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RELATED BY SIMILARITY I: PORISTIC TRIANGLES AND 3-PERIODICS IN THE ELLIPTIC BILLIARD

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Abstract. Discovered by William Chapple in 1746, the Poristic family is a set of variable-perimeter triangles with common Incircle and Circumcircle. By definition, the family has constant Inradius-to-Circumradius ratio. Interestingly, this invariance also holds for the family of 3-periodics in the Elliptic Billiard, though here Inradius and Circumradius are variable and perimeters are constant. Indeed, we show one family is mapped onto the other via a varying similarity transform. This implies that any scale-free quantities and invariants observed in one family must hold on the other.

1. Introduction

The *Poristic family* was discovered by William Chapple in 1746 and was later studied by Euler and Poncelet [1, 3, 10]. It is a 1d of set of variable-perimeter triangles (blue) with fixed Incircle and Circumcircle, Figure 1. By definition its Inradius-to-Circumradius r/R ratio is constant. Interestingly, the same ratio is invariant for the family of constant-perimeter 3-periodics in the Elliptic Billiard [5, 16].

Our main contribution is to show that one family is mapped onto the other via a (varying) similarity transform whose parameters we derive explicitly. Therefore, all scale-free (e.g., area and length ratios) quantities and invariants verified for one family must hold for the other.

In a companion article [15] we show that the Brocard Porism and the Homothetic Poncelet family are also related to each other via a variable similarity transform.

Article Summary: we start with preliminaries in Section 2 and then present the following results:

- Theorem 3.1: The Caustic to the Excentral Triangles of the Poristic Family is the MacBeath Inconic of the Excentrals.
- Theorem 3.2: The Inconic I_3' to the Poristic Excentrals centered on its Circumcenter is a rigidly rotating ellipse.
- Theorem 3.3: The Incenter-centered Circumconic E_1 to the Poristic Triangles is also rigidly rotating and of the same aspect ratio as I_3'.

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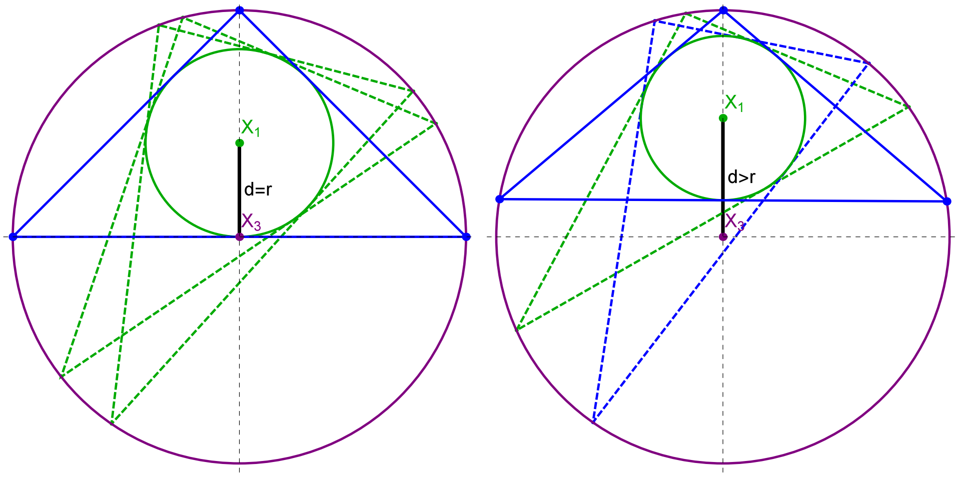

FIGURE 1. Poristic Triangle family (blue): fixed Incircle (green) and Circumcircle (purple). **Left**: With $d = r$, $r/R = \sqrt{2}-1$, all Poristic triangles are acute (dashed green, X_3 is interior) except for the one shown (blue) which is a right-triangle. If $d < r$ (not shown), X_3 will lie in the Incircle (green) and the whole family is acute. **Right**: If $d > r$, X_3 can be both interior or exterior to the triangle, and the family will contain both acute (dashed green) and obtuse (dashed blue) triangles. **Video**: [13, PL#01].

- Theorem 4.1: Poristic and Elliptic Billiard Triangle families are related by a varying similarity transform.

All figures reference illustrative videos in the format [13, PL#nn], where "nn" stands for the position within a playlist. For convenience, all videos mentioned are compiled on Table 4 in Section 5. Table 6 in Appendix A lists all symbols used below.

1.1. **Related Work.** Weaver [19] proved the Antiorthic Axis[1] of this family is stationary. In Appendix B we revisit and add to some of Weaver's original work [19].

Odehnal showed the locus of the Excenters is a circle centered on X_{40} and of radius $2R$ [10]. He also lists several Triangle centers which are stationary. Both circular and elliptic loci are described for Triangle Centers and vertices of derived Triangles. For example, the locus of the Mittenpunkt X_9 is a circle, of known radius and center [10, page 17]; the locus of the vertices of the Tangential Triangle is an ellipse, etc.

Pamfilos studies the family of triangles with fixed circumcircle and 9-point circle [12]. He identifies the poristic intouch triangles as one such family. Since these are homothetic to the excentrals with homothety center X_{57} [6] and since the latter is stationary in the poristic family [10], poristic excentrals will share all properties identified by Pamfilos for the intouch poristics, e.g., fixed circumcircle and 9-point circle, invariant product of cosines, etc. [12].

We previously studied and Circum- and Inconics associated with the family of 3-periodics in the Elliptic Billiard [14], identifying certain semi-axes and focal length ratios to be invariant. The following works prove that the locus of the incenter, barycenter, and circumcenter of elliptic billiard 3-periodics are ellipses [17, 18, 4].

[1]The line passing through the intersections of reference and Excentral sidelines [20].

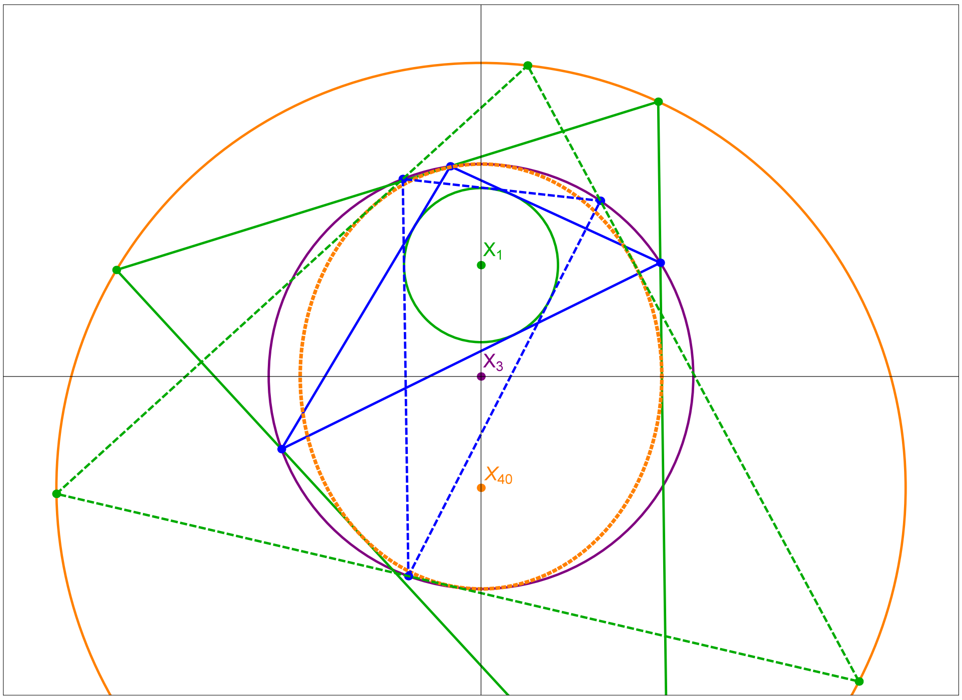

FIGURE 2. The centers of the Incircle (solid green) and Circumcircle (purple) are X_1 and X_3, respectively. A Poristic triangle (solid blue) and its excentral (solid green) with $r/R \simeq 0.3266$. The same are shown (dashed) at a distinct configuration. Odehnal observed the locus of the Excenters is a circle (orange) centered on X_{40} with radius $2R$ [10]. Also shown (dotted orange) is the Caustic to the Excentrals, which is the MacBeath Inconic with centers and foci at $X_i, i = 5, 4, 3$ of the Excentrals and $X_j, j = 3, 1, 40$ of the Poristics. Notice X_{40} is the reflection of X_1 about X_3. **Video**: [13, PL#02].

