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A GENERALIZATION OF PYTHAGORAS ON A SURFACE IN THE SENSE OF TOPONOGOV

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Abstract. By applying Toponogov's sine theorem for an infinitesimal geodesic triangle $\triangle ABC$ on a C^2 regular surface M, which is given in his book [7, Problem 3.7.2] we provide a generalization of the law of cosines for $\triangle ABC$ on M. By replacing in the law of cosines $\angle B = \frac{\pi}{2}$ on M, we obtain a generalization of the Pythagorean theorem for right infinitesimal geodesic triangles, which are slightly bigger than the ones constructed by Gauss on a smooth surface:

$$AC^2 = AB^2 + BC^2 + f(\angle A, \frac{\pi}{2}, AB, BC)o(AC^2)$$

or

$$AC^2 = AB^2 + BC^2 + (\angle A + \angle C - \frac{\pi}{2})^2.$$

where $f(\angle A, \angle B, AB, BC)$ is a rational function w.r. to $\cos\angle A$, $\cos\angle B$, $\sin\angle A$, $\sin\angle B$, AB and BC.

1. Introduction

The law of cosines introduced by Euclid in his Elements (Book II, Proposition 12, 13 in [4]), without using the term cosine, for obtuse angled and acute angled triangles in the Euclidean plane $\mathbb{R}^2$.

In triangle $\triangle ABC$ if angle is obtuse or acute, then

$$AC^2 = AB^2 + BC^2 - 2ABBC\cos\angle B,$$
$$BC^2 = AB^2 + AC^2 - 2ABAC\cos\angle A,$$
$$AB^2 = AC^2 + BC^2 - 2ACBC\cos\angle C.$$

This is the law of cosines (Cosine Theorem) in $\triangle ABC$. By setting, for instance $\angle B = \frac{\pi}{2}$, we derive the Pythagorean theorem

$$AC^2 = AB^2 + BC^2.$$

A generalization of the law of cosines and the law of sines for the two dimensional sphere S^2 and the hyperbolic plane H^2 is given by W. Thurston in his book [6, Chapter 2.4], by using vector calculus.

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W.Thurston considered three unit vectors lying in two dimensional sphere S^2 or to a Lorenz Space $\mathbb{R}^2_1$ in $\mathbb{R}^3$ and defined the dual basis to these vectors and their dot products and applied an inversion w.r to a 3×3 matrix.

This vector process yields the spherical and hyperbolic law of cosines and sines:

Cosine law for a geodesic triangle (arcs of great circles on the sphere) $\triangle ABC$ on the unit sphere $S^2(1)$

$$\cos AC = \cos AB \cos BC + \sin AB \sin BC \cos \angle B,$$
$$\cos BC = \cos AB \cos AC + \sin AB \sin AC \cos \angle A,$$
$$\cos AB = \cos AC \cos BC + \sin AC \sin BC \cos \angle C,$$

Sine law for a geodesic triangle $\triangle ABC$ on the unit sphere $S^2(1)$

$$\frac{\sin AC}{\sin \angle B} = \frac{\sin BC}{\sin \angle A} = \frac{\sin AB}{\sin \angle C}$$

Cosine law for a geodesic triangle $\triangle ABC$ on the hyperbolic plane $H^2(1)$

$$\cosh AC = \cosh AB \cosh BC - \sinh AB \sinh BC \cos \angle B,$$
$$\cosh BC = \cosh AB \cosh AC - \sinh AB \sinh AC \cos \angle A,$$
$$\cosh AB = \cosh AC \cosh BC - \sinh AC \sinh BC \cos \angle C,$$

Sine law for a geodesic triangle $\triangle ABC$ on the hyperbolic plane $H^2(1)$

$$\frac{\sinh AC}{\sin \angle B} = \frac{\sinh BC}{\sin \angle A} = \frac{\sinh AB}{\sin \angle C}.$$

Berg and Nikolaev derived a unified cosine law for the K-plane (a sphere with constant Gaussian curvature $K > 0$ S^2_K and a hyperbolic plane with constant Gaussian curvature $-K < 0$ H^2_K).

