INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 10 (2021), No. 2, 95 - 108

PERSPECTIVE TRIANGLES AND RELATED STRUCTURES

PARIS PAMFILOS

Abstract. We prove a theorem characterizing triangles which are perspective to a given triangle through the existence of a *"real"* conic reciprocating them. It turns out that the question of reality is intimately connected with characteristics of *"bitangent pencils"* of conics that can be interpreted by means of the *"Steiner ellipse"* of the triangle of reference.

1 Three points and their polars

The present article grew out from an attempt to understand the geometry behind a known theorem relating perspective triangles with conics. The theorem due to Chasles ([5, p.64], [1, p.166], [2, vol. II, p.123]) asserts, that if $\{A', B', C'\}$ are three points in general position, and $\{\alpha, \beta, \gamma\}$ their corresponding polars with respect to a conic κ, then triangles $A'B'C'$ and $\alpha\beta\gamma$ are perspective, and their perspector O and perspectrix axis ε are corresponding pol and polar of this conic (see Figure 1). The two triangles $A'B'C'$ and ABC, latter with side-lines $\{\alpha = BC, \beta = CA, \gamma = AB\}$ are *"reciprocal polar"*, in the sense that the side-lines

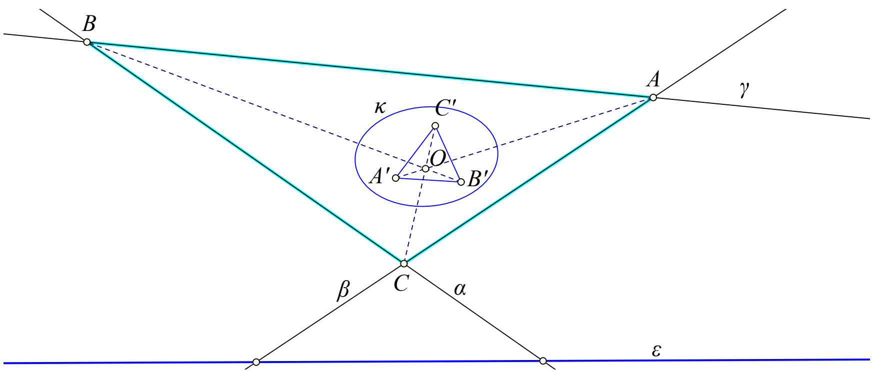

Figure 1: Perspectivity of triangles realized by polarity

of one are the polars of the vertices of the other. We refer to this relation by saying *"the perspectivity of the two triangles is realized by a polarity"*. The converse of this theorem is also known ([4, p.214], [5, p.51]) as a result of the theory of correlations of projective geometry ([5, p.60], [6, p.126]) and more specifically as a property of *"polarities"*, which guarantees that *"every pair of perspective triangles in general positions is realized by a polarity"*.

Keywords and phrases: Triangle, Perspective, Conics, Barycentrics, Tripolar
(2020)Mathematics Subject Classification: 51M15, 51N15, 51N20, 51N25
Received: 21.01.2021. In revised form: 27.03.2021. Accepted: 04.03.2021

The conic κ realizing the perspectivity of the two perspective triangles is called *"fundamental conic of the polarity"* ([13, II, p.282]) and can be real or complex.

Complex conics of the real plane, called also *"virtual"* ([7, p.160]), are defined in projective coordinates by quadratic equations with real coefficients, which have no real solutions, like for example the equation $x^2 + y^2 + z^2 = 0$. The question of complexity of a non-degenerate conic κ expressed through a quadratic equation and a related real symmetric matrix

$$\kappa : \; ax^2 + by^2 + cz^2 + 2pyz + 2qzx + 2rxy = 0 \quad \Leftrightarrow \quad X^t \begin{pmatrix} a & r & q \\ r & b & p \\ q & p & c \end{pmatrix} X = 0,$$

is resolved quantitatively through elementary linear algebra. There it is proved that the preceding example is universal, in the sense that every non degenerate virtual conic can be transformed by a projectivity to this one ([3, II, p.150]). For this it is necessary and sufficient, that the eigenvalues of the related matrix have the same sign. This is, in turn, equivalent to the two conditions on the determinants of the *"upper left submatrices"* ([13, II, p.205], [12, p.331]):

$$\begin{vmatrix} a & r \\ r & b \end{vmatrix} > 0 \quad \text{and} \quad a \cdot \begin{vmatrix} a & r & q \\ r & b & p \\ q & p & c \end{vmatrix} > 0 \,. \tag{1}$$

In the following I study the geometrical content of these two conditions, starting with an elementary proof of the converse to Chasles' theorem and detecting the corresponding conic and its matrix. It turns out that the geometry involves *"bitangent pencils"* ([3, II, p.187], [10]) of conics and the relevant quantitative relations can be reduced to properties of the *"Steiner ellipse"* ([14, p.135]) of the triangle of reference ABC (theorem 5).

