aligned}
& AB - BC + CD - DA \\
= {} & AE_2 - E_1E_2 + E_1B - (BF_1 - F_1F_2 + F_2C) + \\
& CG_2 + G_1G_2 + G_1D - (DH_1 + H_1H_2 + H_2A) \\
= {} & -E_1E_2 + F_1F_2 + G_1G_2 - H_1H_2
\end{aligned}$$

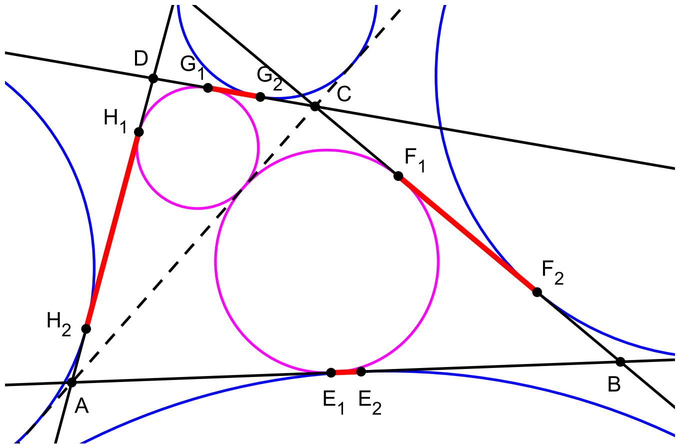


FIGURE 7. $ABCD$ is tangential $\Leftrightarrow$ $-E_1E_2 + F_1F_2 + G_1G_2 - H_1H_2 = 0$

where $E_1B = BF_1$ and $G_1D = DH_1$ according to the two tangent theorem. We have also used that

$$CG_2 = G_1D = DH_1 = H_2A$$

and

$$AE_2 = E_1B = BF_1 = F_2C$$

according to Lemma 2.1. The quadrilateral can have an incircle if and only if $AB - BC + CD - DA = 0$ according to Pitot's theorem, which is equivalent to

$$-E_1E_2 + F_1F_2 + G_1G_2 - H_1H_2 = 0$$

where the negative terms are those where the incircle comes before the tangency point of the excircle when tracing around the quadrilateral in the same direction the vertices are labeled. □

We note that the two incircles in Figure 7 are tangent to each other if and only if $ABCD$ is a tangential quadrilateral according to Theorem 1 in [12]. That theorem was published in Croatian by Jelena Gusić and Petar Mladinić already in 2001 [9], but one direction of the theorem is an old *Sangaku* problem from 1826 according to [3, pp. 34–35]. An excircle version was proved as Theorem 3.1 in [18]. Other pairs of tangent circles will be studied in Section 4 of the present paper.

Next we have a characterization similar to (1) where consecutive line segments are added instead of opposite ones as in Pitot's theorem and most other such characterizations concerning lengths.

**Theorem 2.3.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $J$ and $K$. Assume the vertices are labeled such that $JK$ lies outside of $D$. If the excircle to triangle $BJK$ that is tangent to $JK$ is also tangent to the extensions of $AB$ and $BC$ at $L$ and $M$ respectively, then*

$$LA + AD = DC + CM$$

*if and only if $ABCD$ is a tangential quadrilateral.*

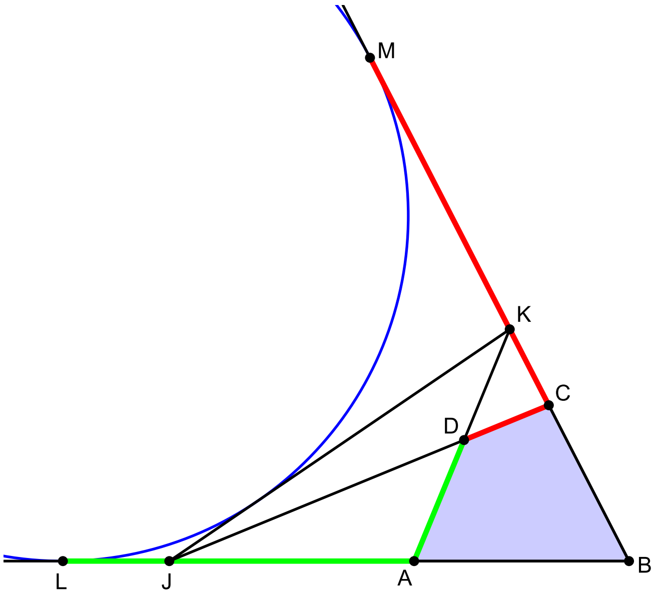


FIGURE 8. $ABCD$ is tangential $\Leftrightarrow$ $LA + AD = DC + CM$

**Proof.** According to Pitot's theorem, $ABCD$ is tangential if and only if

$$\begin{aligned} & AB + CD = BC + DA \\ \Leftrightarrow \quad & (LB - LA) + CD = (BM - CM) + DA \\ \Leftrightarrow \quad & LB + DC + CM = BM + AD + LA \\ \Leftrightarrow \quad & DC + CM = LA + AD \end{aligned}$$

since $LB = BM$ in all convex quadrilaterals according to the two tangent theorem (see Figure 8). □

## 3. Equal line segments

In this section we shall prove eight different characterizations concerning equal line segments in configurations similar to the one in the last theorem, and in almost all of their proofs we are going to apply formulas from Lemma 2.1.

**Theorem 3.1.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $J$ and $K$. Assume the vertices are labeled such that $JK$ lies outside of $D$. Let the incircles in triangles $AJK$ and $CJK$ be tangent to $AJ$ and $CK$ at $L$ and $M$ respectively. Then $AL = CM$ if and only if $ABCD$ is a tangential quadrilateral.*

**Proof.** Applying Lemma 2.1, we have (see Figure 9)

$$2AL = AJ + AK - JK, \qquad 2CM = CJ + CK - JK.$$

Thus

$$2(AL - CM) = AJ + AK - CJ - CK$$

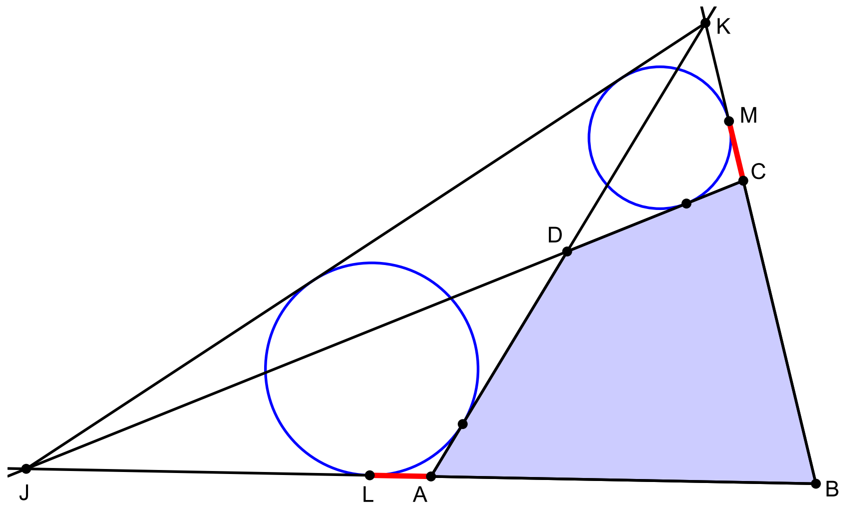


FIGURE 9. $ABCD$ is tangential $\Leftrightarrow$ $AL = CM$

and we have

$$AL = CM \quad \Leftrightarrow \quad AJ + AK = CJ + CK.$$

The equality to the right is a characterization for tangential quadrilaterals according to (2). Hence $AL = CM$ if and only if $ABCD$ is a tangential quadrilateral. □

Note that if the names of the intersections $J$ and $K$ were to interchange compared to in Figure 9, then we still have a valid characterization, since there are two equal tangent lengths at each of $A$ and $C$ in triangles $AJK$ and $CJK$. The same is true in the next necessary and sufficient condition, which is an excircle version of the previous theorem.

**Theorem 3.2.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $J$ and $K$. Assume the vertices are labeled such that $JK$ lies outside of $D$. Let the excircles to triangles $AJK$ and $CJK$ that are tangent to $JK$ be tangent to the extensions of $AJ$ and $CK$ at $L'$ and $M'$ respectively. Then $AL' = CM'$ if and only if $ABCD$ is a tangential quadrilateral.*

**Proof.** This proof is almost the same as the previous one, except that the tangent lengths are this time given by

$$2AL' = AJ + AK + JK, \qquad 2CM' = CJ + CK + JK$$

according to Lemma 2.1 (see Figure 10). The reader can now finish the proof and explain the sign change of $JK$. □

Another excircle version is the following:

**Theorem 3.3.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $J$ and $K$. Assume the vertices are labeled such that $JK$ lies outside of $D$. Let the excircles to triangles $AJK$ and $CJK$ that are tangent to $AK$ and $CJ$ be tangent to the extension*

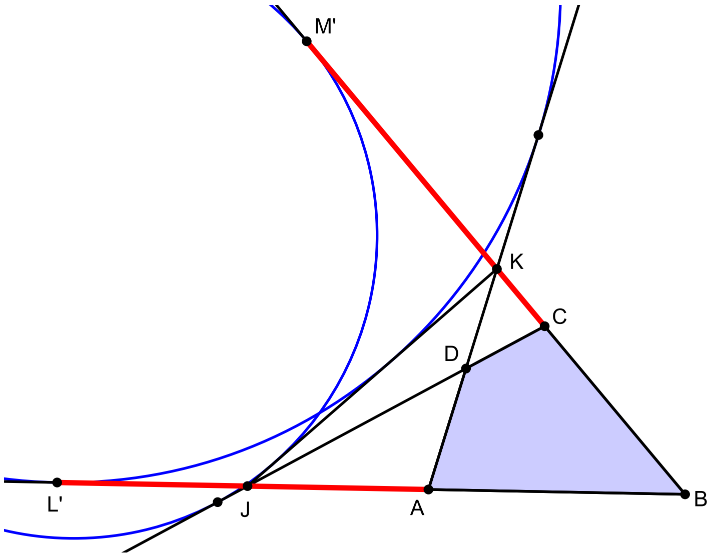


FIGURE 10. $ABCD$ is tangential $\Leftrightarrow$ $AL' = CM'$

*of $JK$ at $O$ and $N$ respectively. Then $JN = KO$ if and only if $ABCD$ is a tangential quadrilateral.*

