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FORGOTTEN PROPERTIES OF THE VAN AUBEL AND BRIDE'S CHAIR CONFIGURATIONS

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Abstract. We present another property of the Van Aubel and bride's chair configurations that went unnoticed in references [4] and [5]. For each case, the main result is presented in the form of a corollary statement. More concise proofs for the six-point circle theorem and corollary than the ones developed in [4] are included in section 3.

1. Van Aubel Configuration

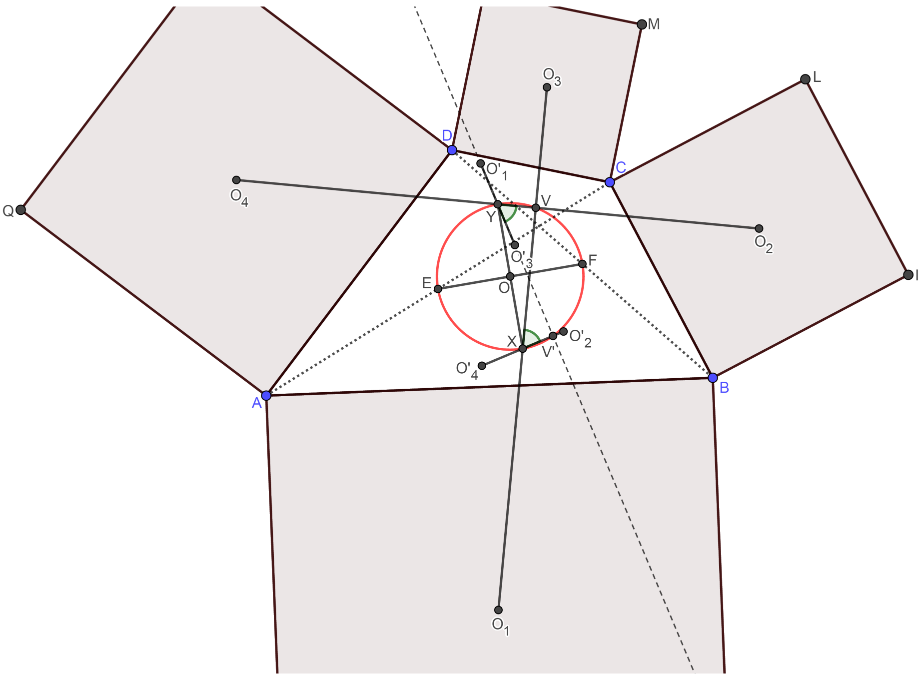

Figure 1. The complete Van Aubel configuration. The internal squares are not shown for the sake of clarity, their centers are represented by the points O'_{i}, with $i = 1, 2, 3, 4$.

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We use the nomenclature defined in [4]. Looking at figure 1, we recall the following:

Theorem 1.1. [1][4] *Each Van Aubel segment joining the centers of the squares constructed externally (internally) to the quadrangle over a pair of opposite sides is bisected by the Van Aubel segment joining the centers of the squares constructed internally (externally) over the other pair of opposite sides.*

For instance, the Van Aubel segment O_1O_3 (obtained joining the centers of the squares constructed externally to the quadrangle over the opposite sides AB and DC) and the Van Aubel segment $O'_2O'_4$ (obtained joining the centers of the squares constructed internally to the quadrangle over the opposite sides BC and AD) bisect each other at point X, as shown in figure 1. The midpoints of the Van Aubel segments are represented as X and Y in figure 1.

The following result holds true:

Corollary 1.1. *The bisecting Van Aubel segments form congruent angles.*

The congruent angles ($\angle O'_2XO_3$ and $\angle O_2YO'_3$) are shown in figure 1 in green.

Proof. Looking at figure 1, as a consequence of Van Aubel's theorem [8][4], segments O_1O_3 and O_2O_4 are orthogonal to each other and, segments $O'_1O'_3$ and $O'_2O'_4$ are also orthogonal to each other. Therefore, the bisecting segments $O'_1O'_3$ and O_2O_4 can be set parallel to the bisecting segments $O'_2O'_4$ and O_1O_3, respectively, via a pure (clockwise or counterclockwise) right angled rotation about point Y. It follows that $\angle O'_2XO_3 = \angle O_2YO'_3$.

