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A structure on the circumcircle

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Abstract. We make a journey through several problems in Triangle Geometry and arrive to a certain structure of points on the circumcircle of a triangle. From points X_{175} and X_{176} as centers of Soddy circles to some (perhaps) new propositions, a nice experience is about to start.

1. Introduction

Let ABC be a triangle. We draw the three internal bisectors, intersecting at the incenter I, the center of the incircle.

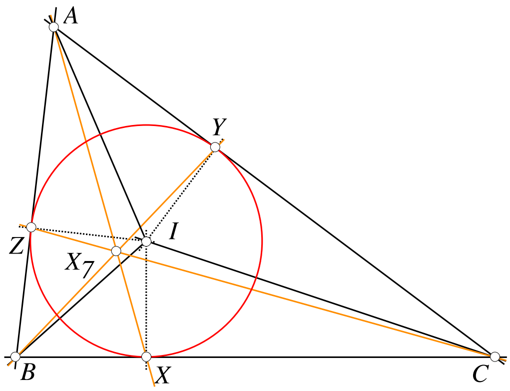

Figure 1

If the incircle touches the sides BC, CA, AB at X, Y, Z respectively and we write $YA = AZ = u$, $ZB = BX = v$, $XC = CY = w$ then, if we call s the semiperimeter of ABC, we have

$$\begin{cases} v+w=a \\ w+u=b \\ u+v=c \end{cases} \Rightarrow 2(u+v+w)=2s \Rightarrow u+v+w=s \Rightarrow \begin{cases} u=s-a \\ v=s-b \\ w=s-c \end{cases}.$$

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Figure 1 shows that I satisfies the property that the pedal triangle XYZ of I is perspective with ABC and the perspector is the Gergonne point X_7 of ABC [1].

2. A problem in a equilateral triangle

The problem shown in Figure 2 is proposed as Problem 1471 in *GoGeometry* website by Antonio Gutierrez.

Problem 1. *Let ABC be a triangle and P a point. Call U, V, W the incenters of the triangles PBC, PCA and PAB. Prove that lines through U, V, W perpendicular to BC, CA, AB are concurrent.*

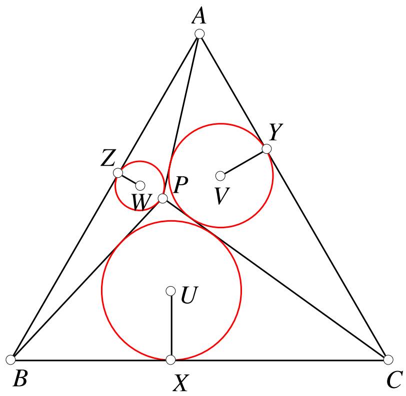

Figure 2

Solution. Call X, Y, Z the contact points of circles (U), (V), (W) and the sidelines BC, CA, AB of ABC. By using the formulas obtained in the previous section, we have

$$BX = \frac{a + PC - PB}{2}, CY = \frac{a + PA - PC}{2}, AZ = \frac{a + PB - PA}{2}$$

and

$$XC = \frac{a + PB - PC}{2}, YA = \frac{a + PC - PA}{2}, ZB = \frac{a + PA - PB}{2}.$$

Then from the identity

$$\begin{aligned}&(a+u)^2 + (a+v)^2 + (a+w)^2 - (a-u)^2 - (a-v)^2 - (a-w)^2\\ =&4a(u+v+w)\end{aligned}$$

we easily get

$$BX^2 + CY^2 + AZ^2 = XC^2 + YA^2 + ZB^2$$

and our problem is solved by the converse of Carnot Theorem (see [1]).

Problem 2 (Carnot Theorem)**.** *Let ABC be a triangle and Q a point. If perpendiculars QX, QY, QZ are drawn from Q to BC, CA, AB, respectively, then we have $BX^2+CY^2+AZ^2 = XC^2+YA^2+ZB^2$, and conversely.*

[1]The locus of points satisfying this property is the Darboux cubic, catalogued in CTC ([3]) as $K004$, containing the incenter I (X_1 in the Encyclopedia of Triangle Centers). Here are some values of n for which n lies on Darboux cubic: 1, 3, 4, 20, 40, 64, 84, 1490, 1498, 2130, 2131, 3182, 3183, 3345, 3346, 3347, 3348, 3353, 3354, 3355, 3472, 3473, 3637. The curve also goes through the excenters.

3. The Soddy circles

From $AY = AZ = s - a$, $BZ = BX = s - b$, and $CX = CY = s - c$, the circles centered at A, B, C with radii $s - a$, $s - b$ and $s - c$ are pairwise tangent to each other.