2. Preliminaries

When referring to Triangle Centers we use Kimberling's notation [7]. Given a generic triangle, let $d = |X_1X_3|$. Let the origin be placed at X_3, with x running along X_1X_3 and y along $(X_3 - X_1)^t$. Note: in all of our figures, for compactness, x is shown vertical. First proven by Chapple [1] though known as a theorem by Euler is the relation:

$$d = \sqrt{R(R - 2r)} \tag{1}$$

Let ρ denote the invariant ratio r/R. For d to be real in (1), $R/r \geq 2$, i.e.:

$$\rho = r/R \in (0, 1/2]$$

Proposition 2.1. *The Poristic family will contain obtuse triangles if $d > r$.*

This stems from the fact that when $d < r$, X_3 is always within the Poristic triangles, Figure 1.

3. Conic Invariants

Some of the results in this section were obtained with the aid of a Computer Algebra System (CAS).

3.1. **Excentral Caustic.** Let I_5' be the MacBeath Inconic [20] to the Excentral Triangles, with center and foci on the latter's[2] X_5', X_4', X_3', i.e., X_3, X_1, X_{40}, of the Poristic family, Figure 2.

Let μ_5' and ν_5' be the major and minor semiaxes of I_5'.

Theorem 3.1. *$\mu_5' = R$ and $\nu_5' = \sqrt{R^2 - d^2}$ are invariant and I_5' is stationary, i.e., it is the Caustic to the family of Excentral Triangles.*

Proof. The sides of the excentral triangle $\ell_i', i = 1, 2, 3$ are defined in (3). Observe the translation $(d, 0)$ in the parametrization of the vertices $P_i(t), i = 1, 2, 3$ given by equation (2) and so $X_3 = (d, 0)$. It is straightforward to verify these are tangent to the ellipse:

$$\frac{(x-d)^2}{R^2} + \frac{y^2}{R^2 - d^2} = 1$$

with center $X_3 = (d, 0)$ and foci $X_{40} = (0, 0)$ and $X_1 = (2d, 0)$.
Applying (1) to $\mu_5'/\nu_5' = R/\sqrt{R^2 - d^2}$ obtain:

Corollary 3.1. *The aspect ratio of I_5' is given by:*

$$\frac{\mu_5'}{\nu_5'} = \frac{1}{\sqrt{2\rho}}$$

Let C' be the circle centered on X_3 and of radius $2R$. Let I_5 be the ellipse centered n X_5 and with foci on X_4 and X_3.

Corollary 3.2. *The conic pair (C', I_5) is associated with a $N = 3$ Poncelet family with stationary X_5.*

This stems from the fact that this pair is the Excentral and Caustic to the Poristic family (taken as reference triangles).

3.2. **Excentral X_3-Centered Inconic.** Let I_3' be the Inconic to the Excentral Triangles centered on their stationary X_3 (X_{40} of the Poristic family). Let μ_3' and ν_3' be the major and minor semiaxes of I_3'.

Theorem 3.2. *$\mu_3' = R + d$ and $\nu_3' = R - d$ are invariant over the Poristic family, i.e., I_3' rigidly rotates about X_{40}.*

Proof. See Appendix C.
As before, applying (1) to $\mu_3'/\nu_3' = (R + d)/(R - d)$ yields:

Corollary 3.3. *The aspect ratio of I_3' is invariant and given by:*

$$\frac{\mu_3'}{\nu_3'} = \frac{1 + \sqrt{1 - 2\rho}}{\rho} - 1$$

Proposition 3.1. *The non-concentric conic pair (C', I_3') is associated with a $N = 3$ Poncelet family with stationary X_3.*

This fact was made originally in [14]:

Remark 3.1. *I_3' contains X_{100}.*

[2] X_i' refers to Triangle Centers of the Excentral Triangle.

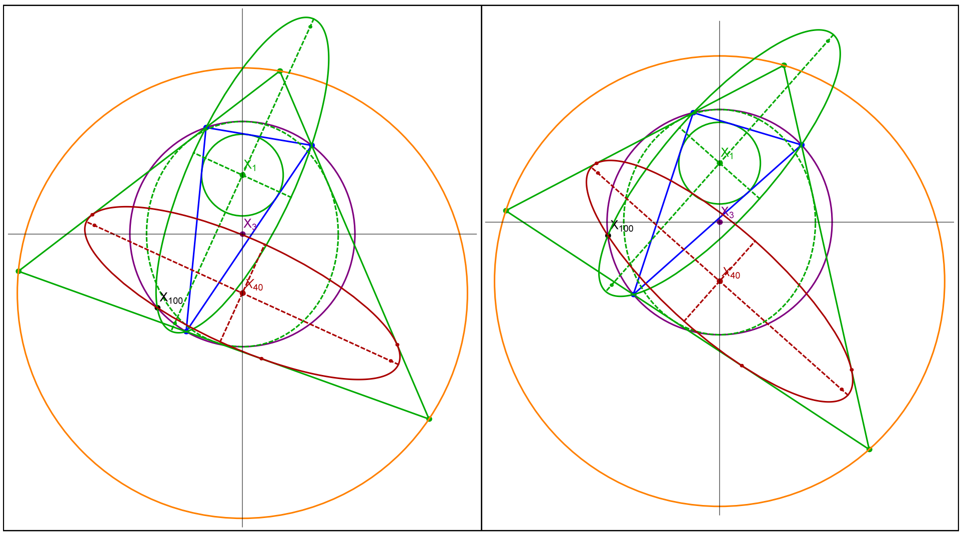

FIGURE 3. Inconic Invariants: two configurations shown of the Poristic Triangle family (blue). The Incircle (green) and Circumcircle (purple) are fixed, and $r/R = 0.3627$. The Excentral Caustic I_5' (dashed green) is the (stationary) MacBeath Inconic with center and foci at $X_i = 5, 4, 3$ of the Excentral, i.e., $X_j, j = 3, 1, 40$ of the Poristic (blue) triangles. The ratio of its semi-axes $\mu_5'/\mu_5 = 1/\sqrt{2\rho}$. Also shown is I_3', the Inconic to the Excentrals centered on its X_3, i.e., X_{40} of the Poristic family (one of the foci of I_5'). Over the family, its semiaxes are invariant at $R+d$ and $R-d$, i.e., this is a rigidly-rotating ellipse about X_{40}. Also shown is E_1 (green ellipse), the X_1-centered circumconic, an 90°-rotated copy of I_3'. **Video:** [13, PL#03]

3.3. **The X_1-Centered Circumconic.** Let E_1 be the Circumconic the Poristic triangles centered on X_1.
Let η_1 and ζ_1 be the major and minor semiaxes of E_1.

Theorem 3.3. *$\eta_1 = R + d$ and $\zeta_1 = R - d$ are invariant over the Poristic family, i.e., E_1 rigidly rotates about X_1.*

Proof. See Appendix D

Corollary 3.4. *The aspect ratio of E_1 is invariant and identical to the aspect ratio of I_3'.*

Proposition 3.2. *E_1 contains X_{100}.*

Proof. E_1 is the set of trilinear triples $p : q : r$ such that:

$$E_1 : (s_2 + s_3 - s_1)/p + (s_1 + s_3 - s_2)/q + (s_1 + s_2 - s_3)/r = 0.$$

In trilinear coordinates $X_{100} = [1/(s_2 - s_3) : 1/(s_3 - s_1) : 1/(s_1 - s_2)]$ and so $E_1(X_{100}) = 0$.