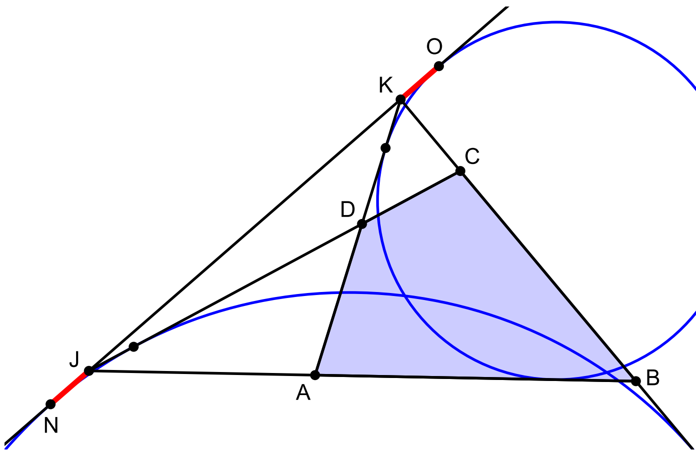


FIGURE 11. $ABCD$ is tangential $\Leftrightarrow$ $JN = KO$

**Proof.** Here we have (see Figure 11)

$$2KO = AJ + AK - JK, \qquad 2JN = CJ + CK - JK, \tag{3}$$

which are exactly the same expressions as those for the lengths of $AL$ and $CM$ in the proof of Theorem 3.1. Thus the end of the argument is identical to that proof.

If the names of the intersections $J$ and $K$ were to interchange compared to in Figure 11, then we get different excircles, but we still have a valid characterization as shown in Figure 12. This is because the lengths of $KO$ and $JN$ are even now given by (3). □

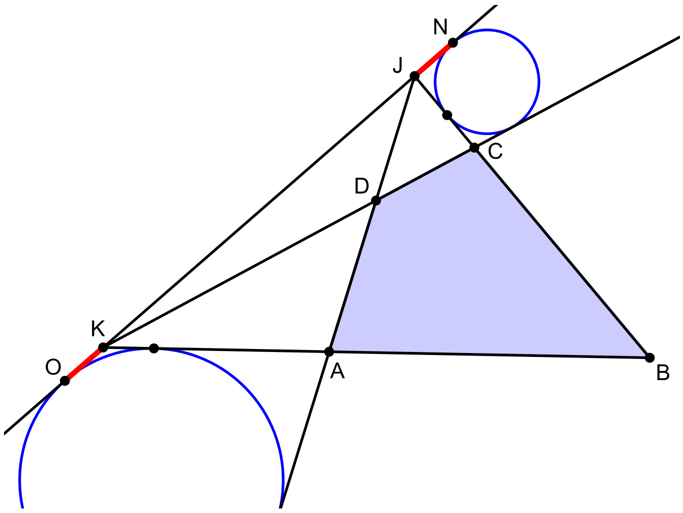


FIGURE 12. A second case where $ABCD$ is tangential $\Leftrightarrow$ $JN = KO$

Next we have a characterization with one incircle and one excircle.

**Theorem 3.4.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $J$ and $K$. Assume the vertices are labeled such that $JK$ lies outside of $D$. Let an excircle to triangle $BJK$ and the incircle in triangle $DJK$ be tangent to $JK$ at $R$ and $S$ respectively. Then $JR = KS$ if and only if $ABCD$ is a tangential quadrilateral.*

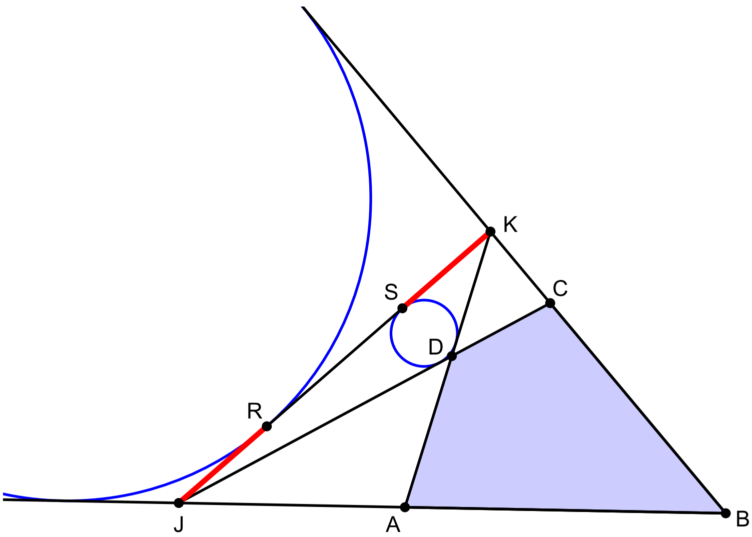


FIGURE 13. $ABCD$ is tangential $\Leftrightarrow$ $JR = KS$

**Proof.** We have that (see Figure 13)

(4) $$2JR = JK + BK - BJ, \qquad 2KS = JK + DK - DJ.$$

Then

$$2(JR - KS) = BK - BJ - DK + DJ$$

and we get

$$JR = KS \quad \Leftrightarrow \quad BJ + DK = BK + DJ.$$

The equality the the right is a characterization of tangential quadrilaterals according to Theorem 4 in [14]; whence so is the equality to the left. □

There is the following variant of the previous theorem:

**Theorem 3.5.** *In a convex quadrilateral $ABCD$, let an excircle to triangle $ABC$ and the incircle to triangle $ACD$ be tangent to diagonal $AC$ at $R'$ and $S'$ respectively. Then $AR' = CS'$ if and only if $ABCD$ is a tangential quadrilateral.*

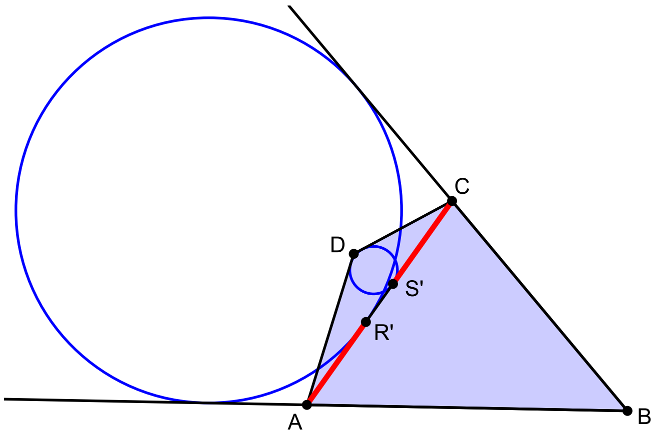


FIGURE 14. $ABCD$ is tangential $\quad\Leftrightarrow\quad AR' = CS'$

**Proof.** According to Lemma 2.1 (see Figure 14), we have

$$2CS' = CD + AC - DA, \qquad 2AR' = BC + AC - AB.$$

Hence

$$2(CS' - AR') = CD - DA - BC + AB$$

and we get that

$$CS' = AR' \quad \Leftrightarrow \quad AB + CD = BC + DA$$

proving the theorem according to Pitot's theorem. □

In the following theorem there are three excircles and one incircle.

**Theorem 3.6.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $J$ and $K$. Assume the vertices are labeled such that $JK$ lies outside of $D$. Let the excircles to triangles $AJK$ and $CJK$ that are tangent to $AJ$ and $CK$ be tangent to the extension of $JK$ at $N'$ and $O'$ respectively, and let an excircle to triangle $BJK$ and the incircle in triangle $DJK$ be tangent to $JK$ at $R$ and $S$ respectively. Then $RN' = SO'$ if and only if $ABCD$ is a tangential quadrilateral.*

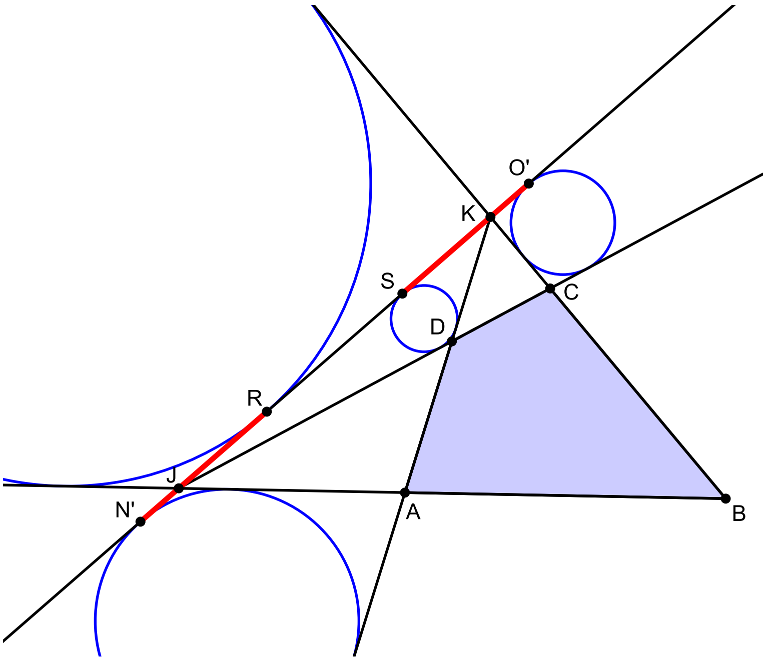


Figure 15. $ABCD$ is tangential $\Leftrightarrow$ $RN' = SO'$

**Proof.** We have

$$2JN' = AJ + AK - JK, \qquad 2KO' = CJ + CK - JK$$

and using (4), we get (see Figure 15)

$$2RN' = 2JR + 2JN' = BK - BJ + AJ + AK,$$
$$2SO' = 2KS + 2KO' = DK - DJ + CJ + CK.$$

Thus

$$\begin{aligned} 2(SO' - RN') &= DK + CD + CK - BK + AB - AK \\ &= DK + CD + CK - BC - CK + AB - AD - DK \\ &= AB + CD - BC - DA \end{aligned}$$

and we get that

$$SO' = RN' \quad \Leftrightarrow \quad AB + CD = BC + DA$$

completing the proof according to Pitot's theorem. □

Next we have a configuration very similar to the one in Theorem 3.1, but with different triangle incircles.