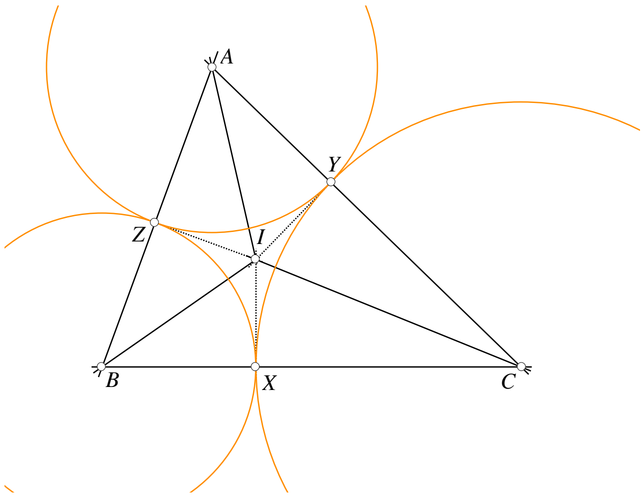

Figure 3

Now we want to construct the circles touching externally (or internally) the three circles (A), (B), (C). These are known as the Soddy circles (see Figure 4).

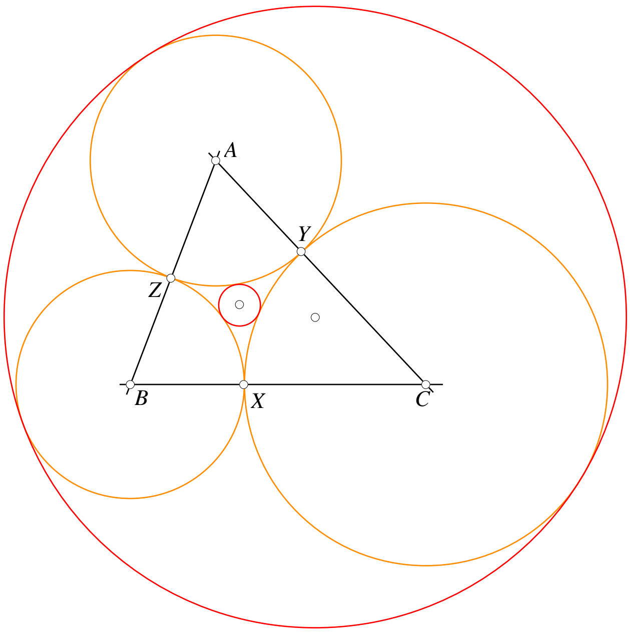

Figure 4

To do that, we use an inversion with center X. In this way the circles (B) and (C) will be inverted in two lines perpendicular to BC. The radius of inversion is chosen in such a way that the circle (A) is fixed, that is, the radius of inversion is the length of the tangent from the point X to the circle (A).

The circles (B) and (C) are inverted in the lines tangent to (A) and perpendicular to BC (see Figure 5).

We construct the circle DEF congruent to (A), also tangent to the same lines tangent to (A) and perpendicular to BC, and farther than (A) from BC. If D', E', F' are the inverse of D, E, F, the circle $D'E'F'$ touches (A), (B), (C) externally.

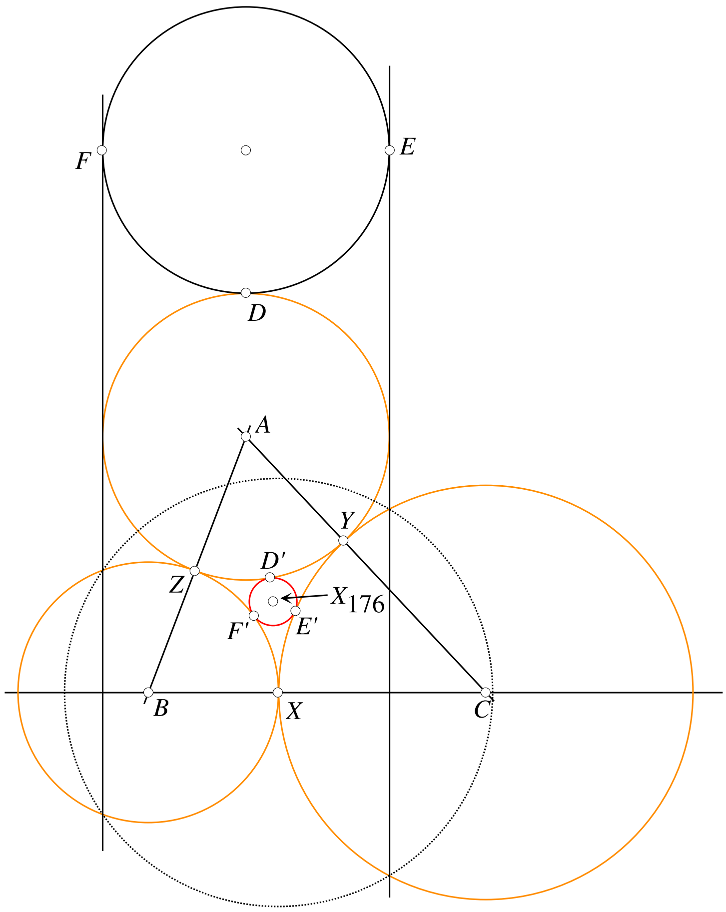

FIGURE 5

The center of this circle, known as *inner Soddy circle* is catalogued as X_{176} in ETC. In barycentric coordinates we have

$$X_{176} = \left(2a + \frac{S}{s-a} : 2b + \frac{S}{s-b} : 2c + \frac{S}{s-c}\right).$$