Two configurations for I_5', E_1, I_3' are shown in Figure 3.

3.4. X_{10}**-circumconic.** Let η_{10}, ζ_{10} be the major, minor semi-axes of E_{10}, the X_{10}-centered Circumconic. The locus of X_{10} over the Poristic family is a circle centered on X_{1385} with radius $R/4 - r/2$ [10, page 56]. Let η_5' and ζ_5' be the major, minor semi-axes of E_5', the Circumconic to the Excentrals centered on their X_5 (i.e., X_3 of the Poristics). Referring to Figure 4:

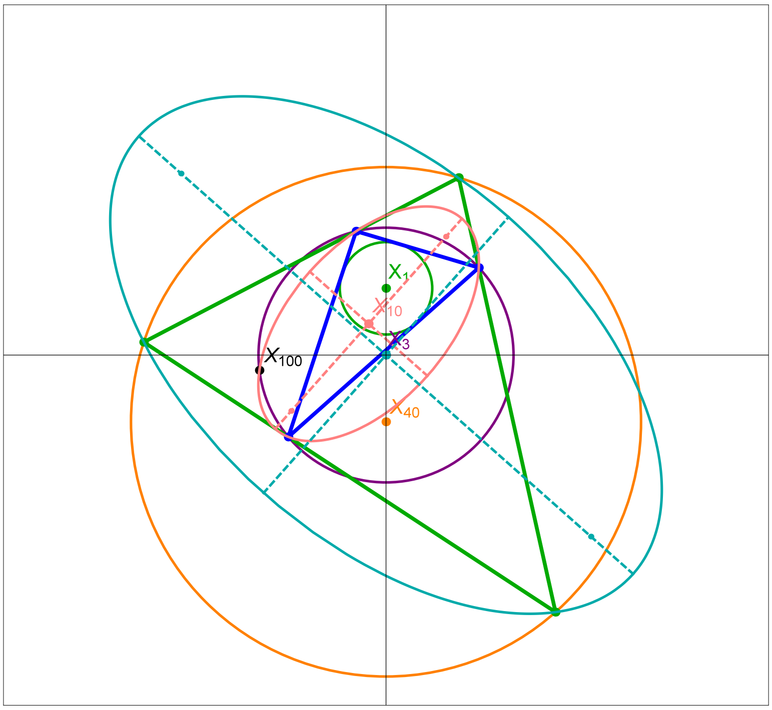

FIGURE 4. The Circumconic E_{10} (pink) to the Poristic triangles (blue) is centered on the Spieker Center X_{10}. Its aspect ratio is invariant over the Poristic family and equal to that of the Circumconic to the Excentral E_5' (light blue), centered on its X_5 (X_3 of the Poristics). **Video**: [13, PL#04]

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Proposition 3.3. *η_{10}/ζ_{10} is invariant and equal to η_5'/ζ_5'. These are given by:*

$$\frac{\eta_5'}{\zeta_5'} = \frac{\eta_{10}}{\zeta_{10}} = \sqrt{\frac{R+d}{R-d}}.$$

Proof. We used a similar approach: generate candidate ratio at isosceles configuration and verify with CAS the ratio is independent of t.

4. Connection with Elliptic Billiards

The Circumbilliard to a generic triangle is the Circumconic which renders the triangle a 3-periodic orbit, i.e., it will be centered on X_9 [14]. Consider the Circumbilliard of a Poristic triangle, Figure 5 and let its semiaxes be denoted by a_9, b_9.

Proposition 4.1. *The perimeter $L(t)$ of a Poristic triangle is given by:*

$$L(t) = \frac{\left(3\,R^2 - 4\,dR\cos t + d^2\right)\sqrt{3\,R^2 + 2\,dR\cos t - d^2}}{R\sqrt{R^2 - 2\,dR\cos t + d^2}}$$

Proof. Follows directly computing $L(t) = |P_1 - P_2| + |P_2 - P_3| + |P_3 - P_1|$ using equation (2) and the relation $r = (R^2 - d^2)/2R$. The long expressions involving square roots were manipulated using a CAS.

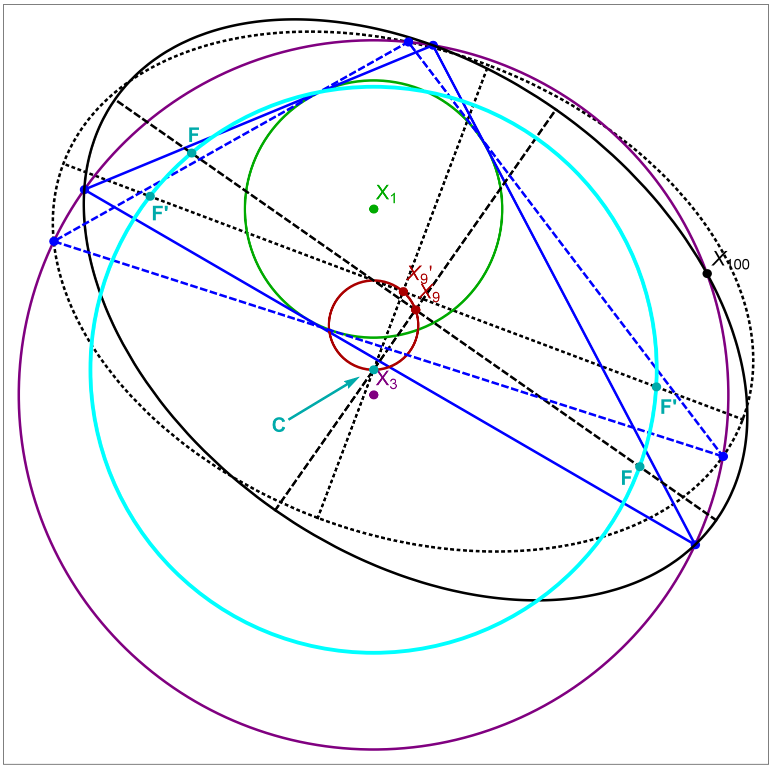

FIGURE 5. Two Poristic triangles (blue and dashed blue) are shown. Also shown are their Circumbilliards (black and dotted black), centered on $X_9(t)$. The locus of X_9 is a circle (red) [10]. It turns out the locus of the CB foci F (cyan) is also a circle centered at C and of radius r_9 (see Proposition 4.3). F' denote the foci of the CB of the second (dashed blue) Poristic triangle. **Video**: [13, PL#05].

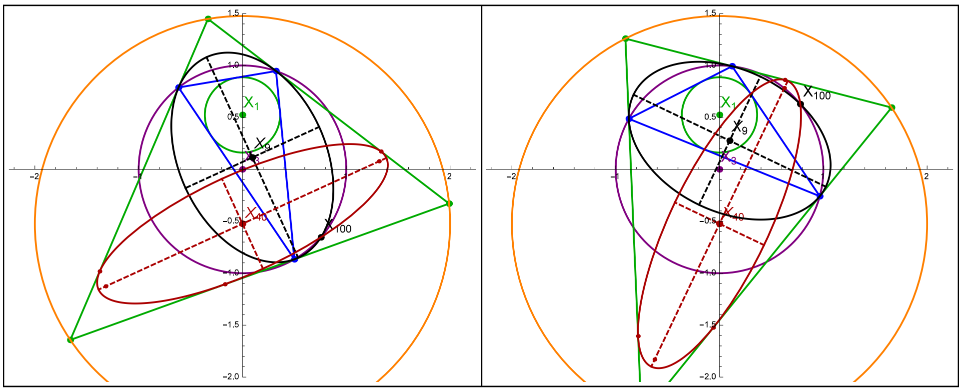

FIGURE 6. Two configurations (left and right) of the Poristic family (blue) for $R = 1, r = 0.36266$. The Incircle and Circumcircle appear green and purple. The Excentral Triangle (green) is shown inscribed in the circular locus (orange) of its vertices [10]. Also shown is I_3' (red, inconic to the Excentrals centered on its Circumcenter) and E_9, the Circumbilliard to the Poristic triangles (black). Over the family, (i) E_9, I_3', E_1 (latter not shown) have invariant aspect ratios, with the latter two identical; (ii) their axes remain parallel; (iii) all meet the Circumcircle at X_{100}. **Video**: [13, PL#06,07].