Moreover, $\angle O'_2XO_3$ coincides with $\angle V'XV$ and $\angle O_2YO'_3$ coincides with $\angle VYV'$, by construction. As V, V', X and Y lie on the six-point circle for the quadrangle [4], $\angle V'XV$ and $\angle VYV'$ are angles at the circumference that subtend the same arc VV'.

The following statement holds true as a consequence of the Van Aubel's theorem [8][4] and corollary 1.1:

Remark 1.1. *Parallelograms $O_1O'_2O_3O'_4$ and $O_2O'_1O_4O'_3$ are congruent.*

The parallelograms sides are not represented in figure 1 for the sake of clarity.

Remark 1.2. *Incidentally, it is worth noticing that the relative position of the Van Aubel points V and V' on the six-point circle is dictated by the angle at the center $\angle VOV'$ which measures twice the angle at the circumference $\angle O'_2XO_3$ ($\angle O_2YO'_3$), as a consequence of the inscribed angle theorem* [3].

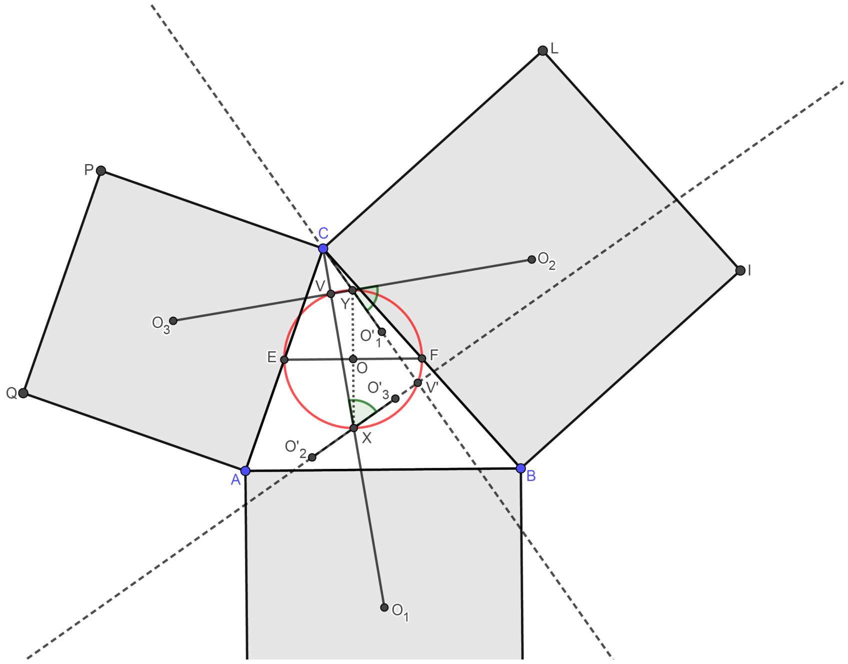

Figure 2. The bride's chair configurations (internal and external) relative to the triangle vertex C - The internal squares are not shown for the sake of clarity, their centers are represented by the points O'_{i}, with $i = 1, 2, 3$.

2. Bride's Chair Configurations

We use the nomenclature defined in [5]. Looking at figure 2, we recall the following:

Theorem 2.1. [5] *Each segment of any external (internal) bride's chair segment pair that joins a vertex of the triangle to the center of the square constructed on the opposite side, is bisected by the segment of the internal (external) bride's chair segment pair that joins the centers of the squares constructed on the other two sides.*

For instance, looking at figure 2, segment O_1C (obtained joining the triangle vertex C with the center of the square constructed externally to the triangle over the opposite side AB) and segment $O'_2O'_3$ (obtained joining the centers of the squares constructed internally to the triangle over the sides BC and AC) bisect each other at point X, as shown in figure 2.
On the other hand, segment O'_1C (obtained joining the triangle vertex C with the center of the square constructed internally to the triangle over the opposite side AB) and segment O_2O_3 (obtained joining the centers of the squares constructed externally to the triangle over the sides BC and AC) bisect each other at point Y, as shown in figure 2.
The midpoints of the bride's chair segment pair relative to the vertex C are represented as X and Y in figure 2.