The circle touching internally the three circles (A), (B), (C) is constructed in a similar way, using another circle congruent to (A) but on the other side (see Figure 7).

The center of this circle, known as *outer Soddy circle*, is catalogued as X_{175}. In barycentric coordinates we have

$$X_{175} = \left(2a - \frac{S}{s-a} : 2b - \frac{S}{s-b} : 2c - \frac{S}{s-c}\right).$$

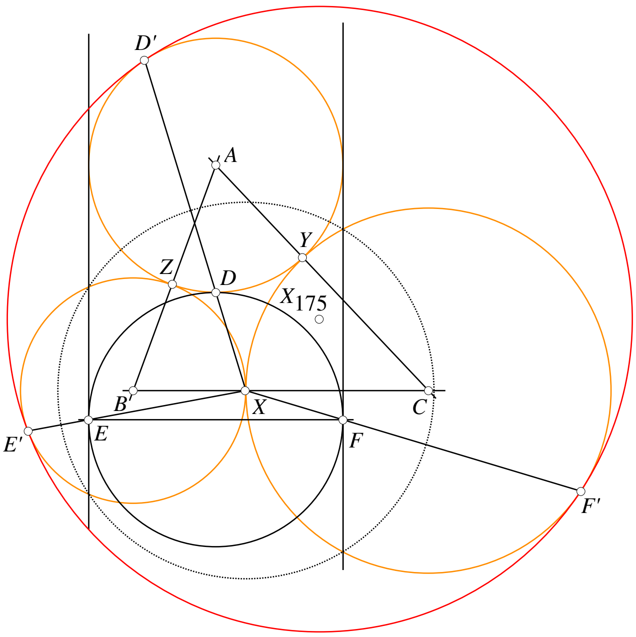

FIGURE 6

The points X_{175} and X_{176} are harmonic conjugates of each other with respect X_1 and X_7.

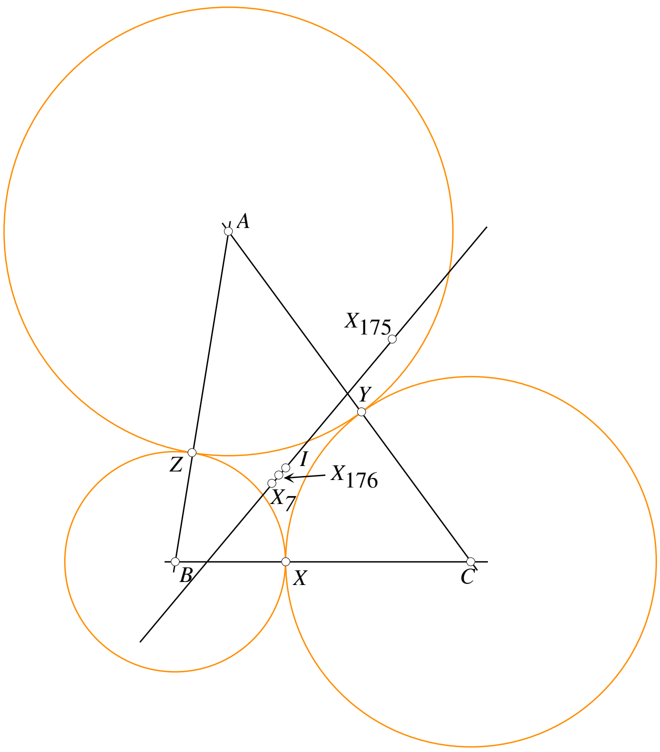

FIGURE 7

4. A property of X_{176}

Let P be the point X_{176}. We have called D', F', E' the contact points of inner Soddy circle and circles (A), (B), (C). We also observe that X, E', F' are the contact point with BC, CP, PB and the incircle (U) of PBC.