**Theorem 3.7.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides $AB$, $CD$ and $BC$, $DA$ intersect at $J$ and $K$ respectively. Assume the vertices are labeled such that $JK$ lies outside of $D$. Let the incircles in triangles $ADJ$ and $CDK$ be tangent to $AJ$ and $CK$ at $H''$ and $G''$ respectively. Then $AH'' = CG''$ if and only if $ABCD$ is a tangential quadrilateral.*

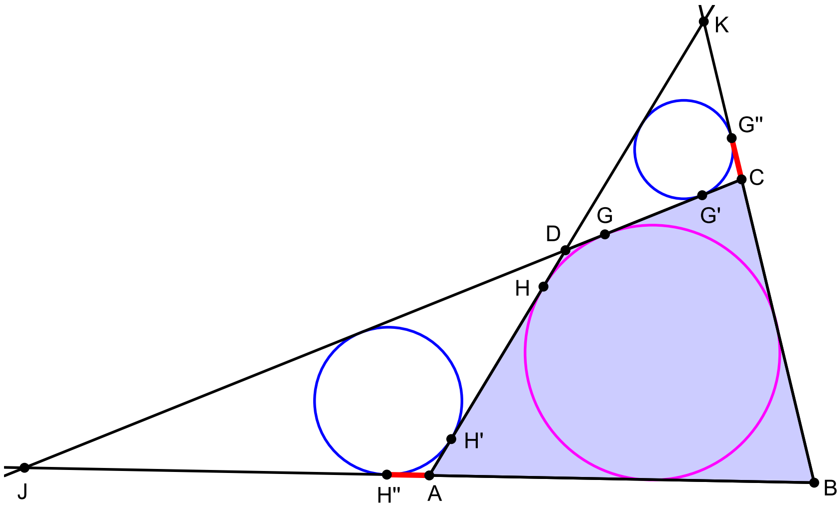


Figure 16. $ABCD$ is tangential $\Leftrightarrow$ $AH'' = CG''$

**Proof.** Suppose $H$ and $G$ are the points where the excircles to triangles $ADJ$ and $CDK$ are tangent to $AD$ and $CD$ respectively (see Figure 16). We have that $ABCD$ is a tangential quadrilateral if and only if the angle bisectors at $A$, $C$, $D$ are concurrent (Theorem 1 in [15]), which is equivalent to that the excircles to triangles $ADJ$ and $CDK$ are identical, which in turn is equivalent to

$$DH = DG \quad \Leftrightarrow \quad AH' = CG' \quad \Leftrightarrow \quad AH'' = CG''.$$

Note that the purple circle in Figure 13 is the two identical excircles to triangles $ADJ$ and $CDK$, which is also the incircle in $ABCD$ when the quadrilateral is tangential. Otherwise these excircles are different, and there can be no incircle in $ABCD$ (since Theorem 1 in [15] is not satisfied). □

The following is an excircle version of the previous theorem.

**Theorem 3.8.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides $AB$, $CD$ and $BC$, $DA$ intersect at $J$ and $K$ respectively. Assume the vertices are labeled such that $JK$ lies outside of $D$. Let the excircles to triangles $ADJ$ and $CDK$ that are tangent to $AJ$ and $CK$ be tangent to the extensions of $CD$ and $AD$ at $P$ and $Q$ respectively. Then $JP = KQ$ if and only if $ABCD$ is a tangential quadrilateral.*

**Proof.** If the excircle and incircle to triangle $ADJ$ are tangent to $AJ$ at $P'$ and $H''$ respectively, and the excircle and incircle to triangle $CDK$ are tangent to $CK$ at $Q'$ and $G''$ respectively (see Figure 17), then $JP = JP' = AH''$ and $KQ = KQ' = CG''$. Applying the previous theorem, we get that $JP = KQ$ is a characterization of tangential quadrilaterals. □

## 4. Tangent circles

Here we shall study two characterizations concerning tangent excircles, and we continue to apply formulas from Lemma 2.1.

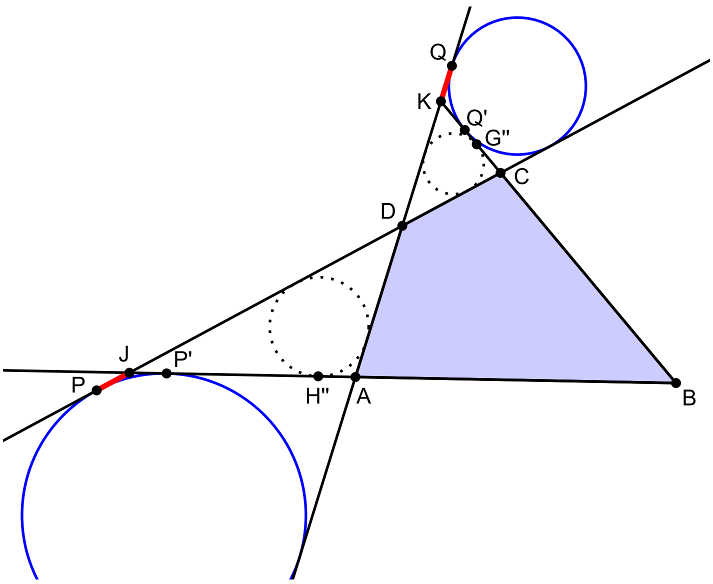


FIGURE 17. $ABCD$ is tangential $\Leftrightarrow$ $JP = KQ$

**Theorem 4.1.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $J$ and $K$, and assume the vertices are labeled such that $JK$ lies outside of $D$. The four excircles to triangles $AJK$ and $CJK$ that are not tangent to $JK$ are tangent to each other in pairs on the extension of $JK$ if and only if $ABCD$ is a tangential quadrilateral.*

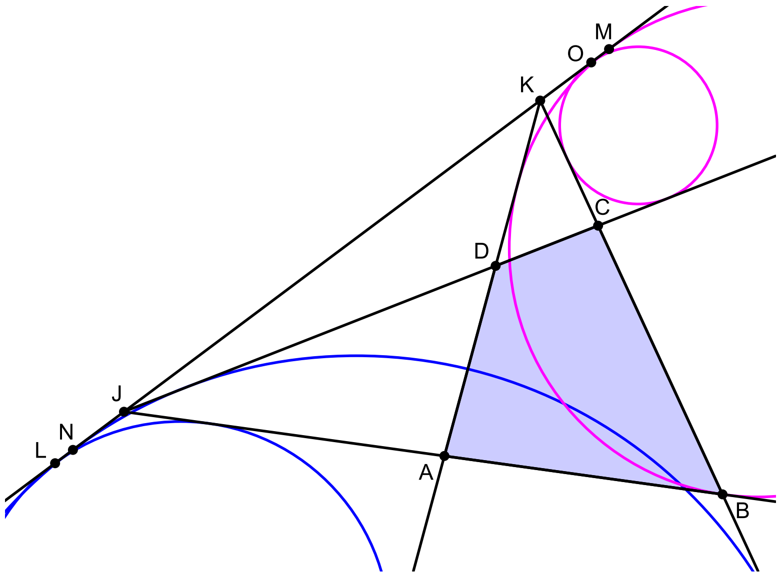


FIGURE 18. $LN = 0$ $\Leftrightarrow$ $ABCD$ is tangential $\Leftrightarrow$ $MO = 0$

**Proof.** Let the excircle to triangle $AJK$ that is tangent to $AJ$ be tangent to the extension of $JK$ at $L$, and the excircle to triangle $CJK$ that is tangent to $CJ$ be tangent to the extension of $JK$ at $N$ (see Figure 18). Then

$$2JL = AJ + AK - JK, \qquad 2JN = CJ + CK - JK$$

and we have

$$2(JL - JN) = AJ + AK - CJ - CK.$$

The two excircles are tangent to the line $JK$ at the same point $L \equiv N$ if and only if $JL = JN$, which is equivalent to $AJ + AK = CJ + CK$, which in turn is equivalent to that $ABCD$ is a tangential quadrilateral according to (2).

The tangency of the two purple excircles at $M \equiv O$ is proved in the same way. □

While the previous theorem was about tangency points for the excircles to triangles $AJK$ and $CJK$, the next deals with triangles $BJK$ and $DJK$. Before we proceed, let us remind the reader that Theorem 4 in [14] states that a convex quadrilateral $ABCD$, where the extensions of opposite sides intersect at $J$ and $K$, is tangential if and only if

$$AJ + CK = AK + CJ$$

when $JK$ is outside one of the vertices $A$ or $C$. (That theorem was formulated differently and used other notations, but this is an equivalent way of expressing it.) It was also remarked after the theorem that when $JK$ is outside of $B$ or $D$ (but again, formulated differently), then the correct characterization of tangential quadrilaterals is instead

$$BJ + DK = BK + DJ \tag{5}$$

which we shall use now when proving our next theorem.