We denote by

$$\kappa = \begin{cases} \sqrt{K} & \text{if } K > 0, \\ i\sqrt{-K} & \text{if } K < 0. \end{cases}$$

The unified cosine law for $\triangle ABC$ is given by:

$$\cos(\kappa PQ) = \cos(\kappa PR)\cos(\kappa RQ) + \sin(\kappa PR)\sin(\kappa RQ)\cos(\angle R), \tag{1.1}$$

for $R \in \{A, B, C\}$.

By replacing $\angle R = \frac{\pi}{2}$ in (1.1), we obtain the Pythagorean theorem on surfaces with constant Gaussian curvature (S^2_K, H^2_K).

$$\cos(\kappa PQ) = \cos(\kappa PR)\cos(\kappa RQ)$$

If the radius $R \to +\infty$ of the sphere S^2_K, then we derive the law of cosines in $\mathbb{R}^2$.

C. F. Gauss succeeded in deriving the law of cosines for an infinitesimal geodesic triangle $\triangle ABC$ on a smooth surface ([2, Chapter VII],[3, Livre VI, Chapter VIII]). By introducing normal coordinates w.r. to a vertex, for instance at B and by assuming that $AB = AB_0$ and $BC = BC_0$, (AB_0, BC_0 are planar linear segments), the following generalization of the law of cosines is given:

$$AC^2 = AB^2 + BC^2 - 2ABBC \cos \angle B - \frac{1}{3} K h^2 AC^2$$

or

$$AC^2 = AB^2 + BC^2 - 2ABBC\cos\angle B - \frac{2}{3}KSABBC\sin\angle B$$

or

$$AC^2 = AB^2 + BC^2 - 2ABBC\cos(\angle B - \frac{KS}{3}),$$

where

$$KS = \angle A + \angle B + \angle C - \pi,$$

where h is the height of triangle $\triangle ABC$ from B and K is the Gaussian curvature of the space in the direction of the planar element of the triangle.

By replacing $\angle B = \frac{\pi}{2}$, we obtain:

$$AC^2 = AB^2 + BC^2 - \frac{2}{3}ABBC(\angle A + \angle C - \frac{\pi}{2}).$$

If we consider as an infinitesimal number $o(AC) = (\angle A + \angle C - \frac{\pi}{2}) << 1$, then multiplied by $-\frac{2}{3}ABBC$ yields

$$AC^2 = AB^2 + BC^2 + (\angle A + \angle C - \frac{\pi}{2}).$$

This formula may be considered as a generalization of Pythagoras on a surface for infinitesimal right geodesic triangles in the sense of Gauss. Thus, we consider the following problem.

Problem 1. *Can we derive a generalization of Pythagoras for bigger infinitesimal geodesic triangles than the ones introduced by Gauss on a C^2 regular surface M having geodesics without self intersections?*

V. Toponogov introduced an important generalization for infinitesimal geodesic triangles on M, which are bigger than the infinitesimal geodesic triangles in the sense of Gauss. In this paper, we obtain a positive answer w.r to Problem 1, by using Toponogov's sine theorem for infinitesimal geodesic triangles on M and by generalizing the law of cosines for infinitesimal geodesic triangles on M. As a special case, we derive the theorem of Pythagoras for right infinitesimal geodesic triangles on a C^2 regular surface M. The law of cosines on M is a generalization of a historical triangle comparison theorem, which in essence serves to confirm Gauss's early realization that curvature and angle excess of triangles are related in [5].

2. Understanding Toponogov's law of sines on a surface

We denote by $\triangle ABC$ an infinitesimal geodesic triangle on a regular surface of class C^2 in $\mathbb{R}^3$, by AB, BC, AC the length of the infinitesimal geodesic arcs, $AC = \delta$ and we set $\angle A \equiv \alpha$, $\angle B \equiv \beta$, and $\angle C \equiv \gamma$.

We continue by analyzing the proof given by V. Toponogov in [7, Problem 3.7.2, Solution].

We note that an arc length parameterization $c(s)$ counting from the vertex A to C shall be used. Let $\sigma(s)$ be a geodesic through $c(s)$, such that $\gamma \equiv \angle(\sigma(s), AC)$ and $B(s) = \sigma(s) \cap AB$, $t(s) = AB(s)$, $l(s) = A(s)B(s)$ and $\beta(s) = \angle AB(s)C$.