Theorem 4.1. *The 3-periodic family is the image of the Poristic family under a one-dimensional family of similarity transformations (rigid rotation, translation, and uniform dilation).*

Proof. Let $\Delta(t) = \{P_1(t), P_2(t), P_3(t)\}$ be a Poristic triangle given by (2) translated by $(-d, 0)$ and consider the circumellipse $E_9(t)$ centered on $X_9(t) = (x_9(t), y_9(t))$ with $a_9(t)$ and $b_9(t)$ the major, minor semiaxes.

Odehnal showed that the locus of the Mittenpunkt X_9 is a circle whose radius is $Rd^2R/(9R^2 - d^2)$ and center is $X_1 + (X_1 - X_3)(2R - r)/(4R + r) = d(3R^2 + d^2)/(9R^2 - d^2)$ [10, page 17]. In fact, using the characterization of X_9 as the intersection of lines passing through the vertices of the excentral triangle and the medium points of the triangle $\Delta(t)$, it follows that $X_9(t)$ is parametrized by:

$$X_9(t) = \left[\frac{d\left(4\,d\cos^2 t(R\cos t - d) - r(3\,d\cos t + R) - r^2\right)}{(4\,R + r)(d\cos t - R + r)}, \frac{4Rd^2\sin t\left(R^2 - (2\,R\cos t - d)^2\right)}{(R^2 + d^2 - 2\,dR\cos t)(9\,R^2 - d^2)}\right]$$

Let $\theta(t)$ be the angle between $a_9(t)$ and the line X_1X_3, Figure 6. Using the vertices $P_1(t)P_2(t)P_3(t)$, translated by $(-d, 0)$ and the center $X_9(t)$ we can obtain the equation of the Circumellipse $E_9(t)$. Developing the calculations it follows that the angle of rotation $\theta(t)$ is given by:

$$\tan\theta(t) = \frac{(1 - \cos t)(R + d - 2R\cos t)}{(2R\cos t + R - d)\sin t}$$

Consider the following transformation:

$$\begin{aligned} x &= L(t)(\cos\theta(t)u + \sin\theta(t)v + x_9(t)) \\ y &= L(t)(-sin\theta(t)u + \cos\theta(t)v + y_9(t)). \end{aligned}$$

By construction, the family of Poristic triangles $\Delta(t)$ is the image of the 3-periodic family of the elliptic billiard defined by:

$$\begin{aligned} E(u,v) &= \frac{u^2}{a_9^2} + \frac{v^2}{b_9^2} - 1 = 0 \\ a_9 &= \quad L(t)\frac{R\sqrt{3\,R^2 + 2\,dR - d^2}}{9\,R^2 - d^2} = L(t)\frac{\sqrt{2}\sqrt{\rho + 1 + \sqrt{1 - 2\,\rho}}}{2\,\rho + 8}, \\ b_9 &= \quad L(t)\frac{R\sqrt{R - d}}{\sqrt{3\,R + d}(3\,R - d)} = L(t)\frac{\sqrt{2}\sqrt{\rho + 1 - \sqrt{1 - 2\,\rho}}}{2\,\rho + 8} \\ c_9 &= \quad \sqrt{a_9^2 - b_9^2} = L(t)\frac{2R\sqrt{dR}}{9R^2 - d^2}. \end{aligned}$$

Therefore, the similarity transform is given by $\theta(t), X_9(t), L(t)$.

Corollary 4.1. *The ratios $a_9(t)/L(t)$, $b_9(t)/L(t)$, and $c_9(t)/L(t)$ are invariant over the Poristic family.*

Proposition 4.2. *The aspect ratio of the Circumbilliard is invariant over the Poristic family and given by:*

$$\frac{a_9(t)}{b_9(t)} = \sqrt{\frac{(R + d)\,(3R - d)}{(R - d)\,(3\,R + d)}} = \sqrt{\frac{\rho^2 + 2(\rho + 1)\sqrt{1 - 2\rho} + 2}{\rho(\rho + 4)}}$$

Proof. The following expression for r/R was derived for the 3-periodic family of an a, b Elliptic Billiard [5, Equation 7]:

$$\rho = \frac{r}{R} = \frac{2(\delta - b^2)(a^2 - \delta)}{c^4}$$

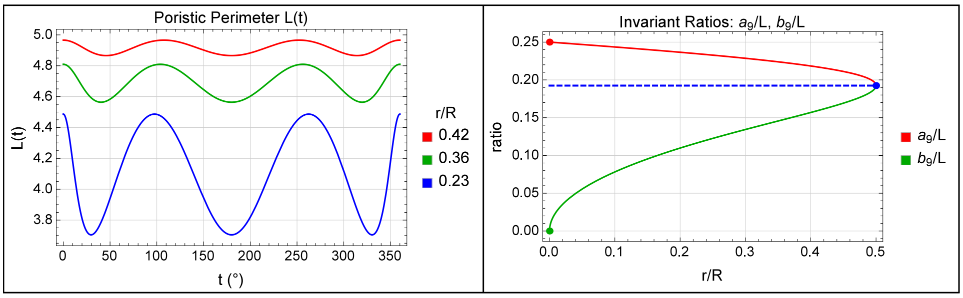

FIGURE 7. **Left**: Perimeter of Poristic triangles vs. parameter t (of one tangent to Incircle) for various value of r/R; **Right**: Invariant Circumbilliard semi-axis ratios $a_9(t)/L(t), b_9(t)/L(t)$ vs $r/R \in [0, 1/2]$. The dashed blue line represents their limit values of $\sqrt{3}/9 \simeq 0.19245$ when $r/R = 1/2$ (equilateral triangles). The red and green dots show that as $r/R \to 0$, $a_9 \to 1/4$ and $b_9 \to 0$.

where $\delta = \sqrt{a^4 - a^2b^2 + b^4}$, and $c^2 = a^2 - b^2$. Solving the above for a/b yields the result.

Figure 7 illustrates the variable perimeter and invariant aspect ratio for the CB of the Poristic family for various values of r/R.

Corollary 4.2. *The axes of the I_3' are parallel to Circumbilliard's.*

This stems from the fact that E_3' for 3-periodics has parallel axes to the CB [14] and the fact that it will be preserved under the similarity transform.

Corollary 4.3. *The axes of the E_{10} and E_6' are parallel to Circumbilliard's axes.*

This stems from the fact that the axes of E_6' are parallel to those of E_{10}, and that the latter has parallel axes to the Circumbilliard [14], see Figure 8.

Corollary 4.4. *The aspect ratio for I_5' and I_3' is the invariant and the same for both Poristic and Billiard families.*

This also stems from the fact that these are true for the EB [14] and that the aspect ratios are preserved by the similarity transform.

Let F be the Feuerbach Hyperbola and J_{exc} be the Excentral Jerabek Hyperbola, Figure 9. Let their focal lengths be γ and γ'.

Corollary 4.5. *The focal length ratio $\gamma'/\gamma = \sqrt{2/\rho}$ is invariant and the same for both Poristic and Billiard families.*

Again, this ratio is invariant for 3-periodics [14] and must be also invariant for the Poristic family.