**Theorem 4.2.** *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at $J$ and $K$, and assume the vertices are labeled such that $JK$ lies outside of $D$. The two excircles to triangles $BJK$ and $DJK$ that are tangent to $JK$ are tangent to each other on $JK$ if and only if $ABCD$ is a tangential quadrilateral.*

**Proof.** Let the two considered excircles to triangles $BJK$ and $DJK$ be tangent to $JK$ at $P$ and $Q$ respectively (see Figure 19). Then

$$2JP = JK + BK - BJ, \qquad 2JQ = JK + DK - DJ$$

so we get

$$2(JP - JQ) = BK - BJ - DK + DJ.$$

Hence

$$P \equiv Q \quad \Leftrightarrow \quad JP = JQ \quad \Leftrightarrow \quad BJ + DK = BK + DJ$$

which is equivalent to that $ABCD$ is a tangential quadrilateral according to (5). □

This characterization has an incircle version that was proved as Theorem 5 in [14].

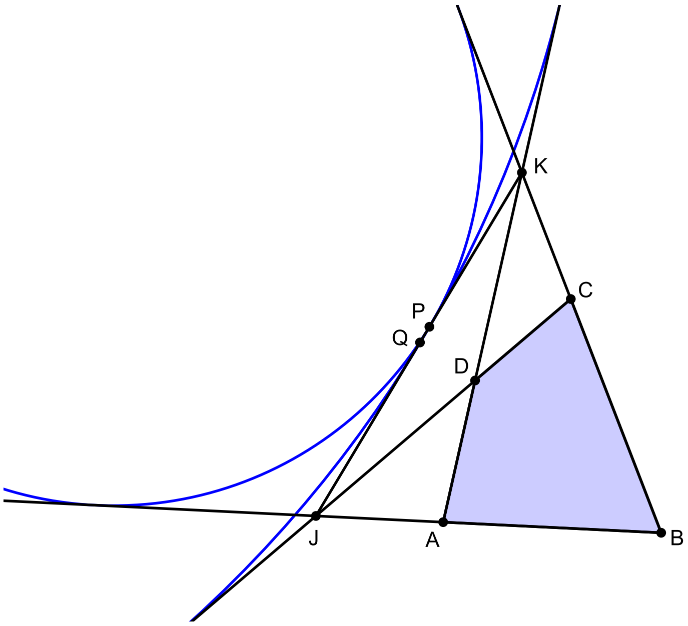


FIGURE 19. $ABCD$ is tangential $\Leftrightarrow$ $PQ = 0$

## 5. Partitions

In a tangential quadrilateral $ABCD$ it is very well known that if the incircle is tangent to the sides $AB$, $BC$, $CD$, $DA$ at $E$, $F$, $G$, $H$ respectively, then $BE = BF$, $CF = CG$, $DG = DH$ and $AH = AE$. In fact, this is a direct consequence of the two tangent theorem. As an extension of this we prove the following partition characterization for the sides:

**Theorem 5.1.** *In a convex quadrilateral $ABCD$ where the angle bisectors at $A$ and $C$ intersect at the interior point $I$, let $E$, $F$, $G$, $H$ be the projections of $I$ on the sides $AB$, $BC$, $CD$, $DA$ respectively. Then*

$$\frac{BE}{BF} = \frac{DG}{DH}$$

*if and only if $ABCD$ is a tangential quadrilateral.*

**Proof.** The intersection $I$ of two angle bisectors in a tangential quadrilateral is the incenter, and $E$, $F$, $G$, $H$ are the points where the incircle is tangent to the sides. As stated before, the necessary condition is a direct consequence of the two tangent theorem.

Conversely, suppose the equality holds in a convex quadrilateral. Let $EI = HI = x$ and $FI = GI = y$ (these distances are equal in pairs due to congruent triangles, see Figure 20). Applying the Pythagorean theorem in right triangles $BEI$ and $BFI$, and also in $DGI$ and $DHI$, we have

$$BE^2 + x^2 = BF^2 + y^2, \qquad DH^2 + x^2 = DG^2 + y^2.$$

Thus

$$BF^2 - BE^2 = x^2 - y^2 = DG^2 - DH^2.$$

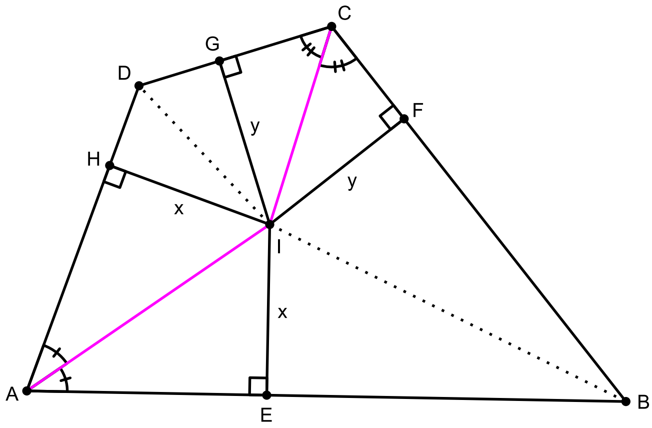


FIGURE 20. $ABCD$ is tangential $\Leftrightarrow$ $\frac{BE}{BF} = \frac{DG}{DH}$

Inserting

$$BE = BF \cdot \frac{DG}{DH}$$

we get, after clearing the denominator, that

$$BF^2(DH^2 - DG^2) = DH^2(DG^2 - DH^2).$$

This is factorized as

$$(BF^2 + DH^2)(DH + DG)(DH - DG) = 0$$

with the single solution $DH = DG$. Then triangles $DHI$ and $DGI$ are congruent (RHS), so $x = y$, which means that $ABCD$ is a tangential quadrilateral according to Theorem 1.1. □

Another partition of the sides that is a necessary and sufficient condition of tangential quadrilaterals was proved as Theorem 5 in [10].

The following characterization is an area partition which nails one basic feature of tangential quadrilaterals.

**Theorem 5.2.** *A convex quadrilateral can be partitioned into three kites if and only if it is a tangential quadrilateral.*

**Proof.** Suppose without loss of generality that we have $AB > DA$ and $BC > CD$ in a tangential quadrilateral $ABCD$. Let $E$ and $F$ be points on $AB$ and $BC$ respectively such that $AE = DA$ and $FC = CD$ (see Figure 21). Then

$$AE + EB + CD = BF + FC + DA \quad \Rightarrow \quad EB = BF$$

according to Pitot's theorem. Since the angle bisectors at $A$ and $C$ intersect at the incenter $I$, we get that triangles $AEI$ and $ADI$ are congruent, as are $CFI$ and $CDI$. This implies that $EI = FI$, and thus $AEID$, $CDIF$ and $BFIE$ are all kites.

Conversely, in a convex quadrilateral $ABCD$ that is partitioned into three kites (see Figure 21), we have

$$AB + CD = AE + EB + CD = DA + BF + FC = DA + BC$$

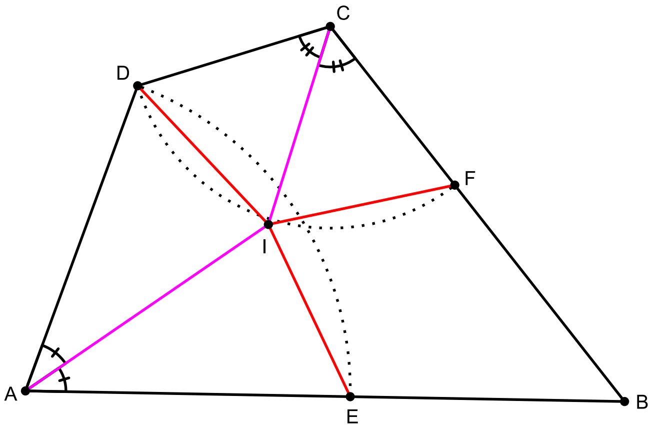


Figure 21. $ABCD$ is tangential $\Leftrightarrow$ $AEID$, $CDIF$, $BFIE$ are kites

confirming that $ABCD$ is tangential according to the converse of Pitot's theorem. □

Note that when the original quadrilateral is a rhombus or a square, then two of the kites are concave and the third kite is a rhombus (which is a special case of a kite).

Next we have another characterization regarding area partitioning. A similar formulation of the 'if' part of this theorem was a problem with a solution given in [2, pp. 68, 189–190]. Our proof is not the same as the one in that book.

**Theorem 5.3.** *Suppose that in a convex quadrilateral $ABCD$ there is a line $L$ that cuts the quadrilateral into two quadrilaterals with equal areas and equal perimeters, and also that the angle bisectors to two opposite vertex angles intersect at an interior point $I$. Then $L$ goes through $I$ if and only if $ABCD$ is a tangential quadrilateral.*

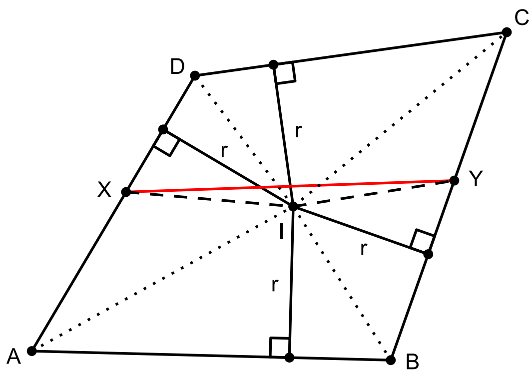


Figure 22. $ABCD$ is tangential $\Leftrightarrow$ $I \in XY$

**Proof.** ($\Rightarrow$) In a tangential quadrilateral, let without loss of generality $L$ intersect $DA$ and $BC$ at $X$ and $Y$ respectively, and assume that the incenter

$I$ lies inside quadrilateral $ABYX$. From the equal perimeters, we get

$$AX + AB + BY = CY + CD + DX \tag{6}$$

and the equal areas yields (see Figure 22)

$$\tfrac{1}{2}AX \cdot r + \tfrac{1}{2}AB \cdot r + \tfrac{1}{2}BY \cdot r + [IXY] = \tfrac{1}{2}CY \cdot r + \tfrac{1}{2}CD \cdot r + \tfrac{1}{2}DX \cdot r - [IXY]$$

where $r$ is the inradius and $[IXY]$ denote the area of triangle $IXY$. The latter equality is equivalent to

$$\tfrac{1}{2}r(AX + AB + BY - CY - CD - DX) = -2[IXY].$$

By (6), the left hand side is equal to zero; hence we get $[IXY] = 0$ which means that $I$ lies on the line segment $XY$.