By taking into account that $A(s) = c(s)$ and by applying the first variational formula of the length of geodesics. w.r to the arc length s, yields:

$$\frac{dl}{ds} = \cos(\gamma) + \cos(\beta(s))\frac{dt}{ds}$$

We note that the physical parameter s corresponds to the parametrization on AC and not on AB. Therefore, in the first variational formula of the length of geodesics ([7, Lemma 3.5.1]), we need to set $\cos(\beta(s))\frac{dt}{ds}$, instead of $\cos(\beta(s))$. Then, he uses the following lemma:

Lemma 1. *Take on M an infinitesimal geodesic triangle $\triangle ABC$ and assume that a region D bounded by $\triangle ABC$ is homeomorphic to a disk. The following two formulas connects the angles of $\triangle ABC$ with the Gaussian curvature K the region D and the Landau symbol $o(AC)$:*

$$\angle A + \angle B + \angle C - \pi = \int\int_D K dS, \tag{2.1}$$

where

$$\int\int_D K dS = o(s). \tag{2.2}$$

Lemma 1 is a special case of a classical theorem of Gauss-Bonnet for an infinitesimal region D. By applying Lemma 1 w.r. to $\triangle AB(s)C(s)$, we get:

$$\beta(s) = \pi - \alpha - \gamma + o(s) \tag{2.3}$$

$$\cos(\beta(s)) = \cos(\pi - \alpha - \gamma + o(s))$$

$$\cos(\beta(s)) = -\cos(\alpha + \gamma) + o(s)\sin(\alpha + \gamma) + o(o(s)).$$

This is an expansion of Taylor series with respect to $(\pi - \alpha - \gamma)!$ and we get:

$$\cos(\beta(s)) = -\cos(\alpha + \gamma) + o(s)\sin(\alpha + \gamma) + o(s)$$

$$\frac{dl}{ds} = \cos(\gamma) - \cos(\alpha + \gamma)\frac{dt}{ds} + o(s)\sin(\alpha + \gamma)\frac{dt}{ds} + o(s)\frac{dt}{ds} \tag{2.4}$$

The function $t(s)$ is not supposed to be linear function w.r. s. Thus, dt/ds does not equal to a constant number, but it is continuous on the interval $[0, \delta]$, therefore it is bounded on $[0, \delta]$ by a constant number $C_{1,AB}$. Integrating both parts of the inequality $(\int_0^\delta dt/dsds) \leq C_{1,AB} \int_0^\delta ds)$ and we get $AB \leq C_{1,AB}AC$.

Therefore, by integrating (2.4) w.r. to s from 0 to δ, yields:

$$l(\delta) = BC = \cos(\gamma)\delta - \cos(\alpha + \gamma)AB + (\sin(\alpha + \gamma) + 1)o(\delta)$$

($C_{1,AB}$=AB/AC after integrating with respect to s)

or

$$BC + \cos(\alpha + \gamma)AB = \cos(\gamma)\delta + (\sin(\alpha + \gamma) + 1)o(\delta) \tag{2.5}$$

Similarly, we derive that:

$$AB = \cos(\alpha)\delta - \cos(\alpha + \gamma)BC + (\sin(\alpha + \gamma) + 1)o(\delta)$$

or

$$\cos(\alpha+\gamma)BC + AB = \cos(\alpha)\delta + (\sin(\alpha+\gamma)+1)o(\delta) \tag{2.6}$$

The solution of the linear system of (2.5), (2.6) w.r. to AB, BC yields V. Toponogov's sine theorem for infinitesimal geodesic triangles on a surface:

Theorem 1 (The Sine Theorem of Toponogov)**.** [7, Problem and Solution 3.7.2]

$$AB = \frac{\sin(\gamma)\delta}{\sin(\alpha+\gamma)} + \frac{o(\delta)(1+\sin(\alpha+\gamma))}{(1+\cos(\alpha+\gamma)} \tag{2.7}$$

and

$$BC = \frac{\sin(\alpha)\delta}{\sin(\alpha+\gamma)} + \frac{o(\delta)(1+\sin(\alpha+\gamma))}{1+\cos(\alpha+\gamma)}. \tag{2.8}$$

3. The cosine theorem on a surface

By using Toponogov's theorem for an infinitesimal geodesic triangle $\triangle ABC$ on M, we obtain a generalization of the law of cosines for infinitesimal geodesic triangles on a surface M.