The work is done in barycentric coordinates ([14, p.25], [9]) w.r.t. the triangle of reference ABC. $\{X, Y, ...\}$ denote column-vectors of such coordinates and $\{X^t, Y^t, ...\}$ denote their corresponding transposed row-vectors. Every conic can be described by a symmetric matrix M and the related quadratic form $f(X) = X^t \cdot M \cdot X = 0$. The matrix M defines also the *"bilinear form"* $F(X, Y) = X^t \cdot M \cdot Y$, whose properties are intimately related to those of the quadratic form f. The symbol f of a conic will also be used to denote its equation $f(X) = 0$ in barycentric coordinates.

2 A conic determination

Theorem 1. *Given three lines α, β, γ and three points A', B', C' in general position, such that triangles $A'B'C'$ and $\alpha\beta\gamma$ are perspective, there is a unique conic κ such that α, β, γ are correspondingly the polars of A', B', C' with respect to this conic.*

Proof. To prove the theorem consider the barycentric coordinate system, for which the given lines α, β, γ are coordinate axes. The existence of the conic is then equivalent to the existence of a symmetric matrix M representing the conic and satisfying the equations

$$MX_i = \lambda_i e_i, \quad i = 1, 2, 3.$$

In this X_i are respectively the column vectors of coordinates of the given points $\{A', B', C'\}$, $\{e_i\}$ are the standard unit column vectors representing the side-lines $\{\alpha, \beta, \gamma\}$ of the triangle ABC and and $\{\lambda_i\}$ are constants. Denoting by X the matrix whose columns are the X_i and by Λ the diagonal matrix of $\{\lambda_i\}$, the equation takes the matrix form

$$MX = \Lambda \quad \Leftrightarrow \quad M = \Lambda X^{-1}.$$

Setting for the diagonal matrix $N = \Lambda^{-1}$ the symmetry of the matrix

$$M = N^{-1}X^{-1} = (XN)^{-1} \quad \text{is equivalent to} \quad (XN)^t = XN \quad \Leftrightarrow \quad NX^t = XN,$$

which, written explicitly, is equivalent to the system of three equations:

$$\frac{\nu_1}{\nu_2} = \frac{x_{21}}{x_{12}}, \quad \frac{\nu_2}{\nu_3} = \frac{x_{32}}{x_{23}}, \quad \frac{\nu_3}{\nu_1} = \frac{x_{13}}{x_{31}}. \tag{2}$$

The assumption of perspectivity means that there is a point $O(o_1, o_2, o_3)$, the perspective center with absolute barycentrics $\{o_i\}$, such that the points $\{X_i\}$ representing $\{A', B', C'\}$ may be expressed as linear combinations

$$X_i \;=\; O + t_i e_i, \quad \text{with constants } t_i,\ i = 1, 2, 3\,. \tag{3}$$

Equation (2) implies then that the two vectors are dependent $(\nu_1, \nu_2, \nu_3) \;=\; k(o_1, o_2, o_3)$ and the matrix takes the form of a multiple of

$$XN \;\cong\; \begin{pmatrix} o_1(o_1 + t_1) & o_1 o_2 & o_1 o_3 \\ o_2 o_1 & o_2(o_2 + t_2) & o_2 o_3 \\ o_3 o_1 & o_3 o_2 & o_3(o_3 + t_3) \end{pmatrix}.$$

The matrix $M \;=\; (XN)^{-1}$ becomes then equal, up to a constant multiple, to

$$M \;\cong\; \begin{pmatrix} -\dfrac{t_2t_3 + o_2t_3 + o_3t_2}{o_1} & t_3 & t_2 \\ t_3 & -\dfrac{t_3t_1 + o_3t_1 + o_1t_3}{o_2} & t_1 \\ t_2 & t_1 & -\dfrac{t_1t_2 + o_1t_2 + o_2t_1}{o_3} \end{pmatrix}. \tag{4}$$

□

By combining the Chasles' theorem and the preceding theorem we obtain the well known characterization of the perspectivity of two triangles by means of a conic, as this is expressed by the following theorem.

Theorem 2. *Two triangles ABC and $A'B'C'$ are perspective if and only if there is a conic κ such that lines $\alpha = BC, \beta = CA, \gamma = AB$ are respectively the polars of points A', B', C' with respect to the conic.*

Remark 1. Given the triangle of reference ABC, we denote by $\tau_{O,T}$ the triangle $A'B'C'$ defined by O and the vector of parameters $T(t_1, t_2, t_3) \in \mathbb{R}^3$ through the relations (3) and by $\kappa_{O,T}$ the conic defined by the point O of the plane and $T \in \mathbb{R}^3$ through its matrix in equation (4). Since we assumed absolute barycentrics $(o_1 + o_2 + o_3 = 1)$, the diagonal entries of this matrix can be written in the form $\{-((o_1 + o_2 + o_3)t_2t_3 + o_2t_3 + o_3t_2)/o_1, \dots\}$, showing that the matrix is homogeneous of degree 0 w.r.t. O. This is not the same w.r.t. the variables $\{t_i\}$, which define different triangles $\{\tau_{O,T}\}$ and different conics $\{\kappa_{O,T}\}$ for the same O but different vectors of parameters $T' \neq T$.