($\Leftarrow$) In a convex quadrilateral $ABCD$ where $L$ goes through the intersection $I$ of two opposite angle bisectors, (6) is still valid according to the assumptions. Suppose $I$ is at a distance $x$ from $AB$ and $DA$, and at a distance $y$ from $BC$ and $CD$. The equal areas now yields

$$\tfrac{1}{2}AX \cdot x + \tfrac{1}{2}AB \cdot x + \tfrac{1}{2}BY \cdot y = \tfrac{1}{2}CY \cdot y + \tfrac{1}{2}CD \cdot y + \tfrac{1}{2}DX \cdot x$$

which is equivalent to

$$\tfrac{1}{2}x(AX + AB - DX) = \tfrac{1}{2}y(CY + CD - BY).$$

Using (6), we get $x = y$, which means that $ABCD$ is a tangential quadrilateral according to Theorem 1.1. □

## 6. Concurrences

First we have the following characterization regarding concurrent circles.

**Theorem 6.1.** *In a convex quadrilateral $ABCD$ where the angle bisectors at $A$ and $C$ intersect at the interior point $I$, let the circle with center $B$ and radius equal to the shortest of the two sides $AB$ and $BC$ intersect the longest of these two sides at $E$, and let the circle with center $D$ and radius equal to the shortest of the two sides $CD$ and $DA$ intersect the longest of these two sides at $F$. The circle with center $I$ and radius equal to the shortest of the two distances $AI$ and $CI$ intersects the first two circles at $E$ and $F$ if and only if $ABCD$ is a tangential quadrilateral.*

**Proof.** In a tangential quadrilateral, $BI$ and $DI$ are angle bisectors at $B$ and $D$ respectively, so triangles $CBI$ and $EBI$ are congruent (SAS) as are triangles $CDI$ and $FDI$ (see Figure 23). Hence $EI = CI = FI$, which proves that the circle with center $I$ goes through $E$ and $F$.

Conversely, if the circle with center $I$ goes through $E$ and $F$ in a convex quadrilateral, then triangles $CBI$ and $EBI$ are congruent (SSS) as are triangles $CDI$ and $FDI$. This implies that $\angle CBI = \angle EBI$ and $\angle CDI = \angle FDI$, so the angle bisectors of all four vertex angles intersect at $I$. Then $ABCD$ is a tangential quadrilateral. □

It is a well-known property for tangential quadrilaterals that the two line segments connecting opposite points of tangency of the incircle on the four sides and the two diagonals are all concurrent. No less than nine different proofs of this theorem can be found in [28]. Next we will prove that there

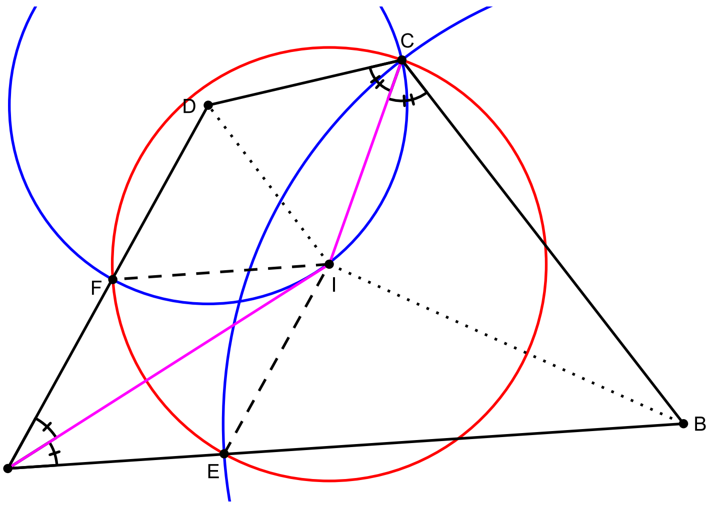


Figure 23. $ABCD$ is tangential $\Leftrightarrow$ Circles are concurrent at $E$ and $F$

is also a true converse of that theorem which gives another characterization of tangential quadrilaterals. Our proof of the direct theorem is basically the same as proof 5 in [28].

**Theorem 6.2.** *In a convex quadrilateral $ABCD$ where the angle bisectors at $B$ and $D$ intersect at the interior point $I$, let $E$, $F$, $G$, $H$ be the projections of $I$ on the sides $AB$, $BC$, $CD$, $DA$ respectively. Then $EG$, $FH$, $BD$ are concurrent if and only if $ABCD$ is a tangential quadrilateral.*

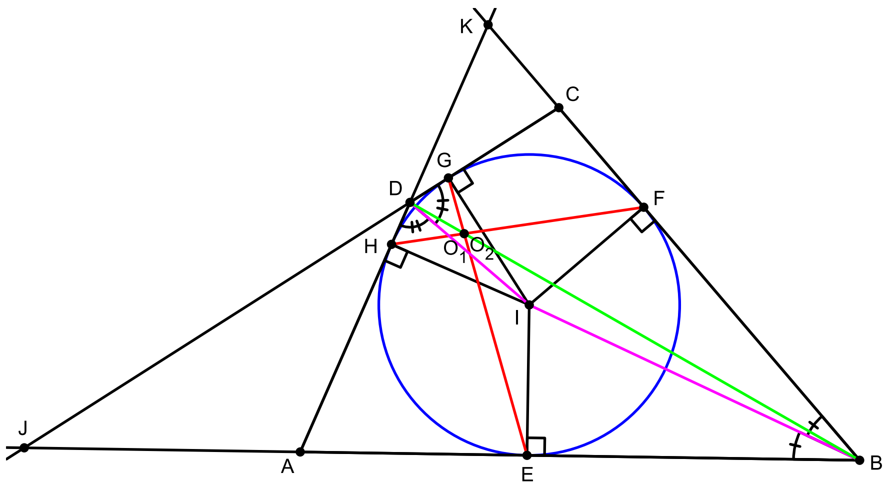


Figure 24. When $ABCD$ is tangential, $EG$, $FH$, $BD$ are concurrent

**Proof.** ($\Rightarrow$) The intersection $I$ of two angle bisectors in a tangential quadrilateral is the incenter, and $E$, $F$, $G$, $H$ are the points where the incircle is tangent to the sides. Suppose the extensions of $AB$ and $CD$ intersect at $J$, and the extensions of $BC$ and $DA$ intersect at $K$. Also suppose that $EG$ and $FH$ intersect $BD$ at $O_1$ and $O_2$ respectively (see Figure 24). Then applying Menelaus's theorem (with non-directed distances) in triangles $BDJ$

and $BDK$ with transversals $EG$ and $FH$ respectively yields

$$\frac{BO_1}{O_1D}\cdot\frac{DG}{GJ}\cdot\frac{JE}{EB}=1=\frac{BO_2}{O_2D}\cdot\frac{DH}{HK}\cdot\frac{KF}{FB}.$$

Here $GJ = JE$, $KF = HK$, $DG = DH$ and $EB = FB$ according to the two tangent theorem. Thus

$$\frac{BO_1}{O_1D}=\frac{BO_2}{O_2D}$$

which means that $O_1$ and $O_2$ divide $BD$ in the same ratio, so they are the same point. Hence $EG$, $FH$, $BD$ are concurrent at $O_1 = O_2$.

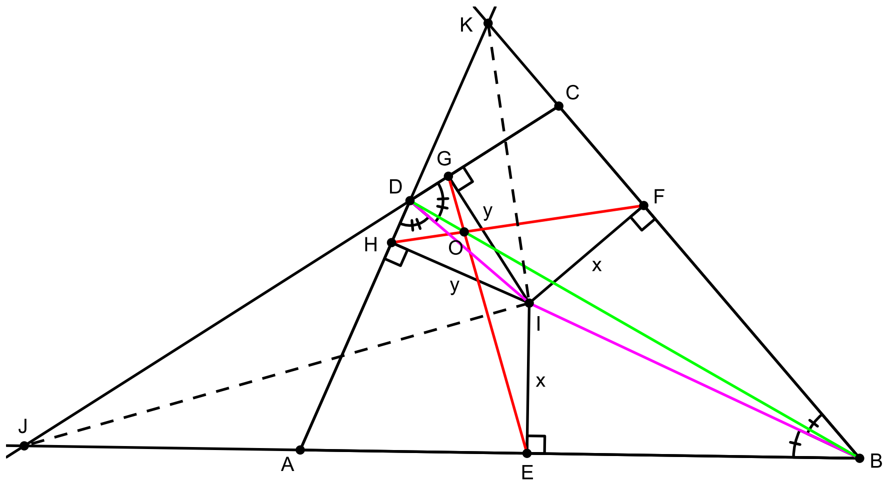


FIGURE 25. When $EG$, $FH$, $BD$ are concurrent, $ABCD$ is tangential

($\Leftarrow$) In a convex quadrilateral where $EG$, $FH$, $BD$ are concurrent at a point $O$, Menelaus's theorem applied in triangles $BDJ$ and $BDK$ yields

$$\frac{BO}{OD}\cdot\frac{DG}{GJ}\cdot\frac{JE}{EB}=1=\frac{BO}{OD}\cdot\frac{DH}{HK}\cdot\frac{KF}{FB}.$$

Triangles $BEI$ and $BFI$ are congruent (AAS), as are $DGI$ and $DHI$ (see Figure 25), so this equality is simplified as

$$JE=\frac{GJ\cdot KF}{KH}. \tag{7}$$

Now let $EI = FI = x$ and $GI = HI = y$. Applying the Pythagorean theorem in right triangles $KHI$ and $KFI$, and also in $JEI$ and $JGI$, we have

$$KH^2+y^2=KF^2+x^2, \qquad JE^2+x^2=GJ^2+y^2.$$

Thus

$$KH^2-KF^2=x^2-y^2=GJ^2-JE^2.$$

Inserting (7), we get after clearing the denominator that

$$KH^2(KH^2-KF^2)=GJ^2(KH^2-KF^2)$$

which is factorized as

$$(KH+KF)(KH-KF)(KH+GJ)(KH-GJ)=0.$$

Then either $KH = KF$ or $KH = GJ$.