The analogous result of corollary 1.1 holds true:

Corollary 2.1. *The bisecting bride's chair segments form congruent angles.*

The congruent angles ($\angle O'_3XC$ and $\angle O_2YO'_1$) are shown in figure 2 in green. The proof is essentially the same as the one presented for corollary 1.1 and can be left to the readers.
The following statement follows as a consequence of theorem 2.1 of reference [5] and corollary 2.1:

Remark 2.1. *Parallelograms* $O_1O'_3CO'_2$ *and* $O_2CO_3O'_1$ *are congruent.*

Remark 2.2. *Again, the angle at the center* $\angle VOV'$*, which measures twice the angle at the circumference* $\angle O'_3XC$ ($\angle O_2YO'_1$)*, dictates the relative position of the bride's chair intersection points* V *and* V' *on the six-point circle.*

Remark 2.3. *Last, the center* O *of the six-point circle through* E*,* X*,* V'*,* F*,* Y *and* V *coincides also with the midpoint of the median of triangle* ABC *from vertex* C*.*

Analogous results hold true for the A-vertex and B-vertex bride's chair configurations.

3. Concise Proofs for the Six-Point Circle Theorem for the Quadrangle and Corollary

The six-point circle theorem for the quadrangle [4] could have been proven, more concisely, in the following way:

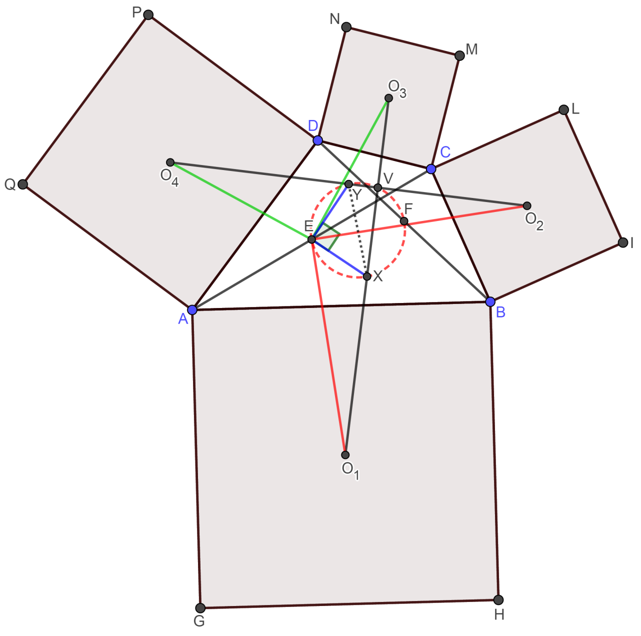

Figure 3. Sketch for a concise proof of the six-point circle theorem for the quadrangle.

Proof. Looking at figure 3 and recalling the proof of Van Aubel's theorem developed in [4], triangles ΔO_1EO_3 and ΔO_2EO_4 are congruent and the latest can be obtained from a right angle counterclockwise rotation around E of ΔO_1EO_3. With X and Y the midpoints of segments O_1O_3 and O_2O_4, the medians EX and EY are drawn. Thanks to the forementioned relation between the two triangles, $\angle XEY$ is a right angle, as shown in the figure. According to Van Aubel's theorem, $\angle XVY$ is also a right angle. Applying the converse of Thales' theorem, we deduce that E, X, Y and V are concyclic points and XY is a diameter of the circle. Similarly, F also lie on the circle through X, V, Y and E. But also $\angle XV'Y$ is a right angle (see figure 1), as a consequence of Van Aubel's theorem applied to the internal configuration and theorem 1.2 presented in [4]. Again, applying the converse of Thales' theorem, V' lies on the circle with XY as diameter. So, points E, X, V', F, V and Y are concyclic.

And another proof of the corollary follows:

Proof. Again, thanks to the relation between triangles ΔO_1EO_3 and ΔO_2EO_4, we have $EX = EY$ other than $\angle XEY = 90°$. So, ΔXEY is an isosceles right-angled triangle. Similarly, ΔXFY is an isosceles right-angled triangle (right-angled at F). So $EXFY$ is a square. EF and XY are the diagonals of this square, so they are orthogonal diameters of the six-point circle. The $EXFY$ square is represented in figure 4, in blue.