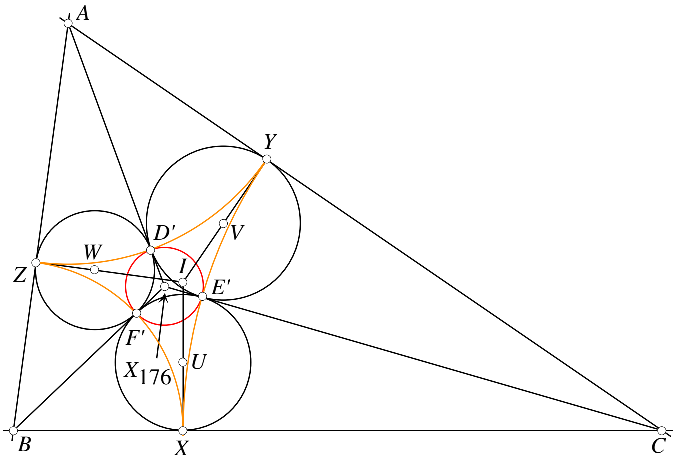

Figure 8

We have:

$$U = \left(a : b + \frac{S}{s-b} : c + \frac{S}{s-c}\right), \text{ etc.}$$

$$UX = \frac{2(s-a)(s-b)(s-c)}{S + 2a(s-a)}, \text{ etc.}$$

5. The case of X_{175}.

This is the figure for $P = X_{175}$. In this case, the lines XU, YV, ZW concur at X_{40}, the Bevan point

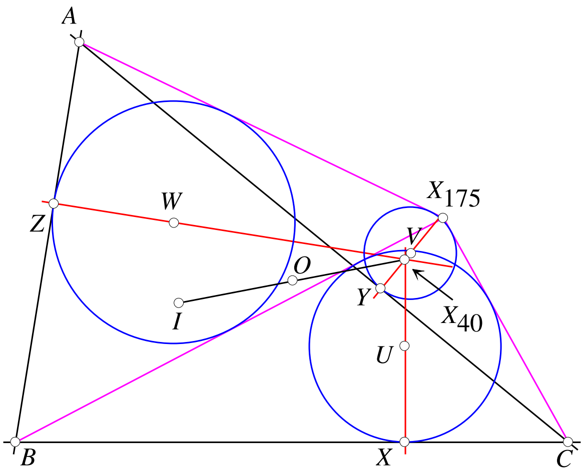

Figure 9

6. A locus problem

This property of X_{176} leads to a locus problem:

Problem 3. *Let ABC a triangle. For any point P, call U, V, W the incenters of PBC, PCA, PAB, respectively, and X, Y, Z the orthogonal projections of U, V, W on BC, CA, AB, also respectively. Find the locus of points P such that lines XU, YV, ZW are concurrent.*

By using barycentric coordinates we find a cubic $\mathcal{K}$ through X_n for $n = 3, 105, 175, 176, 516, 3513, 3514$. This cubic has been catalogued as X_{1175} in CTC.

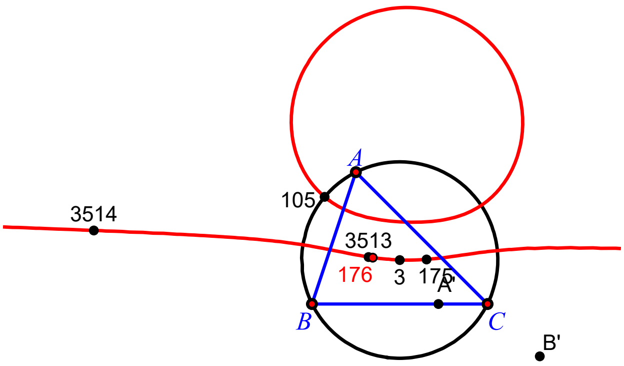

FIGURE 10

We see that X_{105} lies on the circumcircle. The cubic $\mathcal{K}$ interesects the circumcircle at other three points A', B', C'. They are the isogonal conjugates of the points

$$A'' = \big(a(ab + ac - b^2 - c^2) : b(c - a)(s - b) : -c(a - b)(s - c)\big),$$
$$B'' = \big(-a(b - c)(s - a) : b(ab + bc - c^2 - a^2) : c(a - b)(s - c)\big),$$
$$C'' = \big(a(b - c)(s - a) : -b(c - a)(s - b) : c(ac + bc - a^2 - b^2)\big).$$

To answer this question, we observe that X_{105}, as a point on the circumcircle, is the isogonal conjugate of some infinite point. What if we can identify X_{105} more precisely as the isogonal conjugate of the infinite point of the trilinear polar of some point?

7. A vision of points on the circumcircle

We know that the points of the circumcircle are isogonal conjugates of infinite points, therefore, given a point on the circumcircle, it is natural wondering about the direction of its isogonal point. In turn, we can associate an infinite point to the infinite point of the trilinear polar of some point on the circumcircle.

A little research with barycentric coordinates gives the following general result.

