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FROM ANY TWO DIRECTLY SIMILAR FIGURES, PRODUCE A NEW ONE

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Abstract. In this short paper, we consider a theorem which states that if corresponding points of two directly similar figures are joined and divided in the same proportion, the points of division form another figure directly similar to the original two figures. The paper provides a purely geometrical proof of this theorem.

1. Introduction

The following theorem appears in a paper by Duane DeTemple and Sonia Harold [1], as well as in a very interesting paper connected to van Aubel's theorem by Michael de Villiers [2].

Theorem 1: If F and F be any two directly similar figures with points P in F corresponding to points P' in F', and the lines PP' are divided in the ratio of $r : 1 - r$, that is, at points $P'' = (1 - r)P + rP'$, then the new figure F'' formed by the points P'' will be directly similar to F and F' (see fig.1).

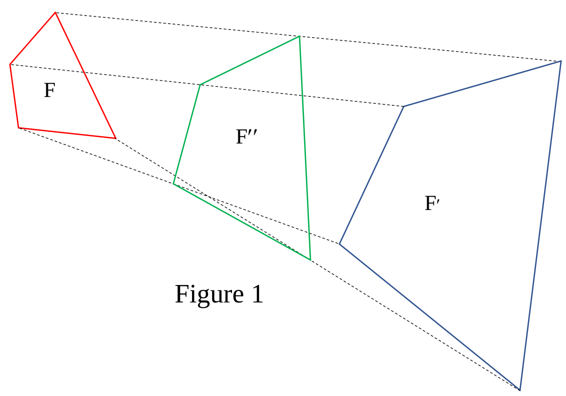

Figure 1

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To get a feeling for the use of the theorem, it is worth looking at a special case (considered in both [1] and [2]) in which the two figures share a common vertex. In figure 2, for example, we have two squares $ABCD$ and $CB'D'C'$ with A corresponding to C (so that $C = A'$), B to B', C to C', D to D'. The segments AC, BB', CC' and DD' are bisected (so that $r = 1/2$) at A'', B'', C'', D''. Because of the special configuration A'' and C'' are the respective centers of the two squares. Theorem 1 then tells us that $A''B''C''D''$ is also a square.

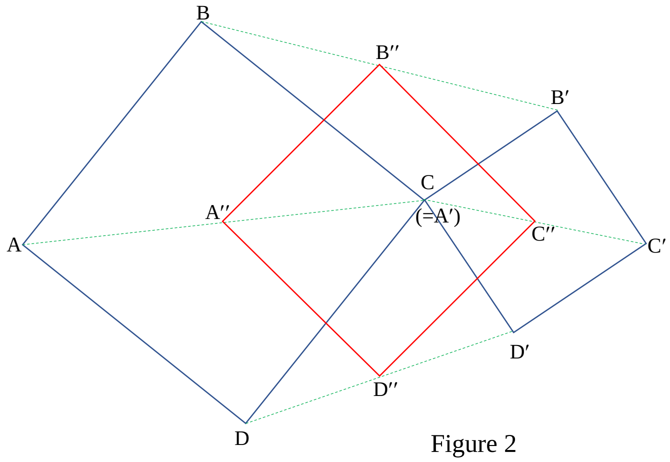

Figure 2

Now, consider just the sides $A''D''$ and $C''D''$. Let us rotate the two squares through an angle of 180° about D'' so that we form a parallelogram DCD' with squares on each of its sides (fig.3). If A'' is mapped onto P and C'' onto Q, then it follows immediately that $QD''C''$ and $PD''A''$ are two equal perpendicular lines that bisect each other at D'', so that A'', C'', P, Q are the vertices of a square. We have thus proven Thébault's theorem: *The quadrilateral formed by joining the centers of squares constructed on the sides of a parallelogram is a square.*

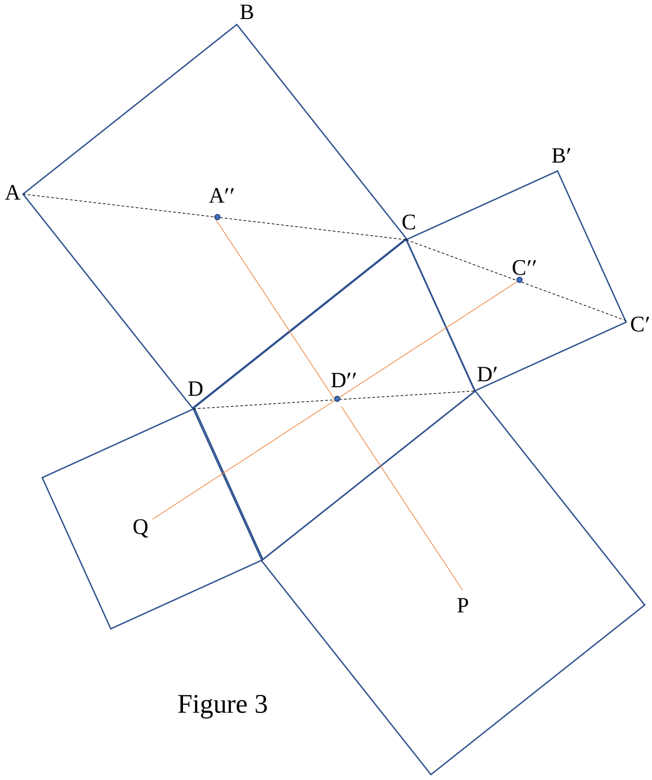

Figure 3

2. Two Proofs of Theorem 1 and a Visualization

The proof of Theorem 1 suggested in [1] uses complex numbers. The proof is very simple, especially if we recall that any two directly similar figures are related by a spiral transformation, that is, the combination of a rotation and dilation [3].