In the first case it follows that $HI = FI$ since triangles $KHI$ and $KFI$ are congruent (RHS). Hence $ABCD$ is tangential according to Theorem 1.1.

In the second case, we get $JE = KF$ from (7) and thus $JB = KB$. From $KH = GJ$ it follows that $KD = DJ$, and since $BD$ is a common side, we have that triangles $BDJ$ and $BDK$ are congruent (SSS). But $\angle ADJ = \angle CDK$, so triangles $ADJ$ and $CDK$ are also congruent (ASA). Hence triangles $ABD$ and $CBD$ are congruent (SSS), making $ABCD$ a kite, which is a special case of a tangential quadrilateral. □

We note that a similar characterization was proved as Theorem 6.3 in [19], which states that the lines $EF$, $GH$, $AC$ are concurrent (outside the quadrilateral) if and only if $ABCD$ is a tangential quadrilateral.

The setup for the next theorem is more or less the same as in Theorem 6 in [15] (with slightly changed notations), but we formulate it in a different way and give a different proof. The idea we use to prove that the points $J_3$, $I_1$ and $P$ are collinear in a tangential quadrilateral (the first half of the proof) comes from the solution of problem 6 for grade 9 in [31, p. 13]. That problem was proposed to the Sharygin Geometry Olympiad by F. Ivlev.

**Theorem 6.3.** *In a convex quadrilateral $ABCD$, let $I_1$, $I_2$, $I_3$, $I_4$ be the incenters in triangles $ABC$, $BCD$, $CDA$, $DAB$ respectively, and let $J_1$, $J_2$, $J_3$, $J_4$ be the excenters to the same triangles that are opposite one of the diagonals. The lines $I_1J_3$, $I_2J_4$, $I_3J_1$, $I_4J_2$ are concurrent at the diagonal intersection if and only if $ABCD$ is a tangential quadrilateral.*

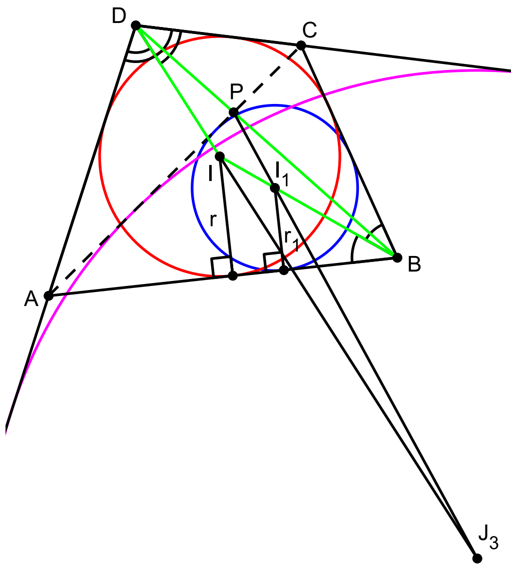


Figure 26. When $ABCD$ is tangential, $J_3$, $I_1$, $P$ are collinear

**Proof.** First we prove that the points $J_3$, $I_1$ and $P$ are collinear in a tangential quadrilateral, where $P$ is the diagonal intersection, by applying the converse to Menelaus's theorem in triangle $IBD$ with transversal $PI_1$, where $I$ is the incenter in $ABCD$. Thus we shall show that (see Figure 26)

$$\frac{DP}{PB}\cdot\frac{BI_1}{I_1I}\cdot\frac{IJ_3}{J_3D}=1. \tag{8}$$

To this end, we have

$$\frac{I_1I}{BI_1}=\frac{BI-BI_1}{BI_1}=\frac{BI}{BI_1}-1=\frac{r}{r_1}-1=\frac{K}{T_1}\cdot\frac{s_1}{s}-1$$

where $r$, $K$, $s$ are the inradius, area and semiperimeter respectively of $ABCD$, and $r_1$, $T_1$, $s_1$ are the corresponding quantities in triangle $ABC$. Then

$$\begin{aligned}\frac{I_1I}{BI_1}&=\frac{BD}{BP}\cdot\frac{AB+BC+CA}{AB+BC+CD+DA}-1\\&=\frac{DP(AB+BC+CA)-BP(CD+DA-CA)}{BP(AB+BC+CD+DA)}\end{aligned}$$

since $\frac{K}{T_1}=\frac{BD}{BP}$. In the same way, by applying the formula $r_3=\frac{T_3}{s_3-CA}$ for the exradius of triangle $CDA$ (where $T_3$ is the area and $s_3$ the semiperimeter), we get

$$\frac{IJ_3}{J_3D}=\frac{DP(AB+BC+CA)-BP(CD+DA-CA)}{DP(AB+BC+CD+DA)}.$$

Using the last two expressions we see that (8) is satisfied, confirming that $J_3$, $I_1$ and $P$ are collinear.

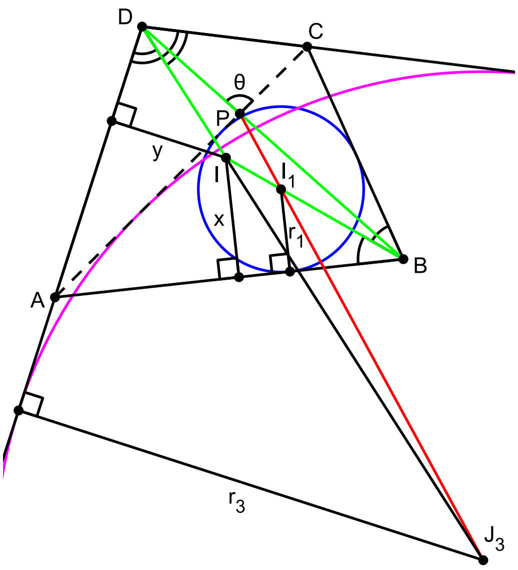


FIGURE 27. A convex quadrilateral where $J_3$, $I_1$, $P$ are collinear

Next we prove the following converse: A convex quadrilateral where $J_3$, $I_1$ and $P$ are collinear is a tangential quadrilateral. Let the angle bisectors at $B$ and $D$ intersect at an interior point $I$, which is at a distance $x$ from

$AB$ and $BC$, and at a distance $y$ from $CD$ and $DA$. We shall prove that $x = y$. By similar triangles (see Figure 27), we have

$$\frac{I_1I}{BI_1} = \frac{x - r_1}{r_1}, \qquad \frac{J_3I}{DJ_3} = \frac{r_3 - y}{r_3}.$$

The inradius $r_1$ in triangle $ABC$ is given by

$$r_1 = \frac{T_1}{s_1} = \frac{AC \cdot BP \sin\theta}{AB + BC + CA} \tag{9}$$

where $\theta$ is the measure of one of the angles between the diagonals, and the radius $r_3$ in the excircle tangent to $AC$ in triangle $CDA$ is given by

$$r_3 = \frac{T_3}{s_3 - CA} = \frac{AC \cdot DP \sin\theta}{AD + DC - CA}. \tag{10}$$

Since $J_3$, $I_1$ and $P$ are collinear, we get from Menelaus's theorem applied in triangle $IBD$ with transversal $PI_1$ that

$$1 = \frac{DP}{PB} \cdot \frac{BI_1}{I_1I} \cdot \frac{IJ_3}{J_3D} = \frac{DP}{PB} \cdot \frac{r_1}{r_3} \cdot \frac{r_3 - y}{x - r_1} = \frac{AD + DC - CA}{AB + BC + CA} \cdot \frac{r_3 - y}{x - r_1}.$$

Whence

$$(AB + BC + CA)(x - r_1) = (AD + DC - CA)(r_3 - y)$$

which we, using (9) and (10), rewrite as

$$(AB + BC)x + (AD + DC)y + CA(x - y) = CA \sin\theta(BP + DP). \tag{11}$$

It holds that the area $K$ of the quadrilateral satisfies

$$(AB + BC)x + (AD + DC)y = 2K = CA \cdot BD \sin\theta.$$

Thus from (11) we have $CA(x - y) = 0$, implying $x = y$, so $ABCD$ is a tangential quadrilateral according to Theorem 1.1.