Proposition 7.1. *Given any point P on the circumcircle, there exists exactly another point Q on the circumcircle such that the infinite point of the trilinear polar of Q is the isogonal conjugate of P.*

Solution. Q is the second intersection of the circumcircle and the line joining P and the centroid G of ABC. In barycentric coordinates, if $P = (u : v : w)$ then we have

$$Q = \left(\frac{a^2}{u(v - w)} : \frac{b^2}{v(w - u)} : \frac{c^2}{w(u - v)}\right).$$

In our case we have $P = X_{105}$, then $Q = X_{100}$. To say it again, X_{105} is the isogonal conjugate of the infinite point of trilinear polar of X_{100}.

Now we give an interpretation of points A', B', C'.

Proposition 7.2. *The isogonal conjugates of A', B', C' are the infinite points of the sidelines of the cevian triangle of X_{100}.*

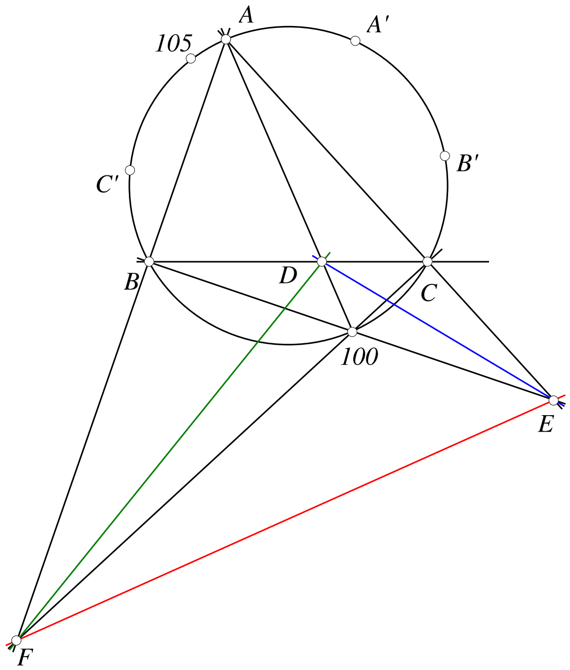

Figure 11

8. Some remarks by Bernard Gibert

- The cubic $\mathcal{K}$ is a circular cubic self-inverse in the circumcircle (O) of ABC.
- It meets the sidelines of ABC at X, Y, Z cevian triangle of X_{927} on (O).
- The second intersections A', B', C' with (O) of the lines through X_{105} and X, Y, Z respectively are on the cubic.
- It follows that the cubic is a isocubic with respect to $A'B'C'$ with pivot X_{105} and isopivot O.
- See $K112$, $K336$, $K436$ which are analogous pKs with respect to ABC.

9. A general property

We can generalize the structure described Proposition 1, Proposition 2 and Gibert's remarks:

Proposition 9.1. *Let ABC be a triangle with centroid G and P a point on the circumcircle. Let Q be the second intersection of line GP and the circumcircle and DEF the cevian triangle of Q. Call A', B', C' the isogonal conjugates of the infinite points of lines EF, FD and DE and X, Y, Z the intersections of PA', PB', PC' and lines BC, CA, AB, respectively. Then the triangle XYZ is the cevian triangle of some point R on the circumcircle.*

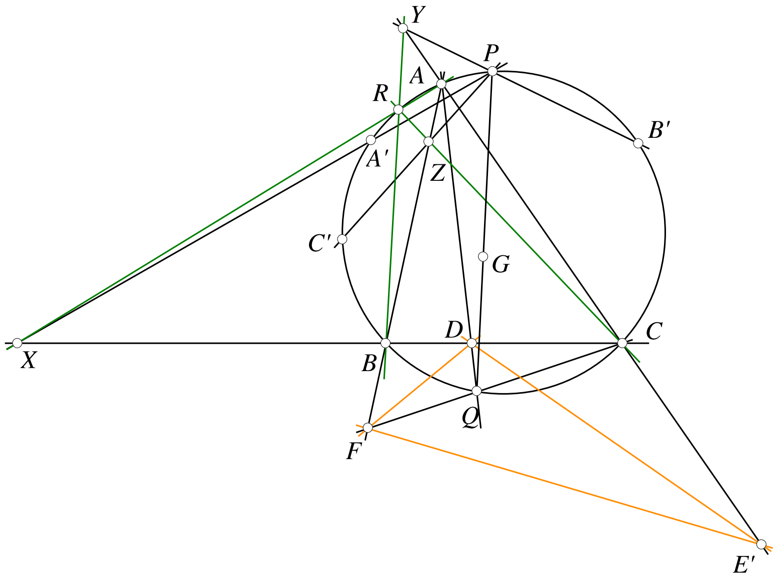

FIGURE 12

Remark. When $P = X_{105}$, we have $Q = X_{100}$ and $R = X_{927}$.