With the aid of CAS, the following can be shown:

Proposition 4.3. *Over the Poristic family, the foci of the Circumbilliard describe a circle with center $[(R-d)d/(3R+d), 0]$ and radius r_9 given by:*

$$r_9 = \frac{4d(R-d)\sqrt{dR}}{(3R-d)\sqrt{(3R-d)(R+d)}}$$

Let η_6', ζ_6' be the major, minor of the E_6', the Circumconic to the Excentrals centered on their X_6 (X_9 of the Poristics. Referring to Figure 8:

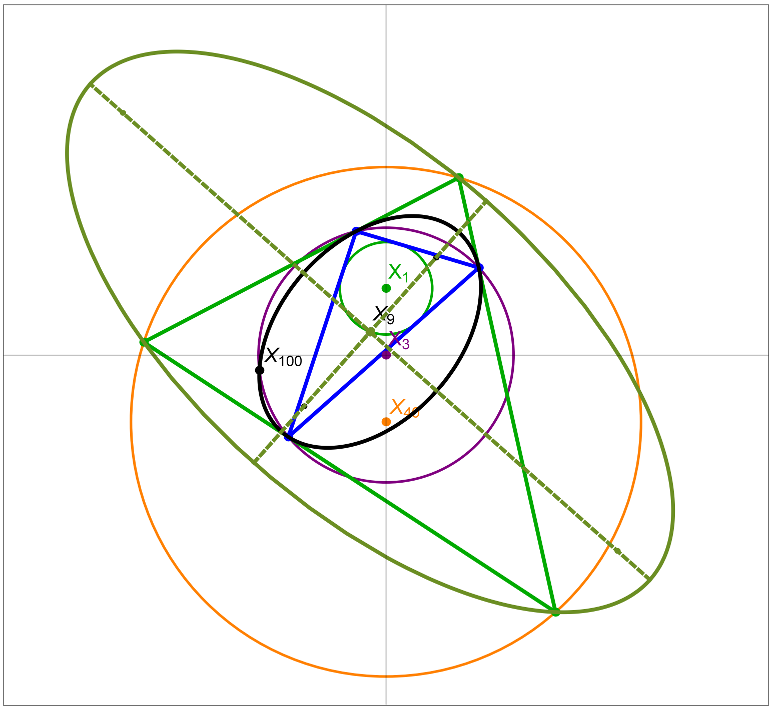

Figure 8. The Circumconic to the Excentral E'_6 (olive green), centered on its X_6 is concentric and axis-parallel to the CB (black). Both conserve their aspect ratio. The locus of the foci of the former is *not* an ellipse, whereas that of the CB is. **Video**: [13, PL#08].

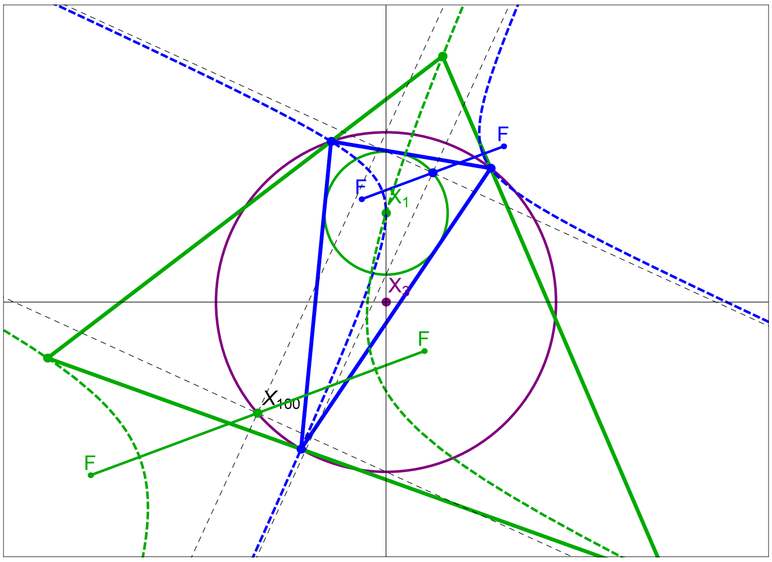

Figure 9. Feuerbach (dashed blue) and Jerabek Circumhyperbolas (dashed green) to a Poristic triangle (blue) and its Excentral (green). Their asymptotes (dashed gray) are parallel (and parallel to the axes of the CB, not shown). Also shown are their foci (blue, green "F"), and their parallel focal axes (solid blue and green). The ratio γ/γ' of their focal lengths is invariant over the family. **Video:** [13, PL#10,11].

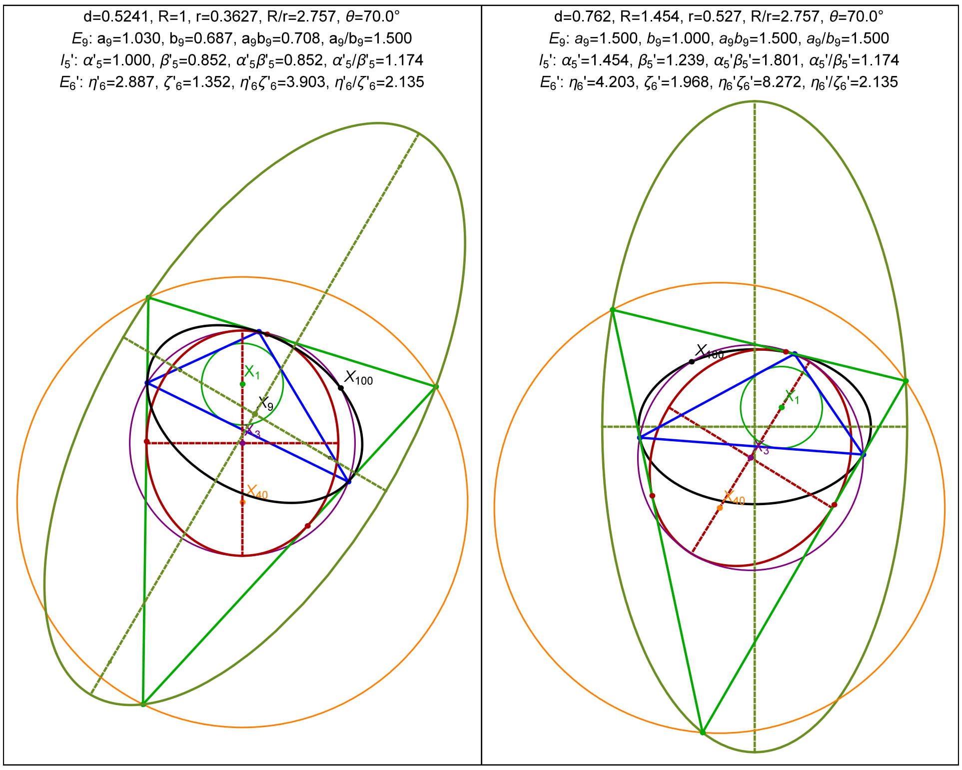

FIGURE 10. **Left:** Poristic triangle (blue), stationary Incircle (green) and Circumcircle (purple). Varying Poristic CB (black), whose aspect ratio is constant. Stationary Excentral MacBeath Inconic and Caustic I_5' (red), circular Excentral locus (orange), and Excentral (MacBeath) Circumconic E_6' (olive green), all with invariant aspect ratios. **Right:** same objects observed on a stationary Elliptic Billiard system: Incircle and Circumcircle are varying (though r/R is invariant). I_5' is moving though its aspect ratio is invariant and equal to its counterpart in the Poristic system. Conversely, E_6' is now stationary and is the locus of the Excenters [4]. Notice the Excentral Circumcircle (orange) is movable. **Video**: [13, PL#09]

Proposition 4.4. *The E_6' is concentric an has parallel axes to the Circumbilliard. Furthermore, its aspect ratio is given by:*

$$\frac{\eta_6'}{\zeta_6'} = \frac{b_9^2+\delta}{a_9}\frac{b_9}{a_9^2+\delta} = \sqrt{\frac{(R+d)(3R+d)}{(3R-d)(R-d)}}$$

$\delta = \sqrt{a_9^4 - a_9^2 b_9^2 + b_9^4}$.

This stems from the fact that E_6' for 3-periodics is the locus of the Excenters, shown to be an ellipse with said aspect ratio [4].