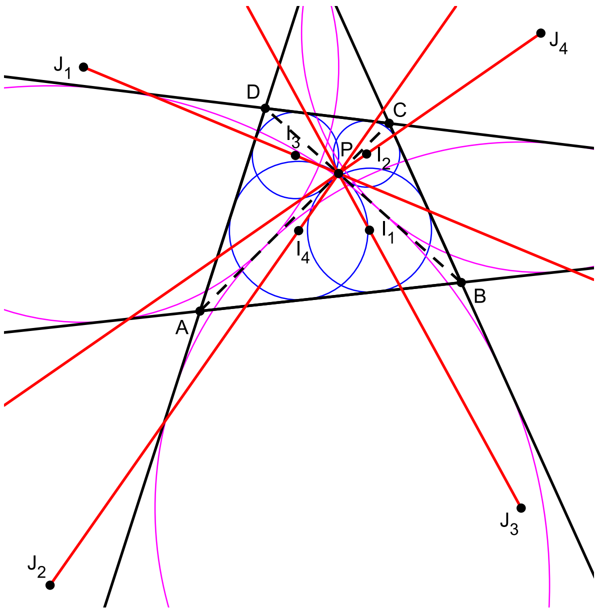


FIGURE 28. $ABCD$ is tangential $\Leftrightarrow$ $I_1J_3, I_2J_4, I_3J_1, I_4J_2$ are concurrent at $P$

So far we have proved that $J_3$, $I_1$, $P$ are collinear if and only if $ABCD$ is a tangential quadrilateral. By symmetry, the same is true for the three points $J_4$, $I_2$, $P$, as well as for $J_1$, $I_3$, $P$, and also for $J_2$, $I_4$, $P$. Hence the lines $I_1J_3$, $I_2J_4$, $I_3J_1$, $I_4J_2$ are concurrent at the diagonal intersection $P$ in a tangential quadrilateral (see Figure 28), and conversely: if $ABCD$ is not tangential, then neither of these lines goes through $P$. □

## 7. Cyclic quadrilaterals

The last characterization is not a discovery of ours. To prove this theorem (both the necessary and sufficient condition) was a problem posted in June 2018 at [26], but at the time of writing this paper, no proof has been given there. The necessary condition is a problem with a longer history than that. It was proposed by Dao Thanh Oai from Vietnam in January 2014 [4]. That post did not get a solution either. The necessary condition then appeared on the shortlist of problems for the Balkan Mathematical Olympiad in 2016, but was not used in that competition. It was however used as problem 4 on the Bulgarian National Olympiad in 2018, proposed by M. Etesami Fard and N. Beluhov. Several solutions can be found at [22]. Another solution was given in the Russian magazine *Kvant* in 2018 [21], where the necessary condition was discussed as a special case of a more general problem. Here we use the main idea from that solution to give a proof of both the necessary and sufficient condition. The proof is based on a triangle lemma.

**Lemma 7.1.** *In a triangle $ABC$ with incenter $I$, let $P$ and $Q$ be points on $AC$ and $BC$ respectively, where one of these points can be on the extension of the corresponding line segment beyond $C$. Then $CPIQ$ is a cyclic quadrilateral if and only if $AP + BQ = AB$.*

**Proof.** Let $S$, $T$, $U$ be the projections of $I$ on the sides $AC$, $BC$, $AB$ respectively.

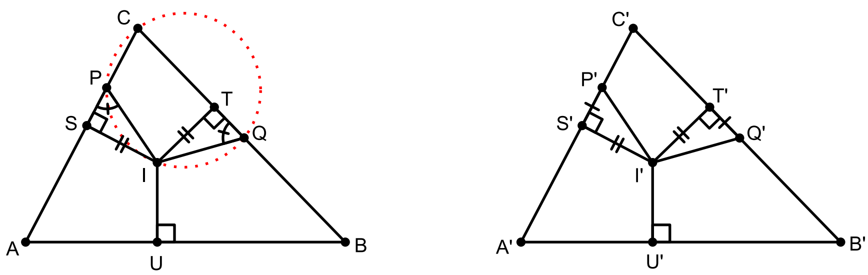


Figure 29. $CPIQ$ is cyclic $\quad\Leftrightarrow\quad AP + BQ = AB$

(⇒) When $CPIQ$ is a cyclic quadrilateral, $\angle SPI = \angle CQI$. Then triangles $SPI$ and $TQI$ are congruent (AAS) since $SI = TI$ are inradii (see the left part of Figure 29). We have

$$AB = AU+BU = AS+BT = AS+TQ+BQ = AS+SP+BQ = AP+BQ$$

since triangles $AIS$ and $AIU$ are congruent (RHS), as are $BIT$ and $BIU$.

(⇐) Given points $P'$ and $Q'$ such that $A'P' + B'Q' = A'B'$, we still have $A'S' = A'U'$ and $B'T' = B'U'$ due to congruent triangles (see the right part of Figure 29). From the assumption we have

$$A'U' + B'U' = A'S' + S'P' + B'T' - T'Q' \quad \Rightarrow \quad 0 = S'P' - T'Q'.$$

Thus triangles $S'P'I'$ and $T'Q'I'$ are congruent (SAS), so $\angle S'P'I' = \angle T'Q'I'$. Hence $C'P'I'Q'$ is a cyclic quadrilateral since an exterior angle is equal to the opposite interior angle (Theorem A.4 in [17]).

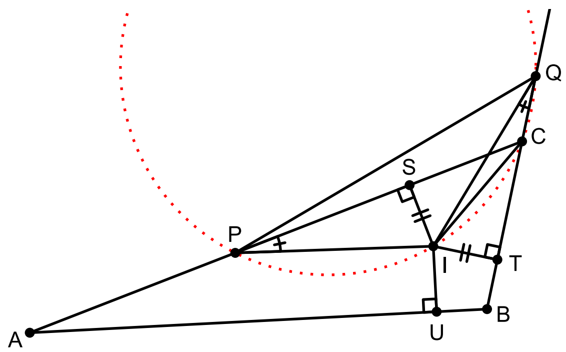


FIGURE 30. $CIPQ$ is cyclic $\Leftrightarrow$ $AP + BQ = AB$

There is another case (see Figure 30) where we have to use a different characterization of cyclic quadrilaterals: The angle between a side and a diagonal in a convex quadrilateral is equal to the angle between the opposite side and the other diagonal if and only if it is a cyclic quadrilateral (Theorem A.1 in [17]). Writing the proof in this case is left as an exercise for the interested reader. □

We conclude this paper by proving the following beautiful characterization, which is about the same configuration with three subtriangle incircles as Theorem 4 in [15].

**Theorem 7.1.** *In a convex quadrilateral $ABCD$, let $E$ be an arbitrary point on side $CD$, and let $I_1$, $I_2$, $I_3$ be the incenters in triangles $ADE$, $ABE$, $BCE$ respectively. Then $EI_1I_2I_3$ is a cyclic quadrilateral if and only if $ABCD$ is a tangential quadrilateral.*

**Proof.** (⇒) In a tangential quadrilateral $ABCD$, draw the circumcircle to triangle $EI_1I_3$ and denote the points where it cuts the segments $CD$, $EA$, $EB$ as $F$, $G$, $H$ respectively (see Figure 31). Applying Lemma 7.1, we have $DF + AG = AD$ and $CF + BH = BC$. Then

$$AG + BH = AD - DF - CF + BC = AD - CD + BC = AB$$

where the last equality is due to Pitot's theorem. We conclude that $EGI_2H$ is a cyclic quadrilateral according to Lemma 7.1, which also means that $EI_1I_2I_3$ is a cyclic quadrilateral since $E$, $F$, $I_1$, $G$, $H$, $I_3$ are on the same circle.

(⇐) In a convex quadrilateral $ABCD$ where $EI_1I_2I_3$ is a cyclic quadrilateral, let its circumcircle cut the segments $CD$, $EA$, $EB$ at $F$, $G$, $H$

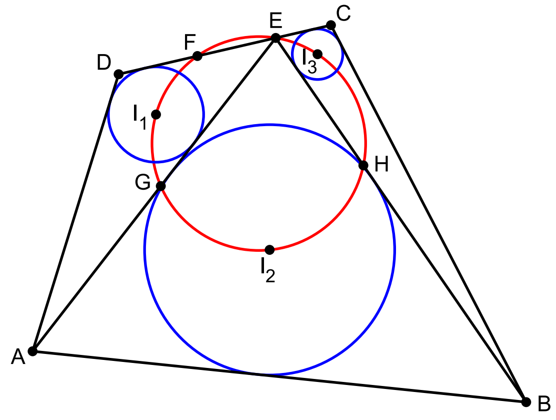


FIGURE 31. $ABCD$ is tangential $\Leftrightarrow$ $EI_1I_2I_3$ is cyclic

respectively (see Figure 31). According to Lemma 7.1, we have

$$DF + AG = AD, \qquad CF + BH = BC, \qquad AG + BH = AB.$$

Hence

$$\begin{aligned} AB + CD &= AB + CF + DF \\ &= AG + BH + BC - BH + AD - AG \\ &= BC + DA \end{aligned}$$

completing the proof according to the converse of Pitot's theorem. □

## 8. Table with the 100 known characterizations

Here we summarize all the 100 characterizations that we know of for when a convex quadrilateral $ABCD$ can have an incircle. We have tried to list them in chronological order of their discoveries (counting both the necessary and sufficient condition), but it can be very difficult to locate the first time a theorem is published, so we make no promises that we have been successful in that attempt. The abbreviations TST and ARO stand for Team Selection Test and All Russian Olympiad.