If $P = (u : v : w)$, then the isogonal conjugate of R has coordinates

$$\left(a^2\left(c^2S_Bv - b^2S_Cw\right) : b^2\left(a^2S_Cw - c^2S_Au\right) : c^2\left(b^2S_Au - a^2S_Bv\right)\right).$$

Now we identify the points P that share the same R:

Proposition 9.2. *Let ABC be a triangle and P a point on the circumcircle. If P' is the second intersection of the line joining P and X_{25}, then P and P' share the same point R.*

In barycentric coordinates, if $P = (u : v : w)$ then $P = (u' : v' : w')$ where

$$\begin{aligned} u' : v' : w' = &p^2S_BS_C(qc^2 + rS_C)(rb^2 + qS_B) \\ &:q^2S_CS_A(ra^2 + pS_A)(pc^2 + rS_C) \ . \\ &:r^2S_AS_B(pb^2 + qS_B)(qa^2 + pS_A). \end{aligned}$$

When $P = X_{105}$ we have $P' = X_{108}$. These two points share the same $R = X_{927}$. As we have said before, for $P = X_{105}$, we have $Q = X_{100}$. For $P = X_{108}$, we have $Q = S$, where S is the isogonal conjugate of X_{3827}. The point S itself is not currently catalogued in *ETC*.

In the figure below, A', B', C' are the isogonal conjugates of the infinite points of the sidelines of X_{100} while A'', B'', C'' are the isogonal conjugates of the infinite points of the sidelines of S.

We see that lines $X_{105}A'$, $X_{108}A''$, $X_{927}A$ concur at the same X on BC, and the same for CA and AB.

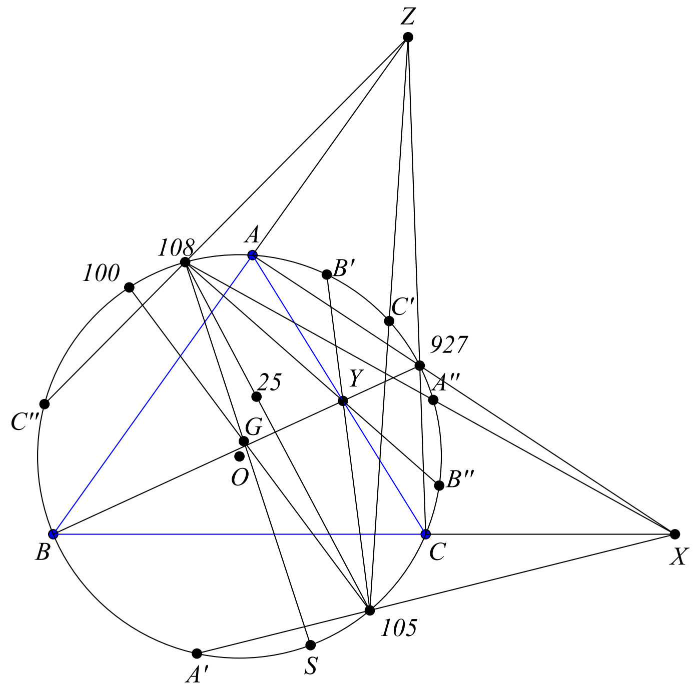

FIGURE 13

Some expamples of points $P = X_n$ for which P' is some X_m both in ETC are listed in the following table:

n	74	98	99	100	101	104	105	106	107	110	111
m	9064	107	2374	15344	9085	9107	108	9088	98	3563	112

We identify some of the points X_n in the following table:

n	Name
94	Isogonal conjugate of X_{50}
98	Tarry point
99	Steiner point
100	Anticomplement of Feuerbach point
101	Ψ(Incenter, Symmedian point)
104	Antipode of X_{100}
105	Λ(Incenter, Symmedian Point)
106	Λ(Incenter, Centroid)
107	Ψ(Symmedian point, Orthocenter)
108	Ψ(Circumcenter, Incenter)
110	Focus of Kiepert parabola
111	Parry point
112	Ψ(Orthocenter, Symmedian point)

For more information, see [4]

10. The relationship between R and the line PP'.

We may want to know about the reverse. If a point R on the circumcircle is given, which line through X_{25} is PP'? In other words, in terms of infinite points, what is the relationship between the infinite point of line PP' and the isogonal conjugate of R?

Proposition 10.1. *The infinite point of line PP' is the image of the isogonal conjugate of R by the homography in the line at infinity maping the infinite points of the sidelines to the infinite points of the cevians of X_{25}.*

We may want to visualize the homography φ from the line at infinity to itself mapping the infinite points of lines BC, CA, AB to the infinity points of cevians of a point P.