Proof (1). Let $z_1, z_2, ...$ be points in F and $z_1', z_2', ...$ be the corresponding points in the directly similar figure F'. Let us suppose too that F and F' have been situated in the complex plane so that the origin is the center of the spiral transformation, $w \to Ke^{it}z = Hz$ Thus if $z'' = (1-r)z + rz'$ we have:

$$z_j'' - z_k'' = (1-r)(z_j - z_k) + r(z_j' - z_k')$$

$$= (1-r)(z_j - z_k) + rH(z_j - z_k) = (1-r+H)((z_j - z_k) = M(z_j - z_k)$$

where $M = 1-r+H$. Thus, the figure F'' formed by the points z'' is related to those forming F by a spiral transformation, so that F and F'' are directly similar.

But a purely geometric proof can be given in the following way. Again, we rely on the fact that any two directly similar figures are related by a spiral similarity.

Proof (2): Let the two figures be $ABCD$ and $A'B'C'D'$ where, again, A corresponds to A', B to B', etc. And let these be related by a spiral similarity with center O (fig.4).

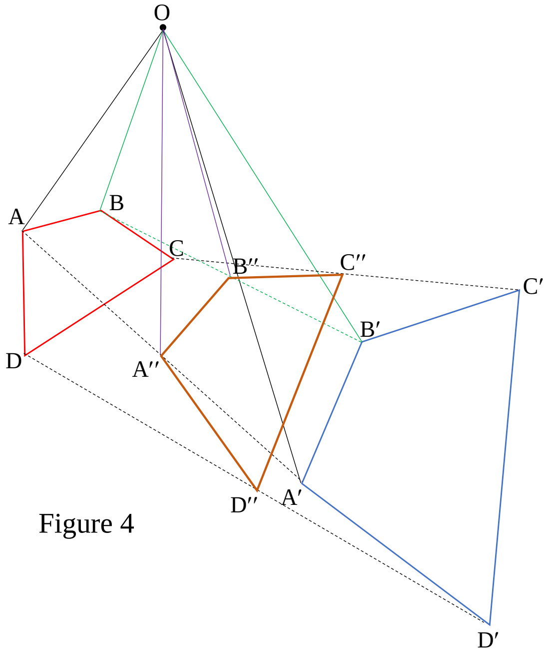

Figure 4

Let AA', BB', CC', DD' be divided each in the same ratio at points A'', B'', C'', D''. We will show that these latter points are the images of A, B, C, D under a spiral similarity with center O.

Proof: By the original spiral similarity between $ABCD$ and $A'B'C'D'$,

$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \frac{OD}{OD'}$$

and

$$\angle AOA' = \angle BOB' = \angle COC' = \angle DOD'$$

Therefore, the triangles, AOA', BOB', COC', DOD' are all similar. Thus, also, since the the sides AA', BB', CC', DD' be divided each in the same ratio at points A'', B'', C'', D'', the triangles $AOA'', BOB'', COC'', DOD''$ are all similar. Accordingly,

$$\frac{OA}{OA''} = \frac{OB}{OB''} = \frac{OC}{OC''} = \frac{OD}{OD''}$$

and

$$\angle AOA'' = \angle BOB'' = \angle COC'' = \angle DOD''$$

Which is what we needed to show

.

One gets a slightly different picture if one thinks of the situation in three dimensions as follows. The directly similar figures F and F' are placed in parallel planes π and π' and corresponding points P and P' on the respective perimeters are joined. If the lines PP' are cut by a third plane π'' parallel to the first two, it will divide the lines PP' in the same ratio (see fig.5). Moreover, the projection of the figure into the plane π we will have precisely the same figure as figure 1. So, one can say that the parallel sections of the ruled surface formed by the lines joining corresponding points of F and F' will all contain figures directly similar to F. A special case, albeit a trivial one, is the fact that parallel planes cut off circular sections from an oblique cone.

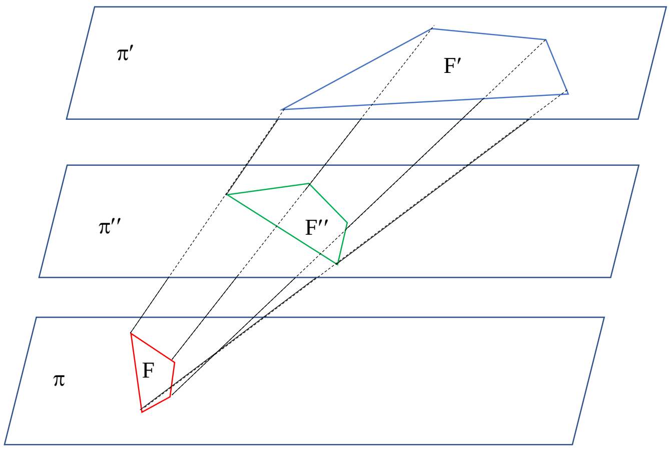

Figure 5

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