5. Conclusion

Table 1 summarizes properties and invariants for the various circum- and inconics mentioned above. A comparison between basic parameters in the Poristic family and 3-Periodics in the Elliptic Billiard appear on Table 2. Finally, shape invariances of conics in either family are compared on Table 3 and illustrated in Figure 10.

conic	poristic	EB	X_{100}	ctr	note
E_1	axes	ratio	y	X_1	center on F_{med}
E_9	ratio	axes	y	X_9	(Circum-) EB, center on F_{med}
E_{10}	ratio	ratio	y	X_{10}	center on F_{med}
I_9	ratio	axes	–	X_9	Mandart Inellipse, EB Caustic
E'_3	axes	ratio	y	X_{40}	Excentral Circumcircle
E'_5	ratio	ratio	–	X_3	same ratio as E_{10}
E'_6	ratio	axes	–	X_9	MacBeath Circumconic
I'_3	axes	ratio	y	X_{40}	$90°$-rotated copy of E_1
I'_5	axes	ratio	–	X_3	McBeath Inconic, Excentral Caustic

Table 1. Table of conics, all with mutually parallel axes (except for I'_5). Columns "poristic" and "EB" define whether for that family the aspect ratio is invariant. E_k (resp. I_k) stands for the Circumellipse (resp. Inellipse) centered on X_k. E'_k, I'_k refer to Excentral conics.

qty.	poristic	EB
d	y	–
r	y	–
R	y	–
r/R	y	y
$R{\pm}d$	y	–
$\frac{R+d}{R-d}$	y	y
L	–	y
J	–	y

Table 2. Column "poristic" (resp. EB) indicates if the named quantity is invariant in the given family. Only r/R and $(R+d)/(R-d) = f(r/R)$ are invariant on both.

object	ctr	semiaxes	poristic	EB	note
Incircle	X_1	r	y	–	
Circumcircle	X_3	R	y	–	
I'_5	X_3	$R, \sqrt{R^2-d^2}$	y	–	poristic exc. caustic
E'_6	X_9	$(b_9^2+\delta)/a_9, (a_9^2+\delta)/b_9$	–	y	EB exc locus
Exc. Circumcircle	X_{40}	$2R$	y	–	
Elliptic Billiard	X_9	a_9, b_9	–	y	

Table 3. Various position and axes for conics in each Poristic and 3-periodic (EB) families. A "y" in the "poristic" or "EB" columns indicates shape invariance.

Videos mentioned above have been placed on a playlist [13]. Table 4 contains quick-reference links to all videos mentioned, with column "PL#" providing video number within the playlist.

Acknowledgments

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id	Sec.	Title	youtu.be/...
01	1,B	Poristic family, circular locus of excenters, and Antiorthic axis	DS4ryndDK6Q
02	3	Poristic Circumbilliard (CB) has invariant aspect ratio	yEu2aPiJwQo
03	3	E_1 and I_3' have constant, parallel, and identical semi-axes	0VHBjdHXbJc
04	3	E_{10} and E_5' have axes parallel to the Poristic CB as well as invariant, identical aspect ratio	-4AAUSFxvmo
05	4	Loci of center & foci of Poristic CB are circles	LGgh11LMGGY
06	4	I_3' has constant semi-axes, parallel to those of the Poristic CB	0VHBjdHXbJc
07	4	I_3' and I_5' of 3-Periodics in the EB have invariant aspect ratio.	CHbrZvx1I8w
08	4	E_6' has invariant aspect ratio and its axes coincide with those of the Poristic CB	Fy4T-dmu-8s
09	4	Side-by-side Poristic and Elliptic Billiard Excentral MacBeath Inconic & Circumconics	NvjrX6XKSFw
10	4	F and J_{exc} Circumhyperbolas have invariant focal length ratio over 3-periodic family I	bn1tq6NU_y0
11	4	F and J_{exc} Circumhyperbolas have invariant focal length ratio over 3-periodic family II	Pz4tUijYZCA

Table 4. Videos mentioned in the paper. Column "PL#" indicates the entry within the playlist [13]

Appendix A. Table of Symbols

Tables 5 and 6 lists most Triangle Centers and symbols mentioned in the paper.

Appendix B. Weaver Invariants

B.1. **Antiorthic Axis.** The Antiorthic axis $\mathcal{L}_1$ is stationary, and X_{1155} is stationary intersection of $\mathcal{L}_1$ with $\mathcal{L}_{650} = X_1X_3$, Figure 11. The anthiortic axis is given by:

$$x = \frac{3R^2 - d^2}{2d}$$

B.2. **A possible correction to Weaver's 2nd order invariant.** Assume the origin[3] is on X_3. In [19, Theorem III] it is proposed that a circle C_w centered on $[-R, 0]$ and of radius $\sqrt{Rd(R+d)(R+d+r)}/d$ has the same power with respect to the Antiorthic axis $\mathcal{L}_1$ as the Incircle. We have found this not to be the case. Let $I_1 : (x-d)^2 + y^2 = r^2$ denote the Incircle. Referring to Figure 12(left), let C_w' be circle centered on $[-R, 0]$ and of radius:

[3]In Weaver's paper, the origin is on X_1, so the center of C_w is at $[-R-d, 0]$.

Center	Meaning	Note
X_1	Incenter	Trilinear Pole of $\mathcal{L}_1$, focus of I'_5
X_3	Circumcenter	Focus of I_5
X_4	Orthocenter	Focus of I_5
X_5	Center of the 9-Point Circle	Center of I_5
X_6	Symmedian Point	
X_9	Mittenpunkt	Center of (Circum)billiard
X_{10}	Spieker Point	Incenter of Medial
X_{40}	Bevan Point	Focus of I'_5
X_{100}	Anticomplement of X_{11}	Lies on $E_i, i = 1, 3, 9, 10$ and I'_3
X_{650}	Cross-difference of X_1, X_3	Generates X_1X_3
X_{651}	Isogonal Conjug. of X_{650}	Trilinear Pole of $\mathcal{L}_{650} = X_1X_3$
X_{1155}	Schröder Point	Intersection of X_1X_3 with Antiorthic Axis
$\mathcal{L}_1$	Antiorthic Axis	Line $X_{44}X_{513}$ [8]
$\mathcal{L}_{650}$	OI Axis	Line X_1X_3

Table 5. Kimberling Centers and Central Lines mentioned in paper

Symbol	Meaning	Note
P_i, s_i	Vertices and sidelengths of Poristic triangles	
P'_i	Vertices of the Excentral triangle	
X_i, X'_i	Kimberling Center i of Poristic, Excentral	
a_c, b_c	Semi-axes of confocal Caustic	
r, R, ρ	Inradius, Circumradius, r/R	ρ is invariant
d	Distance $\lvert X_1X_3\rvert$	$\sqrt{R(R-2r)}$
a_9, b_9	Semi-axes of Poristic CB	
δ	Constant associated w/ the CB	$\sqrt{a_9^4 - a_9^2b_9^2 + b_9^4}$
E_i	Circumellipse centered on X_i	Axes parallel to E_9 if X_i on F_{med}
E'_i	Excentral Circumellipse centered on X'_i	
η_i, ζ_i	Major and minor semiaxis of E_i	Invariant ratio for $i = 1, 3, 9, 10$
η'_i, ζ'_i	Major and minor semiaxis of E'_i	Invariant ratio for $i = 3, 5, 6$
I_i	Inellipse on X_i	I_3; MacBeath I_5; Mandart I_9
I'_i	Excentral Inellipse centered on X'_i	I'_3; MacBeath I'_5
μ_i, ν_i	Major and minor semiaxis of I_i	
μ'_i, ν'_i	Major and minor semiaxis of I'_i	Invariant ratio for $i = 3, 5$
F_{exc}	F of Excentral Triangle	Center X_{3659} [9]
J_{exc}	J of Excentral Triangle	Center X_{100}, Perspector X_{649}
F_{med}	F of Medial	Center X_{3035} [9]
λ', λ	Focal lengths of J_{exc}, F	Invariant ratio