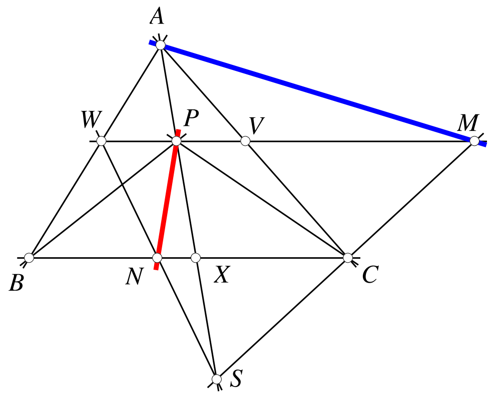

Figure 14

We consider the intersection points V, W of the parallel to BC through P and CA, AB respectively, and the intersection point X of AP and BC.

Since BC, CA, AB meet VW at ∞, V, W and the lines AP, BP, CP meet line BC at X, B, C, the homography φ gives another homography $\psi : VW \to BC$ such that $\psi(\infty) = X$, $\psi(V) = B$ and $\psi(W) = C$.

The line AX is the axis of the homography ψ, because $BW \cap CV = A$ and $BC \cap VX = X$. To find the image of any M on line VW, we use a pair of homologous points (W, C) and we construct $S = MC \cap AP$ and $N = SW \cap BC$. Then we have $N = \psi(M)$ and $\varphi(\infty_{AM}) = \varphi(\infty_{PN})$.

In barycentric coordinates, if $P = (u : v : w)$, and $J = (x : y : z)$ is an infinite point, then

$$\phi(J) = (u(wy - vz) : v(uz - wx) : w(vx - uy).) .$$

11. Equation of the cubic

The cubic $\mathcal{K}$ is now catalogued as

K1175 - Soddy García Capitán cubic

in CTC where many properties found by Bernard Gibert can be seen.