Remark 2. For a given point O, the family of triangles $\{\tau_t\}$ resulting from parameters of the form $\{T = tO = (to_1, to_2, to_3), t \in \mathbb{R}\}$ is worth noticing. These triangles are perspective to ABC by polarities realized by conics κ_t corresponding to the matrices

$$M_t \;\cong\; \begin{pmatrix} \frac{-(2+t)o_2o_3}{o_1} & o_3 & o_2 \\ o_3 & \frac{-(2+t)o_3o_1}{o_2} & o_1 \\ o_2 & o_1 & \frac{-(2+t)o_1o_2}{o_3} \end{pmatrix} \;\cong\; \begin{pmatrix} \frac{-(2+t)}{o_1^2} & \frac{1}{o_1o_2} & \frac{1}{o_1o_3} \\ \frac{1}{o_1o_2} & \frac{-(2+t)}{o_2^2} & \frac{1}{o_2o_3} \\ \frac{1}{o_1o_3} & \frac{1}{o_2o_3} & \frac{-(2+t)}{o_3^2} \end{pmatrix}.$$

For $t = -2$ the diagonal elements of the matrix M_{-2} vanish and κ_{-2} becomes the circumconic of ABC whose *"perspector"* is the point O, whereas the corresponding perspective to ABC triangle $\tau_{-2} = A'B'C'$ is the *"anticevian"* of ABC w.r.t. O, having the conic κ_{-2} tangent to its sides at the vertices of ABC (see Figure 2). For $t = -1$ the corresponding

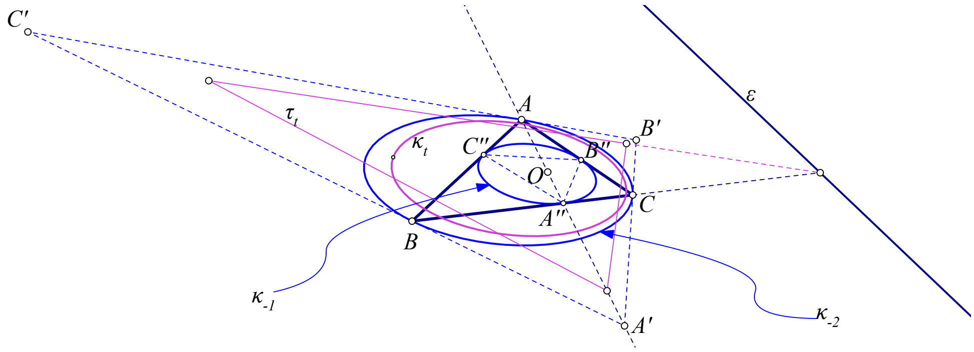

Figure 2: Pencil generated by the in/circum- conic $\{\kappa_{-1}, \kappa_{-2}\}$ with perpector O

conic κ_{-1} coincides with the inscribed in ABC conic with perspector O and corresponding triangle $\tau_{-1} = A''B''C''$ the *"cevian"* triangle of O. For $t = -3$ the conic κ_{-3} is degenerate and coincides with the double line ε^2, line ε being also the *"tripolar"* of O w.r.t. ABC. In

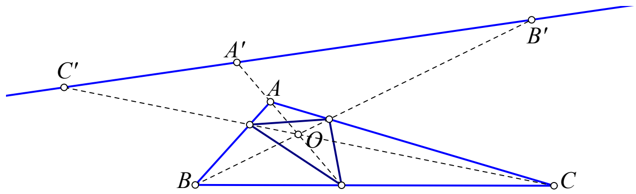

Figure 3: The degenerate triangle $A'B'C'$ on the tripolar of O

this case the correspoding triangle τ_{-3} is also degenerate (see Figure 3). The various conics κ_t coincide with members of the pencil $\mathcal{D}$ of conics generated by $\{\kappa_{-1}, \kappa_{-2}\}$ and the corresponding triangles τ_t have their vertices on the lines $\{OA, OB, OC\}$ and their sides pass through the intersections of lines $\{\varepsilon \cap AB, \varepsilon \cap BC, \varepsilon \cap CA\}$. While all the triangles $\{\tau_t, t \in \mathbb{R}\}$ are real, being degenerate for $t \in \{-3, 0\}$, we'll see below that the corresponding conics $\{\kappa_t\}$ are real and non degenerate only for t varying in the interval $(-3, 0)$.

3 A bitangent pencil of conics

Our basic configuration consists of a fixed point O playing the role of the perspectivity center, a fixed $T(t_1, t_2, t_3) \in \mathbb{R}^3$ and the variable vectors $\{T' = \lambda \cdot T, \lambda \in \mathbb{R}\}$ of parameters representing a family of triangles $\{\tau_\lambda = \tau_{O,\lambda T}\}$ perspective to the triangle of reference ABC with vertices

$$A' = O + \lambda t_1 A, \qquad B' = O + \lambda t_2 B, \qquad C' = O + \lambda t_3 C\,. \tag{5}$$