Table 6. Symbols used in paper

$$r'_w = \left(\frac{d+R}{2R}\right)\sqrt{\frac{(3\,R-d)(4\,R^2 - Rd - d^2)}{d}}$$

Proposition B.1. *C'_w and I_1 have the same power with respect to $\mathcal{L}_1$.*

Proof. Translate the vertices of the Poristic family in (2) by $(-d, 0)$. It is straightforward to show that the Antiorhtic axis $\mathcal{L}_1$ is given by $x =$

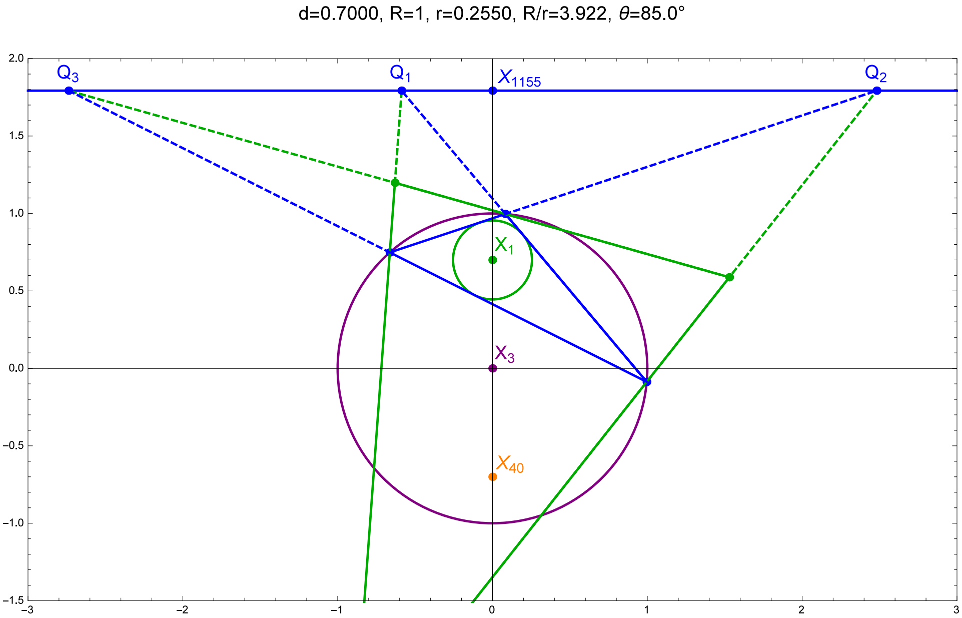

FIGURE 11. A result by Weaver [19] is that over the Poristic family, the Antiorthic Axis L_1 is Invariant. Odehnal observed X_{1155} was one of the many stationary Triangle centers along $L_{663} = X_1X_3$. This point happens to lie at the latter's intersection with L_1. **Video:** [13, PL#01].

$(3R^2 - d^2)/(2d)$. The power P_w of $P_0 = [(3R^2 - d^2)/(2d), 0]$ with respect to the circle $C'_w : (x + R)^2 + y^2 = r'^2_w$ is given by:

$$P_w(P_0, C'_w) = |P_0 - [-R, 0]|^2 - r'^2_w = \frac{(d + R)^2(3R - d)^2}{4d^2} - r'^2_w.$$

Also,

$$P_w(P_0, I_1) = |P_0 - [d, 0]|^2 - r^2 = \frac{\left(R^2 - d^2\right)^2 \left(9\,R^2 - d^2\right)}{4R^2d^2}.$$

Therefore, $\mathcal{L}_1$ is the radical axis of the pair of circles C'_w and I_1 if, and only if,

$$\frac{(d + R)^2(3R - d)^2}{4d^2} - r'^2_w = \frac{\left(R^2 - d^2\right)^2 \left(9\,R^2 - d^2\right)}{4R^2d^2}.$$

Solving the equation above leads to the result.
Additionally, we derive a circle whose power with respect to $\mathcal{L}_\infty$ is equal to the Circumcircle's, Figure 12(right). Let C''_w be a circle centered on $[-R, 0]$ and of radius:

$$r''_w = \sqrt{\frac{(3\,R - d)\,(d + R)\,R}{d}}$$

Proposition B.2. *C''_w has the same same power with respect to $\mathcal{L}_1$ as (i) the Circumcircle C, and (ii) C_e, centered on X_{40} and of radius $2R$ (locus of the Excenters).*

Proof. Translate the verices of the Poristic family in (2) by $(-d, 0)$. Also, $\mathcal{L}_1$ is the radical axis (see [2, Chapter 2]) of the pair of circles

$$C_e: \ (x+d)^2 + y^2 = 4R^2, \ C: \ x^2 + y^2 = R^2.$$

In fact, the power P_w of $P_0 = ((3R^2 - d^2)/(2d), 0)$ with respect to the circles C_e and C is given by:

$$P_w(P_0, C_e) = |P_0 - (-d, 0)|^2 - 4R^2 = \frac{Rr(4R+r)}{R-2r} = P_w(P_0, C).$$

Consider the pair of circles

$$C''_w: (x+R)^2 + y^2 = r_w^2, \ C: \ x^2 + y^2 = R^2$$

Analogously, $P_w(P_0, C'_w) = P_w(P_0, C)$ if, and only if, $r_w^2 = (3\,R - d)\,(d + R)\,R/d$.

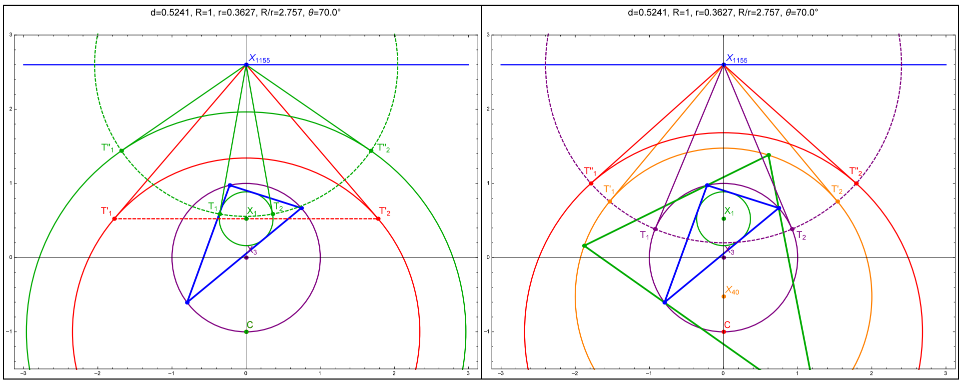

Figure 12. **Left:** Circle C_w proposed in [19, Theorem III] (red) does not have the same power as the Incircle I_1 (green) with respect to $\mathcal{L}_1$ (blue): rather, its tangency points T'_1, T'_2 from from X_{1155} are collinear with X_1. We derived a new equal power circle C'_w (green) of radius r'_w (see text): its tangency points T''_1, T''_2 are concyclic with T_1, T_2. Note: both C_w, C'_w are centered on $C = [-R, 0]$. **Right**: the following three circles have the same power with respect to $\mathcal{L}_1$: (i) the Circumcircle C, (ii) C_e, the X_{40}-centered circular locus of the Excenters of radius $2R$ (orange), and (iii) C''_w, centered on $[-R, 0]$ and of radius r''_w (red), see text. Notice tangency point $T_i, T'_i, T''_i, i = 1, 2$ are concyclic.

Appendix C. I'_3 Axis Invariance

Here we reproduce a proof that the axes of I'_3 are invariant and equal to $R \pm d$, kindly contributed by Odehnal [11].