Theorem 2 (Cosine Theorem)**.** *The law of cosines of an infinitesimal geodesic triangle on M is given by:*

$$AC^2 = AB^2 + BC^2 - 2ABBC\cos(\beta) + f(\alpha,\beta,AB,BC)o(AC^2) \tag{3.1}$$

where $f(\alpha,\beta,AB,BC)$ is a rational function w.r. to $\cos\alpha$, $\sin(\alpha)$, $\cos\beta$, $\sin\beta$, *AB and BC*

or

(3.2)

$$AC^2 = AB^2+BC^2-2ABBC\cos(\beta)+f(\alpha,\beta,AB,BC)(\angle A+\angle B+\angle C-\pi)^2.$$

Proof. We set $m(\alpha,\gamma) \equiv 1+\sin(\alpha+\gamma)$.

From (2.8) of the sine theorem of Toponogov, we get:

$$\sin(\alpha+\gamma)BC = \sin(\alpha)\delta + \frac{o(\delta)\sin(\alpha+\gamma)m(\alpha,\gamma)}{m(\frac{\pi}{2}-\alpha,\frac{\pi}{2}-\gamma)} \tag{3.3}$$

By setting $w(\alpha,\gamma) \equiv \frac{\sin(\alpha+\gamma)m(\alpha,\gamma)}{m(\frac{\pi}{2}-\alpha,\frac{\pi}{2}-\gamma)}$ and by replacing $w(\alpha,\gamma)$ in (3.3), we derive that:

$$\sin(\alpha+\gamma)BC = \sin(\alpha)\delta + w(\alpha,\gamma)o(\delta) \tag{3.4}$$

By squaring both parts of (2.5) and (3.4) and by adding the two derived equations, we obtain that:

(3.5)

$$AB^2 + BC^2 + 2ABBC\cos(\alpha+\gamma) = AC^2 + (m^2(\alpha,\gamma)+w^2(\alpha,\gamma))o(AC^2) + \\ +2(\cos\alpha m(\alpha,\gamma)+\sin\alpha w(\alpha,\gamma))ACo(AC)$$

By using the property of Landau symbol $o(AC^2) = ACo(AC)$, we obtain:

(3.6) $$AB^2 + BC^2 + 2ABBC\cos(\alpha+\gamma) = AC^2 + \\ +(m^2(\alpha,\gamma) + w^2(\alpha,\gamma) + 2(\cos\alpha m(\alpha,\gamma) + \sin\alpha w(\alpha,\gamma))o(AC^2).$$

By replacing $\alpha+\gamma = \pi - \beta + o(AC)$ in $\cos(\alpha+\gamma)$, we obtain:

(3.7)
$$\cos(\alpha+\gamma) = \cos(\pi - \beta + o(AC)) = \cos\beta(1 - o(AC)^2/2) - \sin\beta(o(AC))$$

By replacing (3.7) in (3.6) and taking into account properties of the Landau symbols o, we obtain:

(3.8)
$$AC^2 = AB^2 + BC^2 - 2ABBC\cos(\beta) + f(\alpha,\beta,AB,BC)(\angle A + \angle B + \angle C - \pi)^2.$$

□

Corollary 1. *For $\angle A + \angle B + \angle C = \pi$, we derive the law of cosines in $\mathbb{R}^2$.*

By replacing $\angle B = \frac{\pi}{2}$ in Theorem 2, we derive as a special case of the cosine theorem for infinitesimal geodesic triangles on M the generalized Pythagorean Theorem for right infinitesimal geodesic triangles on a surface M.

Theorem 3 (The generalized Pythagorean Theorem on M)**.** *The generalized theorem of Pythagoras for $\triangle ABC$, and $\angle B = \frac{\pi}{2}$ on a surface is given by:*

$$AC^2 = AB^2 + BC^2 + f(\angle A, \frac{\pi}{2}, AB, BC)o(AC^2)$$

or

$$AC^2 = AB^2 + BC^2 + (\angle A + \angle C - \frac{\pi}{2})^2.$$

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