Figure 4 shows a couple of triangles $\{\tau_1, \tau_\lambda\}$. To these triangles correspond, according to theorem 2, two conics $\{\kappa_1, \kappa_\lambda\}$. The first conic κ_1 is described by the matrix-expression in (4) and the second conic κ_λ, is seen to have a matrix which is a multiple of

$$M_\lambda = \lambda \begin{pmatrix} -\frac{t_2 t_3}{o_1} & 0 & 0 \\ 0 & -\frac{t_3 t_1}{o_2} & 0 \\ 0 & 0 & -\frac{t_1 t_2}{o_3} \end{pmatrix} + \begin{pmatrix} -\frac{o_2 t_3 + o_3 t_2}{o_1} & t_3 & t_2 \\ t_3 & -\frac{o_3 t_1 + o_1 t_3}{o_2} & t_1 \\ t_2 & t_1 & -\frac{o_1 t_2 + o_2 t_1}{o_3} \end{pmatrix}. \tag{6}$$

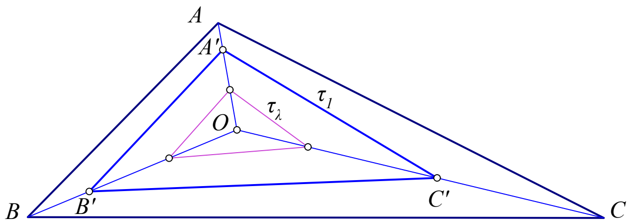

Figure 4: Variable triangle τ_λ perspective to $A'B'C'$ w.r.t. O

In this sum of matrices $M_\lambda = \lambda M' + M''$, the matrix M' represents a conic $\mu_{O,T}$ depending on $\{O, T\}$ w.r.t. which the triangle ABC is self-polar. The second matrix M'' represents a degenerate conic, since the non zero vector of coordinates $O(o_1, o_2, o_3)$ is in the nullspace of M'' : $M'' \cdot O = 0$. Thus, the conic $\nu_{O,T}$ represented by the matrix M'' is a product of two lines, possibly complex. In fact, we'll see below that $\nu_{O,T}$ is the product of the two tangents to $\mu_{O,T}$ from the point O.

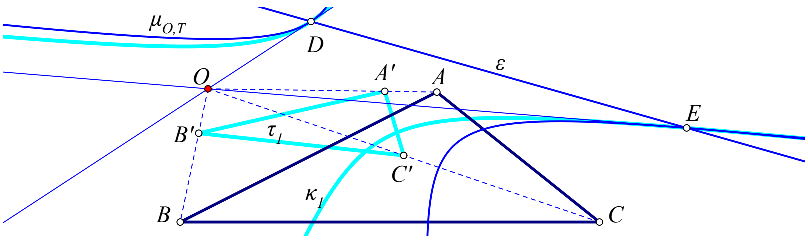

Figure 5: Perspective triangle $\tau_1 = \tau_{O,T}$ and corresponding conic $\kappa_1 = \kappa_{O,T}$

Figure 5 shows a characteristic case for which the conic κ_1, realizing the perspectivity of triangles ABC and $A'B'C'$ by polarity, is real. The figure shows also the two tangents to $\mu_{O,T}$ from O, whose product is represented by the matrix M'' in equation (6) and the polar ε of O w.r.t. $\mu_{O,T}$, which also coincides with the perspectivity axis of the two triangles. We should notice here that the two tangents are complex lines when, either the conic $\mu_{O,T}$ is not satisfied by real points, or O is an inner point of the real non-degenerate conic $\mu_{O,T}$. In this case the only real point satisfying the equation $\nu_{O,T}(X) = 0$ representing their product is point O.

Theorem 3. *The conic $\nu_{O,T}$ represented by the matrix M'' in equation* (6) *is the product of the two tangents from O to the conic $\mu_{O,T}$ represented by the matrix M'.*

Proof. By a well known property of the equation of conics ([11, p.149], [8, II,p.66]), if the conic is represented by a symmetric matrix like M', by the equation $f(X) = X^t \cdot M' \cdot X = 0$, then the equation of the pair of the tangents drawn to it from a point O is described by the corresponding bilinear form $F(X, Y) = X^t \cdot M' \cdot Y$ and the equation

$$F(O, O)F(X, X) - F(O, X)^2 = 0 . \tag{7}$$

Doing the corresponding matrix multiplications we find the relation:

$$F(O, O)F(X, X) - F(O, X)^2 = -t_1 t_2 t_3 (X^t \cdot M'' \cdot X) , \tag{8}$$

which supplies the required proof. □

From all said so far we conclude that the conics $\{\kappa_\lambda = \lambda\mu_{O,T} + \nu_{O,T}\}$ form a pencil $\mathcal{D}$ generated by the conic $\mu_{O,T}$ and the conic $\nu_{O,T}$, of the product of the two tangents to $\mu_{O,T}$ from O. This implies that the pencil is a *"bitangent"* one, having all its members tangent to each other at the two points $\{D, E\}$ lying on ε, where the tangents to $\mu_{O,T}$ contact this conic. In addition all these member-conics have O and ε as respective pol and polar and can be written as combinations $\{p \cdot \nu_{O,T} + q \cdot \varepsilon^2, p, q \in \mathbb{R}\}$, the product of the tangents $\nu_{O,T}$ as well as the conic κ_1 being particular members of the pencil. Depending on the position of O and the nature of the conic $\mu_{O,T}$, the points $\{D, E\}$ on the line ε and the tangents to $\mu_{O,T}$ from O may be real or complex conjugate.