There are a few changes of notations in the table compared with the original sources since we have used the following notations in all characterizations: Opposite sides $AB$, $CD$ and $BC$, $DA$ intersect at $J$ and $K$ respectively (but $K$ have another meaning when used together with the letter $L$, see these sources for details), the diagonals intersect at $P$, and two opposite internal angle bisectors intersect at $I$. Other notations are not explained here, since the purpose of this table is only to get an overview of all characterizations. If you want specifics, please consult the references.

| **No.** | **Year** | **Surname** | **Short description** | **Ref.** |
|---|---|---|---|---|
| 1 | 1815 | Durrande | $AB + CD = BC + DA$ | [27] |
| 2 | 1935 | Simionescu | $AB \cdot CD - BC \cdot DA = AC \cdot BD \sin\theta$ | [24] |
| 3 | 1954 | Fetisov | 4 concurrent internal angle bisectors | [7] |
| 4 | 1954 | Fetisov | 3 concurrent internal angle bisectors | [7] |
| 5 | 1954 | Iosifescu | $\tan\frac{x}{2}\tan\frac{z}{2} = \tan\frac{y}{2}\tan\frac{w}{2}$ | [23] |

| | | | | |
|---|---|---|---|---|
| 6 | 1973 | Grossman | $BJ + BK = DJ + DK$ | [8] |
| 7 | 1973 | Grossman | $AJ - AK = CJ - CK$ | [8] |
| 8 | 1995 | Vaynshtejn | $\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$ | [29] |
| 9 | 1995 | Vasilyev | $\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}$ | [29] |
| 10 | 1996 | Vaynshtejn | 4 cyclic subtriangle incenters | [30] |
| 11 | 1996 | Vaynshtejn | $T'_1T'_3 = T'_2T'_4$ | [30] |
| 12 | 1999 | Pop | $T_{AIB} + T_{CID} = T_{BIC} + T_{DIA}$ | [24] |
| 13 | 2001 | Gusić | 2 tangent subtriangle incircles | [9] |
| 14 | 2002 | [Iran TST] | 4 cyclic subtriangle excenters | [1] |
| 15 | 2008 | [ARO] | 2 incircles visible under the same angle | [5] |
| 16 | 2009 | Minculete | Equation (2) in | [23] |
| 17 | 2009 | Minculete | $\frac{AB}{T_{APB}} + \frac{CD}{T_{CPD}} = \frac{BC}{T_{BPC}} + \frac{DA}{T_{DPA}}$ | [23] |
| 18 | 2009 | Minculete | Proposition 2 (e) in | [23] |
| 19 | 2011 | Josefsson | $ZS + VW = TU + XY$ | [12] |
| 20 | 2011 | Josefsson | $\frac{1}{R_1} + \frac{1}{R_3} = \frac{1}{R_2} + \frac{1}{R_4}$ | [12] |
| 21 | 2011 | Josefsson | Theorem 6 in | [12] |
| 22 | 2011 | Josefsson | Theorem 8 in | [12] |
| 23 | 2011 | Josefsson | Another 4 cyclic subtriangle excenters | [12] |
| 24 | 2011 | Josefsson | $\frac{1}{R_a} + \frac{1}{R_c} = \frac{1}{R_b} + \frac{1}{R_d}$ | [12] |
| 25 | 2011 | Hoehn | $agh + cef = beh + dfg$ | [11] |
| 26 | 2012 | Josefsson | $R_1 + R_3 = R_2 + R_4$ | [13] |
| 27 | 2012 | Josefsson | $R_a \cdot R_c = R_b \cdot R_d$ | [13] |
| 28 | 2014 | Josefsson | $\angle AIB + \angle CID = \pi = \angle AID + \angle BIC$ | [14] |
| 29 | 2014 | Josefsson | Theorem 2 in | [14] |
| 30 | 2014 | Josefsson | Incircles in $AJK$ and $CJK$ are tangent | [14] |
| 31 | 2014 | Josefsson | First $KLMN$ is cyclic (Theorem 6) | [14] |
| 32 | 2014 | Josefsson | Second $KLMN$ is cyclic (Theorem 8) | [14] |
| 33 | 2014 | Hess | Third $KLMN$ is cyclic (Theorem 5) | [10] |
| 34 | 2017 | Josefsson | Concurrent angle bisectors at $B$, $J$, $K$ | [15] |
| 35 | 2017 | Josefsson | Circle cuts out equal chords on 4 sides | [15] |
| 36 | 2017 | Josefsson | Common tangent to 3 incircles | [15] |
| 37 | 2017 | Josefsson | $KN$, $LM$, $AC$ are concurrent | [15] |
| 38 | 2017 | Josefsson | $I_1J_1$, $AC$, $BD$ are concurrent | [15] |
| 39 | 2019 | Josefsson | $\frac{1}{h_a} + \frac{1}{h_c} = \frac{1}{h_b} + \frac{1}{h_d}$ | [16] |
| 40 | 2020 | Dalcín | $V'W' = X'Y'$ | [18] |
| 41 | 2020 | Josefsson | $U'V' \vert\vert T'W'$ | [18] |
| 42 | 2020 | Dalcín | $S_1T_1 \cdot W_1X_1 = U_1V_1 \cdot Y_1Z_1$ | [18] |
| 43 | 2020 | Dalcín | 2 excircles opposite $BD$ are tangent | [18] |
| 44 | 2020 | Dalcín | $S''Z'' = X''Y''$ | [18] |
| 45 | 2020 | Dalcín | $S_3S_4 = V_3V_4$ | [18] |
| 46 | 2020 | Dalcín | $U''V'' \vert\vert W''T''$ | [18] |
| 47 | 2020 | Dalcín | $ST \cdot WX = UV \cdot YZ$ | [18] |
| 48 | 2020 | Dalcín | $S_a \cdot S_c = S_b \cdot S_d$ | [18] |
| 49 | 2020 | Josefsson | $r_a \cdot r_c = r_b \cdot r_d$ | [18] |
| 50 | 2020 | Dalcín | First $EFGH$ is cyclic (Theorem 5.1) | [18] |
| 51 | 2020 | Dalcín | Fourth $KLMN$ is cyclic (Theorem 5.2) | [18] |
| 52 | 2020 | Dalcín | $K'L'M'N'$ is cyclic | [18] |
| 53 | 2021 | Dalcín | Theorem 2.1 in | [19] |
| 54 | 2021 | Dalcín | Theorem 2.2 in | [19] |
| 55 | 2021 | Dalcín | $\pm ZS \pm VW = \pm TU \pm XY$ | [19] |
| 56 | 2021 | Dalcín | $T_1X_1 = V_1Z_1$ | [19] |
| 57 | 2021 | Josefsson | $[S_1T_1W_1X_1] = [Y_1Z_1U_1V_1]$ | [19] |
| 58 | 2021 | Dalcín | $T_2X_2 = U_2Y_2$ | [19] |
| 59 | 2021 | Dalcín | $T_2X_2 = V_2Z_2$ | [19] |
| 60 | 2021 | Dalcín | $[S_2T_2W_2X_2] = [Y_2Z_2U_2V_2]$ | [19] |
| 61 | 2021 | Josefsson | $\angle GEQ = \angle EGQ$ | [19] |

| | | | | |
|---|---|---|---|---|
| 62 | 2021 | Josefsson | 2 internal and 1 external angle bisector | [19] |
| 63 | 2021 | Josefsson | Perpendicular angle bisectors at $E$ & $F$ | [19] |
| 64 | 2021 | Josefsson | Perpendicular angle bisectors at $E'$ & $F'$ | [19] |
| 65 | 2021 | Josefsson | Second $EFGH$ is cyclic (Theorem 5.1) | [19] |
| 66 | 2021 | Josefsson | $M_1M_2M_3M_4$ is cyclic | [19] |
| 67 | 2021 | Dalcín | $M_5M_6M_7M_8$ is cyclic | [19] |
| 68 | 2021 | Dalcín | $O_1O_2O_3O_4$ is cyclic | [19] |
| 69 | 2021 | Dalcín | $S'T'$, $X'W'$, $AC$ are concurrent | [19] |
| 70 | 2021 | Dalcín | $Z'U'$, $Y'V'$, $AC$ are concurrent | [19] |
| 71 | 2021 | Josefsson | $S''T''$, $X''W''$, $AC$ are concurrent | [19] |
| 72 | 2021 | Josefsson | $Z''U''$, $Y''V''$, $AC$ are concurrent | [19] |
| 73 | 2021 | Dalcín | $EF$, $HG$, $AC$ are concurrent | [19] |
| 74 | 2021 | Josefsson | $O_1$, $I$, $O_3$ are collinear | [19] |
| 75 | 2021 | Josefsson | Diagonals of $O_1O_2O_3O_4$ intersect at $I$ | [19] |
| 76 | 2021 | Josefsson | $I$ lies on Newton's line | [19] |
| 77 | 2021 | Josefsson | $I$ is equidistant to 2 opposite sides | [20] |
| 78 | 2021 | Josefsson | $HK = FL$ | [20] |
| 79 | 2021 | Dalcín | $MH + PQ = NO + GL$ | [20] |
| 80 | 2021 | Dalcín | $M'H + PQ' = N'O + GL'$ | [20] |
| 81 | 2021 | Josefsson | $EJ + FK = GJ + HK$ | [20] |
| 82 | 2021 | Dalcín | $\pm E_1E_2 \pm F_1F_2 \pm G_1G_2 \pm H_1H_2 = 0$ | [20] |
| 83 | 2021 | Dalcín | $LA + AD = DC + CM$ | [20] |
| 84 | 2021 | Dalcín | $AL = CM$ | [20] |
| 85 | 2021 | Dalcín | $AL' = CM'$ | [20] |
| 86 | 2021 | Dalcín | $JN = KO$ | [20] |
| 87 | 2021 | Dalcín | $JR = KS$ | [20] |
| 88 | 2021 | Dalcín | $AR' = CS'$ | [20] |
| 89 | 2021 | Dalcín | $RN' = SO'$ | [20] |
| 90 | 2021 | Dalcín | $AH'' = CG''$ | [20] |
| 91 | 2021 | Dalcín | $JP = KQ$ | [20] |
| 92 | 2021 | Dalcín | Excircles to $AJK$ and $CJK$ are tangent | [20] |
| 93 | 2021 | Dalcín | Excircles to $BJK$ and $DJK$ are tangent | [20] |
| 94 | 2021 | Josefsson | $\frac{BE}{BF} = \frac{DG}{DH}$ | [20] |
| 95 | 2021 | Josefsson | Partition of a quadrilateral into 3 kites | [20] |
| 96 | 2021 | Josefsson | Line $L$ goes through $I$ | [20] |
| 97 | 2021 | Josefsson | 2 pairs of concurrent circles | [20] |
| 98 | 2021 | Josefsson | $EG$, $FH$, $AC$ are concurrent | [20] |
| 99 | 2021 | Josefsson | $I_1J_3$, $I_2J_4$, $I_3J_1$, $I_4J_2$ are concurrent | [20] |
| 100 | 2021 | Josefsson | $EI_1I_2I_3$ is cyclic | [20] |

Table 1. Characterizations of tangential quadrilaterals

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