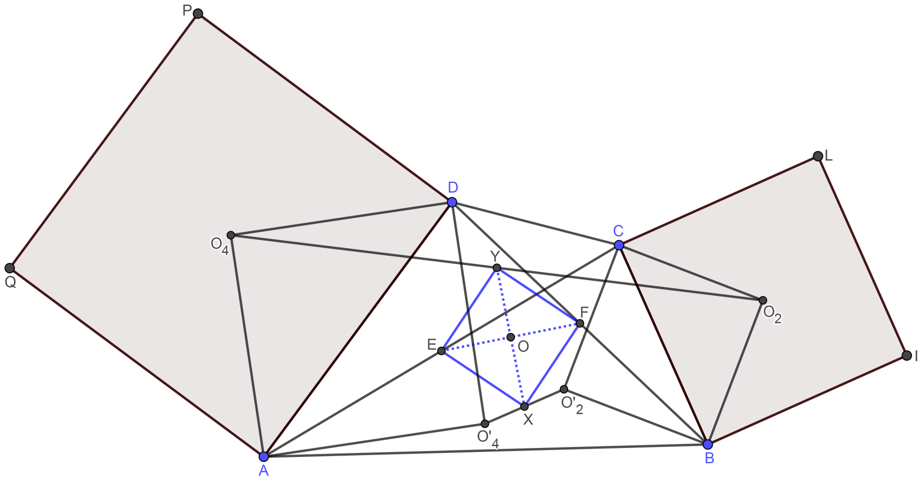

Figure 4. The $EXFY$ square.

It can be noticed that points E, X, F and Y are the midpoints of the segments joining the corresponding vertexes of squares AO'_4DO_4 and CO'_2BO_2: AC, $O'_4O'_2$, DB and O_4O_2. With this, an alternative proof of the corollary is directly obtained applying theorem 4 (its generalized version) presented in [2]. The fact that $EXFY$ is a square was already known quite well before the writing of [4]. For example, it was presented in [6]. We notice that the six-point circle theorem could have also been proved after this result: the concyclic property of the forementioned points would follow applying Van Aubel's theorem for the external and the internal configurations and the converse of Thales' theorem [7].

4. EPILOGUE

The *six-point circle for the quadrangle* could also be referred to as the *six-point circle for the quadrilateral* or the *six-point circle for the quadrigon* or simply, the *six-point circle.* In [4], the word *quadrangle* was employed only for aesthetical reasons. When the formal definitions of quadrangle, quadrilateral and quadrigon according to [9] are used, the proper naming should be the *six-point circle for the quadrigon*: as a quadrangle contains three quadrigons with shared centroid, it is easily seen that the quadrangle configuration contains three concentric six-point circles.

This six-point circle is a particular case of the so-called *QG-Ci2 Thales circles.* Chris van Tienhoven's Encyclopedia of Quadri-Figures (EQF) contains a description of these circles, developed together with Eckart Schmidt, as well as, a dedicated page to the Van Aubel points which are alternatively referred to as the *Outer* and *Inner* Van Aubel points [9].

REFERENCES

[1] Bogomolny, A., *Interactive Mathematics Miscellany and Puzzles*, available at https://www.cut-the-knot.org.

[2] de Villiers, M., *Dual Generalisations of Van Aubel's Theorem*, Math. Gaz., **85 (1998)**, 405-412.

[3] Euclid, *The Elements*, proposition 20, book III.

[4] Pellegrinetti, D., *The Six-Point Circle for the Quadrangle*, International Journal of Geometry, **2 (2019)**, 5-13.

[5] Pellegrinetti, D., *On a Six-Point Circles Family for the Triangle*, International Journal of Geometry, **1 (2020)**, 47-57.

[6] Tabov, J., *Solutions to a Posed Problem*, Mathematics & Informatics Quarterly, **6:4 (1996)**, 213-214.

[7] Weisstein, E.W., *Thales' theorem*, MathWorld–A Wolfram Web Resource, available at http://mathworld.wolfram.com/ThalesTheorem.html

[8] Van Aubel, H., *Note concernant les centres des carrés construits sur les cotés dun polygon quelconque* (French), Nouv. Corresp. Math., **4 (1878)**, 40-44.

[9] Van Tienhoven, C., *Encyclopedia of Quadri - Figures – EQF*, available at https://www.chrisvantienhoven.nl/qg-items/qg-mult-pts/qg-2p6.

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