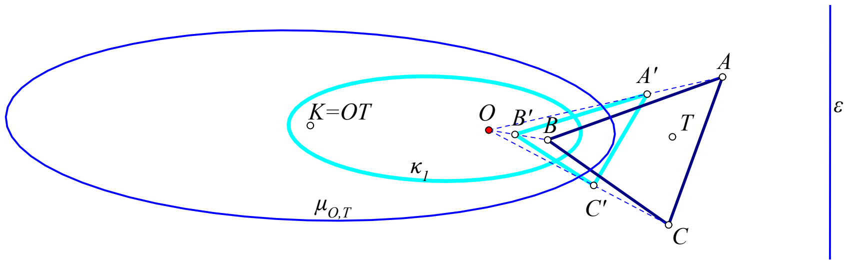

Figure 6: A case of real conic and complex conjugate points $\{D, E\}$

Figure 6 shows a case in which O is inside the conic $\mu_{O,T}$ and there are no real tangents from O to $\mu_{O,T}$, equivalently O is an inner point of the conic and its polar ε does not intersect this conic. In the case shown the points $\{D, E\}$ on ε are complex conjugate and all conics of the pencil $\mathcal{D}$, real or complex, pass through these points, which can easily and explicitly calculated from the data $\{O, T\}$. In the next section we take a closer look at this complexity question.

4 Virtual conics

As we noticed in the introduction, *"virtual"* is a non-degenerate conic defined by an equation with real coefficients for which there is no real point satisfying it. Such a conic carries only complex points X and with each point X also its conjugate $\overline{X}$ satisfies its equation.

Lemma 1. *The conic $\mu_{O,T}$ is virtual if and only if $\{o_1t_1, o_2t_2, o_3t_3\}$ have the same sign.*

Proof. This is trivial, since two diagonal elements representing M', the first two say, which by assumption must have the same sign, satisfy inequalities like the following:

$$\left(\frac{t_2t_3}{o_1}\right) \cdot \left(\frac{t_3t_1}{o_2}\right) > 0 \quad \Leftrightarrow \quad \frac{t_2}{o_1} \cdot \frac{t_1}{o_2} > 0 \quad \Leftrightarrow \quad (o_1t_1)(o_2t_2) > 0. \qquad \square$$

Regarding our particular kind of the conic $\kappa_1 = \mu_{O,T} + \nu_{O,T}$ we have a simple case guarantying that it is virtual.

Theorem 4. *If the coefficients $\{o_1t_1, o_2t_2, o_3t_3\}$ are all positive, then the conic $\kappa_1 = \mu_{O,T} + \nu_{O,T}$ realizing the perspectivity of the triangles $\{ABC, A'B'C'\}$, is virtual.*

Proof. In fact, the matrix M' can be written in the form

$$M' = \begin{pmatrix} \frac{-t_2t_3}{o_1} & 0 & 0 \\ 0 & \frac{-t_3t_1}{o_2} & 0 \\ 0 & 0 & \frac{-t_1t_2}{o_3} \end{pmatrix} = -t_1t_2t_3 \begin{pmatrix} \frac{1}{o_1t_1} & 0 & 0 \\ 0 & \frac{1}{o_2t_2} & 0 \\ 0 & 0 & \frac{1}{o_3t_3} \end{pmatrix} = -t_1t_2t_3M^*.$$

This implies that the equation representing $\kappa_1(X) = \mu_{O,T}(X) + \nu_{O,T}(X)$ has the form

$$\begin{aligned} \kappa_1(X) &= -t_1t_2t_3(X^tM^*X) + X^tM''X = \\ -t_1t_2t_3 &\left(X^tM^*X + \frac{1}{(t_1t_2t_3)^2}[F(O,O)F(X,X) - F(O,X)^2] \right) = \\ &-t_1t_2t_3 \left(F^*(X,X) + [F^*(O,O)F^*(X,X) - F^*(O,X)^2] \right). \end{aligned} \tag{9}$$

The assumption implies, that for all $X \neq 0$ the value of the quadratic form $F^*(X,X) = X^t M^* X > 0$ and by the Schwarz inequality: $F^*(O,O)F^*(X,X) - F^*(O,X)^2 \geq 0$. Hence the parenthesis in the last formula is positive for all X. □

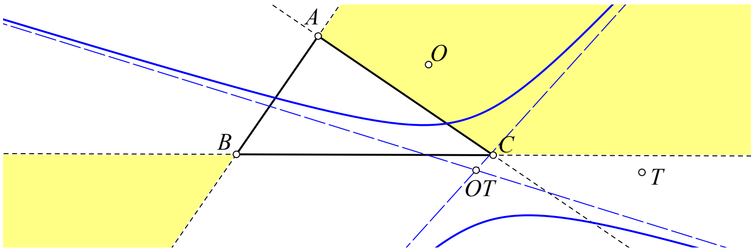

Figure 7: Points $\{O, T\}$ lying in different non-opposite domains

Figure 7 shows a case in which the conic $\mu_{O,T}$ is real. The triangle of reference ABC divides the plane in seven domains through its side-lines. Given the position of the perspector O, the parameters T define a point of the plane and if both $\{O, T\}$ lie in the same outer domain, then $O \cdot T = (o_1 t_1, o_2 t_2, o_3 t_3)$ lies in the inner domain of the triangle and by theorem 4 the corresponding conic $\kappa_1 = \mu_{O,T} + \nu_{O,T}$ is virtual.