Let X_{40} be the origin and the x-axis run along X_3X_1, Figure 13. Parametrize Poristic triangles $\Delta(t) = P_1P_2P_3$ by their tangency point on the Incircle [10]:

$$
\begin{aligned}
P_1 &= [\cos t(d\cos t + r) - \omega \sin t + d, (d\cos t + r)\sin t + \omega \cos t] \\
P_2 &= [\cos t(d\cos t + r) + \omega \sin t + d, (d\cos t + r)\sin t - \omega \cos t] \\
P_3 &= \left[\tfrac{R(2\,dR - (R^2+d^2)\cos t)}{R^2 - 2\,dR\cos t + d^2} + d, \tfrac{R(d^2 - R^2)\sin t}{R^2 - 2\,d\,R\cos t + d^2}\right] \\
\omega &= \sqrt{R^2 - (d\cos t + r)^2}
\end{aligned} \tag{2}
$$

Lemma C.1. *Let $\ell_i : a_ix + b_iy + c_i = 0$ $(i \in \{1, 2, 3\})$ be three tangents lines of a conic $E : Ax^2 + 2Bxy + Cy^2 + D = 0$ centered at $(0, 0)$. Then, the coefficients A, B, C, D are given by:*

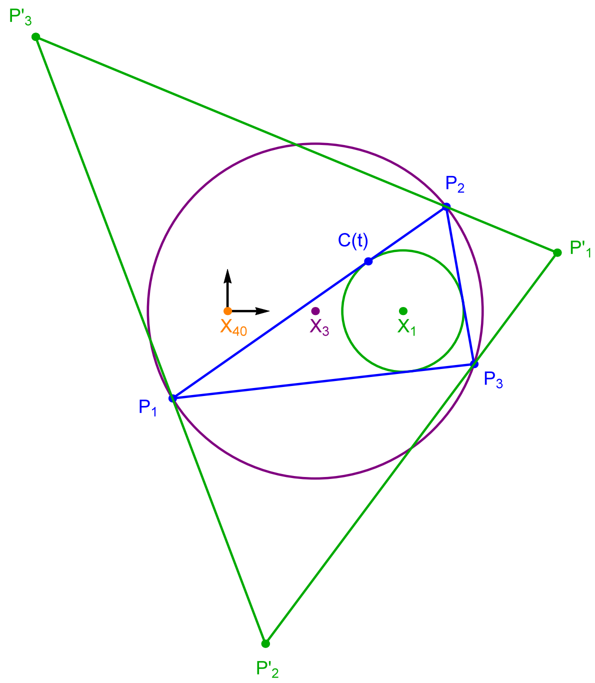

FIGURE 13. Coordinate System used in this Appendix: the origin is on X_{40} and the positive x-axis runs along X_3X_1. The Poristic family is parametrized by a first tangency point $C(t)$ on the Incircle.

$$
\begin{aligned}
A =\ & a_2a_3c_1^2\delta_{23} - a_1a_3c_2^2\delta_{13} + a_1a_2c_3^2\delta_{12}\\
B =\ & \tfrac{1}{2}\left((a_2b_3+a_3b_2)c_1^2\delta_{23} - (a_1b_3+a_3b_1)c_2^2\delta_{13} + (a_1b_2+a_2b_1)c_3^2\delta_{12}\right)\\
C =\ & b_2b_3c_1^2\delta_{23} - b_1b_3c_2^2\delta_{13} + b_1b_2c_3^2\delta_{12}\\
D =\ & \tfrac{1}{4}(\delta_{12}\delta_{13}\delta_{23})^{-1}\left(\delta_{23}c_1+\delta_{13}c_2-\delta_{12}c_3\right)\left(\delta_{23}c_1-\delta_{13}c_2-\delta_{12}c_3\right)\\
& \left(\delta_{23}c_1-\delta_{13}c_2+\delta_{12}c_3\right)\left(\delta_{23}c_1+\delta_{13}c_2+\delta_{12}c_3\right)
\end{aligned}
$$

Here $\delta_{ij} = a_ib_j - a_jb_i$.

Proof. The condition of tangency of ℓ_i $(i=1,2,3)$ with E is given by the discriminant equation

$$(AC-B^2)c_i^2 + (Ab_i^2 - 2Ba_ib_i + Ca_i^2)D = 0.$$

Solving the system leads to the result stated.

Lemma C.2. *The three sides of the excentral triangle* $\Delta'(t) = \{P_1'(t), P_2'(t), P_3'(t)\}$ *are given by the straight lines*

$$
\begin{aligned}
&\ell_1'(t) : ((d\sin t-\omega)\sin t - r\cos t)x - ((d\cos t + r)\sin t - \omega\cos t)y\\
&\qquad +R^2-d^2 = 0\\
&\ell_2'(t) : ((d\sin t+\omega)\sin t - r\cos t)\,x - ((d\cos t + r)\sin t + \omega\cos t)\,y\\
&\qquad +R^2-d^2 = 0\\
&\ell_3'(t) : (R\cos t - d)\,x + R\sin t\, y - 2\,dR\cos t + R^2 + d^2 = 0.
\end{aligned}
\tag{3}
$$

Proof. Direct calculations of the external bisector lines passing through the vertices $P_1(t)$, $P_2(t)$ and $P_3(t)$ given by equation (2).

Proposition C.1. *I_3' is given implicitly by the equation:*

$$
\begin{aligned}
I_3'(x,y,t) \;=\;& ((R^2-d^2)^2 - 8dR^2(R\cos t - d)\sin^2 t)x^2\\
&+\; ((R^2-d^2)^2 - 4dR\cos t((R\cos t-d)^2 - R^2\sin^2 t))y^2\\
&+\; 4dR\sin t(2R\cos t - R - d)(2R\cos t + R - d)xy\\
&-\; (R^2-d^2)^2(R^2+d^2-2dR\cos t) = 0
\end{aligned}
\tag{4}
$$

Proof. Consider the Poristic defined by the circles $(x-d))^2+y^2=R^2$and $(x-2d)^2+y^2=r^2$. By [10] we know that the inconic $I_3'(t)$, tangent to the sides of the excentral triangle $\Delta'(t)$, is centered in $X_{40}=(0,0)$. Applying lemmas C.1 and C.2, and using the Euler relation $R^2-d^2=2rR$, the equation (4) is obtained. This expression was confirmed using CAS.

Consider a rotation of the coordinates of (4) by angle θ defined by:

$$
\tan 2\theta = \frac{\sin t(R^2 - \left(2\,R\cos t - d\right)^2\big)}{\cos t\,((2\,R\cos t - d)^2 - 3\,R^2) + 2\,dR}.
$$

This re-expresses (4) in canonical form:

$$
(R^2+d^2-2dR\cos t)\left((R+d)^2u^2 + (R-d)^2v^2 - (R^2-d^2)^2\right) = 0.
\tag{5}
$$

Clearly, the semiaxis lengths of (5) are $R\pm d$ which is the goal of this proof.

Appendix D. E_1 Axis Invariance

Proposition D.1. *The semiaxes of E_1 are $\eta_1 = R+d$ and $\zeta_1 = R-d$.*

Proof. Using the parametrization of the triangle $P_1(t), P_2(t), P_3(t)$ given by equation (2) and that E_1 pass through the vertices $P_i(t)$ (i=1,2,3) and centered in $X_1=(2d,0)$ it is obtained:

$$
\begin{aligned}
E_1(x,y) =& \left((R^2-d^2)^2 - 4\,dR\cos t(R\cos t - d)^2 - R^2\sin^2 t\right)x^2\\
&+\left((R^2-d^2)^2 - 8\,dR^2(R\cos t - d)\sin^2 t\right)y^2\\
&-4\,dR\sin t\,(2\,R\cos t - R - d)\,(2\,R\cos t + R - d)\,xy\\
&+4d(4dR\cos t((R\cos t-d)^2 - R^2\sin^2 t) - (R^2-d^2)^2)x\\
&+8Rd^2\sin t(2R\cos t + R - d)(2R\cos t - R - d)y\\
&-2dR\cos t(16d^2R\cos t(R\cos t - d) - (R^2-d^2)(R^2+7d^2))\\
&-(R^2-3d^2)(R^2-d^2) = 0
\end{aligned}
$$

Proceeding as in the proof of Theorem 3.2 it is direct to verify that the canonical form of the above equation is $u^2/(R+d)^2 + v^2/(R-d)^2 = 1$.

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