The equation of the cubic $\mathcal{K}$ is

$$\begin{aligned}
&a^4b^4x^3 - 4a^3b^5x^3 + 6a^2b^6x^3 - 4ab^7x^3 + b^8x^3 - 2a^4b^2c^2x^3 + 4a^3b^3c^2x^3 - \\
&2a^2b^4c^2x^3+4a^3b^2c^3x^3-8a^2b^3c^3x^3+4ab^4c^3x^3+a^4c^4x^3-2a^2b^2c^4x^3+4ab^3c^4x^3- \\
&2b^4c^4x^3-4a^3c^5x^3+6a^2c^6x^3-4ac^7x^3+c^8x^3+2a^6b^2x^2y-8a^5b^3x^2y+13a^4b^4x^2y- \\
&12a^3b^5x^2y+8a^2b^6x^2y-4ab^7x^2y+b^8x^2y-2a^6c^2x^2y+4a^5bc^2x^2y-4a^4b^2c^2x^2y+ \\
&8a^3b^3c^2x^2y - 12a^2b^4c^2x^2y + 8ab^5c^2x^2y - 2b^6c^2x^2y + 4a^5c^3x^2y - 8a^4bc^3x^2y + \\
&4a^3b^2c^3x^2y - a^4c^4x^2y + 8a^2bc^5x^2y - 4ab^2c^5x^2y - 4a^2c^6x^2y - 4abc^6x^2y + \\
&2b^2c^6x^2y + 4ac^7x^2y - c^8x^2y + a^8xy^2 - 4a^7bxy^2 + 8a^6b^2xy^2 - 12a^5b^3xy^2 + \\
&13a^4b^4xy^2 - 8a^3b^5xy^2 + 2a^2b^6xy^2 - 2a^6c^2xy^2 + 8a^5bc^2xy^2 - 12a^4b^2c^2xy^2 + \\
&8a^3b^3c^2xy^2 - 4a^2b^4c^2xy^2 + 4ab^5c^2xy^2 - 2b^6c^2xy^2 + 4a^2b^3c^3xy^2 - 8ab^4c^3xy^2 + \\
&4b^5c^3xy^2-b^4c^4xy^2-4a^2bc^5xy^2+8ab^2c^5xy^2+2a^2c^6xy^2-4abc^6xy^2-4b^2c^6xy^2+ \\
&4bc^7xy^2 - c^8xy^2 + a^8y^3 - 4a^7by^3 + 6a^6b^2y^3 - 4a^5b^3y^3 + a^4b^4y^3 - 2a^4b^2c^2y^3 + \\
&4a^3b^3c^2y^3-2a^2b^4c^2y^3+4a^4bc^3y^3-8a^3b^2c^3y^3+4a^2b^3c^3y^3-2a^4c^4y^3+4a^3bc^4y^3- \\
&2a^2b^2c^4y^3+b^4c^4y^3-4b^3c^5y^3+6b^2c^6y^3-4bc^7y^3+c^8y^3-2a^6b^2x^2z+4a^5b^3x^2z- \\
&a^4b^4x^2z-4a^2b^6x^2z+4ab^7x^2z-b^8x^2z+4a^5b^2cx^2z-8a^4b^3cx^2z+8a^2b^5cx^2z- \\
&4ab^6cx^2z + 2a^6c^2x^2z - 4a^4b^2c^2x^2z + 4a^3b^3c^2x^2z - 4ab^5c^2x^2z + 2b^6c^2x^2z - \\
&8a^5c^3x^2z+8a^3b^2c^3x^2z+13a^4c^4x^2z-12a^2b^2c^4x^2z-12a^3c^5x^2z+8ab^2c^5x^2z+ \\
&8a^2c^6x^2z - 2b^2c^6x^2z - 4ac^7x^2z + c^8x^2z - 2a^8xyz + 4a^7bxyz - 2a^6b^2xyz + \\
&4a^5b^3xyz-8a^4b^4xyz+4a^3b^5xyz-2a^2b^6xyz+4ab^7xyz-2b^8xyz+4a^7cxyz- \\
&8a^6bcxyz+4a^4b^3cxyz+4a^3b^4cxyz-8ab^6cxyz+4b^7cxyz-2a^6c^2xyz+6a^4b^2c^2xyz- \\
&8a^3b^3c^2xyz+6a^2b^4c^2xyz-2b^6c^2xyz+4a^5c^3xyz+4a^4bc^3xyz-8a^3b^2c^3xyz- \\
&8a^2b^3c^3xyz+4ab^4c^3xyz+4b^5c^3xyz-8a^4c^4xyz+4a^3bc^4xyz+6a^2b^2c^4xyz+ \\
&4ab^3c^4xyz-8b^4c^4xyz+4a^3c^5xyz+4b^3c^5xyz-2a^2c^6xyz-8abc^6xyz-2b^2c^6xyz+ \\
&4ac^7xyz+4bc^7xyz-2c^8xyz-a^8y^2z+4a^7by^2z-4a^6b^2y^2z-a^4b^4y^2z+4a^3b^5y^2z- \\
&2a^2b^6y^2z-4a^6bcy^2z+8a^5b^2cy^2z-8a^3b^4cy^2z+4a^2b^5cy^2z+2a^6c^2y^2z-4a^5bc^2y^2z+ \\
&4a^3b^3c^2y^2z-4a^2b^4c^2y^2z+2b^6c^2y^2z+8a^2b^3c^3y^2z-8b^5c^3y^2z-12a^2b^2c^4y^2z+ \\
&13b^4c^4y^2z+8a^2bc^5y^2z-12b^3c^5y^2z-2a^2c^6y^2z+8b^2c^6y^2z-4bc^7y^2z+c^8y^2z+ \\
&a^8xz^2 - 2a^6b^2xz^2 + 2a^2b^6xz^2 - b^8xz^2 - 4a^7cxz^2 + 8a^5b^2cxz^2 - 4a^2b^5cxz^2 - \\
&4ab^6cxz^2+4b^7cxz^2+8a^6c^2xz^2-12a^4b^2c^2xz^2+8ab^5c^2xz^2-4b^6c^2xz^2-12a^5c^3xz^2+ \\
&8a^3b^2c^3xz^2 + 4a^2b^3c^3xz^2 + 13a^4c^4xz^2 - 4a^2b^2c^4xz^2 - 8ab^3c^4xz^2 - b^4c^4xz^2 - \\
&8a^3c^5xz^2+4ab^2c^5xz^2+4b^3c^5xz^2+2a^2c^6xz^2-2b^2c^6xz^2-a^8yz^2+2a^6b^2yz^2- \\
&2a^2b^6yz^2+b^8yz^2+4a^7cyz^2-4a^6bcyz^2-4a^5b^2cyz^2+8a^2b^5cyz^2-4b^7cyz^2- \\
&4a^6c^2yz^2+8a^5bc^2yz^2-12a^2b^4c^2yz^2+8b^6c^2yz^2+4a^3b^2c^3yz^2+8a^2b^3c^3yz^2- \\
&12b^5c^3yz^2 - a^4c^4yz^2 - 8a^3bc^4yz^2 - 4a^2b^2c^4yz^2 + 13b^4c^4yz^2 + 4a^3c^5yz^2 + \\
&4a^2bc^5yz^2-8b^3c^5yz^2-2a^2c^6yz^2+2b^2c^6yz^2+a^8z^3-2a^4b^4z^3+b^8z^3-4a^7cz^3+ \\
&4a^4b^3cz^3+4a^3b^4cz^3-4b^7cz^3+6a^6c^2z^3-2a^4b^2c^2z^3-8a^3b^3c^2z^3-2a^2b^4c^2z^3+ \\
&6b^6c^2z^3-4a^5c^3z^3+4a^3b^2c^3z^3+4a^2b^3c^3z^3-4b^5c^3z^3+a^4c^4z^3-2a^2b^2c^4z^3+ \\
&b^4c^4z^3 = 0.
\end{aligned}$$

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