Notice that $O \cdot T$ is the center of the conic $\mu_{O,T}$, since the matrix M' applied to it produces $(1,1,1)$, which are the coefficients of the line at infinity. Figure 8 shows the shapes

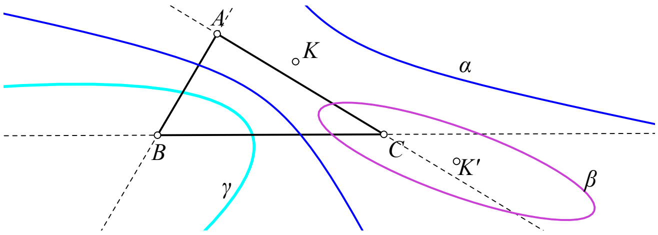

Figure 8: Possible shapes of real conics $\mu_{O,T}$

$\{\alpha, \beta, \gamma\}$ of the possible real conics $\mu_{O,T}$ that may occur, depending on the position of its center $K = O \cdot T$ in the various outer domains defined by the triangle. The conics γ are parabolas occurring for K lying at infinity, i.e. satisfying $o_1 t_1 + o_2 t_2 + o_3 t_3 = 0$.

Thus, in order to have a perspectivity with corresponding real conic κ_1 we have to examine the two alternative to theorem 4 cases: (i) the case of $O \cdot T$ lying in an outer domain, and (ii) the case in which $O \cdot T$, considered as a point of the plane, lies in the inner domain, but has some or all $\{t_i < 0\}$ and also all $\{o_i t_i < 0\}$ and the hypothesis of theorem 4 is not satisfied. Latter can happen when the points $\{O, T\}$ of the plane lie in *"opposite"* outer domains or O is in the inner domain and all $\{t_i\}$ are negative. We have here a subtlety, since $\{T, -T\}$ while they define the same point of the projective plane, they deliver different conics. In the next section we discuss further the example introduced in remark 2 which shows, that if the conic κ_1 corresponding to T is real, then the conic κ_{-1} corresponding to $-T$ is virtual.

In any case we should notice, that whether $\{\mu_{O,T}, \nu_{O,T}\}$ are real or virtual, the pencil $\mathcal{D}$ of conics $\{\kappa_t = t\mu_{O,T} + \nu_{O,T}, t \in \mathbb{R}\}$ they generate contains always real conics and, even more, these real conics cover all the points $\{P \neq O\}$ of the plane. In fact, given such a point P of the plane, either $\mu_{O,T}(P) = 0$ or equation $t\mu_{O,T}(P) + \nu_{O,T}(P) = 0$ determines a unique $t \in \mathbb{R}$ such that the conic $\kappa_t(P) = 0$, which therefore is real, since it contains P. The problem here is to find those $\{t\}$ for which κ_t is real. In order to answer our main problem we must determine in addition, for given $\{O, T\}$, whether $t = 1$ is also among these $\{t \in \mathbb{R}\}$ which produce a real conic κ_t.

5 A pencil of triangles and a pencil of conics

Returning to the example of remark 2 with a fixed perspectivity center O and the pencil $\{\tau_t\}$ of triangles resulting for parameters of the form $T = tO$ and having vertices:

$$\{ A_t(o_1 + to_1, o_2, o_3) \ , \ B_t(o_1, o_2 + to_2, o_3) \ , \ C_t(o_1, o_2, o_3 + to_3) \ \},$$

we notice that for $t > 0$ the corresponding bilinear form $F^*(X,Y) = \frac{1}{t}\left(\frac{x_1y_1}{o_1^2} + \frac{x_2y_2}{o_2^2} + \frac{x_3y_3}{o_3^2}\right)$ is positive definite and by theorem 4 the conic κ_t is virtual. In this case also it follows that

$$F^*(O,O) = \frac{3}{t} \quad \Rightarrow \quad F^*(X,X) + [F^*(O,O)F^*(X,X) - F^*(O,X)^2] = \left(1 + \frac{3}{t}\right)F^*(X,X) - F^*(O,X)^2 \, .$$

Thus, for $t < 0$ follows that $F^*(X,X) < 0$ and if $t < -3$ the last expression is strictly

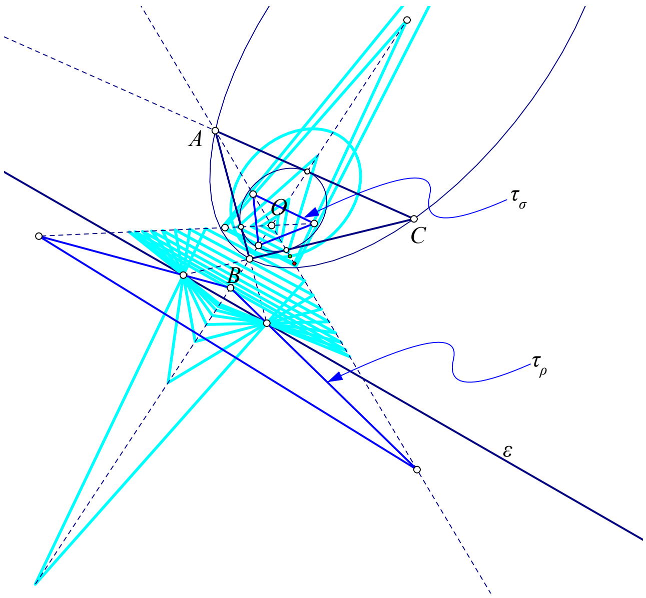

Figure 9: Triangles corresponding to real conics

negative for all X, implying that the conic κ_t for $t < -3$ is also virtual. The pencil $\mathcal{D}$ in this case has real non degenerate conics $\kappa_t = t\mu_{O,T} + \nu_{O,T}$ only for t varying in the interval $(-3, 0)$. This is verified by detecting a real point on the conic κ_t for $t \in (-3, 0)$. For example, by solving the equation $\kappa_t(O + sA) = 0$ for s. The corresponding calculation shows that indeed, if $t \in (-3, 0)$, then there is such a real solution s.

Figure 9 shows some members of the family of triangles $\{\tau_t, t \in (-3, 0)\}$ corresponding to real conics. It shows also two triangles $\{\tau_\rho, \tau_\sigma\}$ corresponding respectively to values of the parameter $\{\rho < -3, \sigma > 0\}$, for which the conic realizing the perspectivity to ABC through polarity is virtual.

Despite this deficiency, regarding the reality of the conics $\{\kappa_t\}$, as we noticed in the preceding section, for every point $P \neq O$ of the plane there is a *"real"* conic κ_t passing through P. More precisely, a characteristic of this example of pencil of conics is that every triangle τ_s, whether it corresponds to a real or imaginary conic realizing its perspectivity to $\tau = ABC$, has a corresponding real conic κ_t passing through its vertices. In fact, assume that the conic κ_t passes through one vertex, $(o_1 + so_1, o_2, o_3)$ say, of such a triangle τ_s. Then it is readily seen that s satisfies the equation

$$(1+s)^2 + 2 - (3+s)^2/(3+t) = 0 \quad \Rightarrow \quad t = -\frac{2s^2}{(s+1)^2 + 2}, \qquad (10)$$

and the same result comes out also for the other two vertices. This implies that $t = t(s)$ is independent of the vertex of τ_s chosen and the conic κ_t passes also from the other two vertices of the triangle. Figure 10 shows the graph of this function. The figure shows also

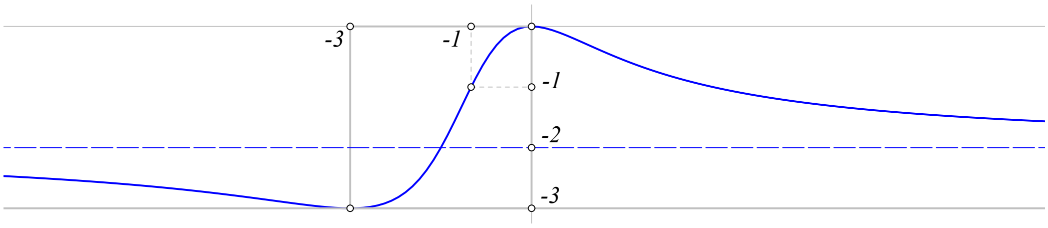

Figure 10: Correspondence between triangles

a fixed point $(-1, -1)$ of this function, which corresponds to the cevian triangle of ABC w.r.t. O corresponding to $s = -1$ and having associated circumscribing it conic κ_{-1} the one inscribed in ABC with perspector O. It is easy to see that the two roots of equation (10) defining two triangles inscribed in the same conic κ_t have a fixed harmonic mean

$$2\frac{s_1 s_2}{s_1 + s_2} = -3\,.$$

Figure 11 shows the pencil of the conics $\{\kappa_t\}$ defined by the point O. The two triangles $\{\tau_s, \tau_{s'}\}$ shown correspond to two values $\{s, s'\}$ satisfying the preceding equation. The distinguished conic κ_t shown, is the corresponding member-conic of the pencil passing through the vertices of these two triangles. It can be shown that the *"harmonic perspectivity"* or *"harmonic homology"* ([5, p.55]) defined by the pair (O, ε) leaves each member conic of the pencil $\mathcal{D}$ invariant and interchanges the two triangles $\{\tau, \tau'\}$, one of them corresponding to a real conic and the other to a virtual one.

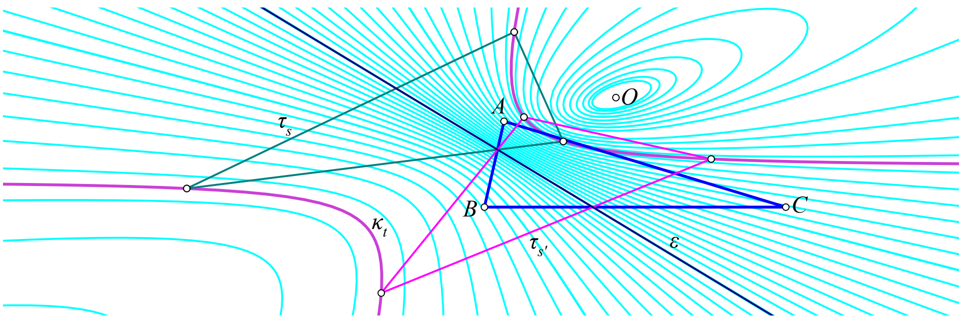

Figure 11: The pencil of conics $\{\kappa_t\}$

6 Domains of reality and Steiner ellipse

Here we consider a given triangle $\tau = ABC$ and fix the position of a point O, which will play the role of perspector, and does not lie on a side-line of the triangle. Using barycentric coordinates w.r.t. ABC, every other triangle $\tau' = A'B'C'$ perspective to τ w.r.t. O can be expressed through the coordinate vectors $\{X_i\}$ of equation (3). From equation (9) we see that the conic κ_1 corresponding to the parameter vector $T(t_1, t_2, t_3)$ can be written:

$$(1 + F^*(O,O))F^*(X,X) - F^*(O,X)^2 = 0 \quad \text{with} \quad F^*(X,Y) = \frac{x_1 y_1}{o_1 t_1} + \frac{x_2 y_2}{o_2 t_2} + \frac{x_3 y_3}{o_3 t_3}$$

$$\Rightarrow \quad \kappa_1 : \left(1 + \frac{o_1}{t_1} + \frac{o_2}{t_2} + \frac{o_3}{t_3}\right)\left(\frac{x_1^2}{o_1 t_1} + \frac{x_2^2}{o_2 t_2} + \frac{x_3^2}{o_3 t_3}\right) - \left(\frac{x_1}{t_1} + \frac{x_2}{t_2} + \frac{x_3}{t_3}\right)^2 = 0\,. \quad (11)$$

To further investigate these conics we simplify somewhat the related equations by setting $\{t_i = o_i s_i\}$ in equation (11). Clearly κ_1 is a real conic precisely when the conic κ_1' resulting by replacing in its equation $\{x_i = o_i y_i\}$:

$$\kappa_1' \ : \ \left(1+\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\right)\left(\frac{y_1^2}{s_1}+\frac{y_2^2}{s_2}+\frac{y_3^2}{s_3}\right)-\left(\frac{y_1}{s_1}+\frac{y_2}{s_2}+\frac{y_3}{s_3}\right)^2 = 0\,, \tag{12}$$

is a real conic. This is equivalent to considering our initial problem for the special position $O = G(1,1,1)$, where G is the centroid of the triangle $\tau = ABC$ and the vector of parameters $S(s_1,s_2,s_3)$ defining the perspective to τ triangle

$$\tau' = A'B'C', \quad \text{with} \quad A' = G + s_1 A, \quad B' = G + s_2 B, \quad C' = G + s_3 C.$$

The initial configuration and this one are related by the simple projective transformation $\{t_i = o_i s_i\}$, which fixes the vertices of the triangle of reference and maps point G to O. By means of this transformation every relation between the $\{s_i\}$ corresponds to a relation between the $\{t_i\}$ and vice versa. We denote the conic $\sum y_i^2/s_i = 0$ by μ_0 and the line $\sum y_i/s_i = 0$ by ε_0. We use also the expressions

$$\Pi = s_1 s_2 s_3\,,\ \Sigma = s_1 s_2 + s_2 s_3 + s_3 s_1\,,\ \Sigma_1 = 1+\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3} = \frac{1}{\Pi}(\Pi+\Sigma). \tag{13}$$

Our aim now is to find conditions for S, guarantying that the conic $\kappa_1' = \Sigma_1\mu_0 - \varepsilon_0^2$ is real. The pencil $\mathcal{D}$ we met in section 3 is now replaced by the one generated by $\{\mu_0, \varepsilon_0^2\}$. We arrive to our main theorem through a few lemmata formulating simple properties of the conics of this pencil. A key role play the intersection points $\{D,E\}$ of $\{\mu_0,\varepsilon_0\}$ which are common to all members of $\mathcal{D}$. Thus, if they are real, then all members of the pencil will be real. The Steiner ellipse appears in the next lemma concerning the location of S.

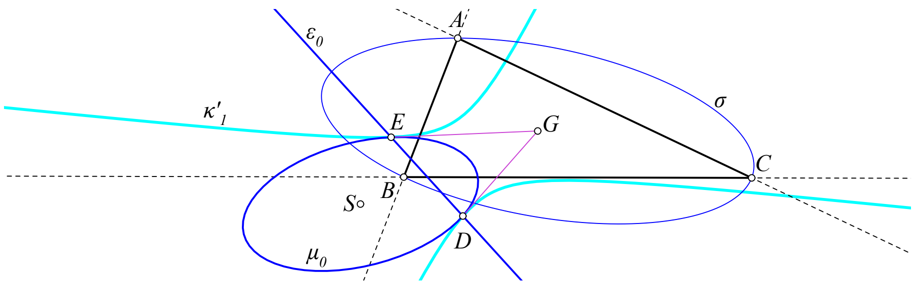

Figure 12: A case for which S is outside the Steiner ellipse

Lemma 2. *The intersection points* $\{D,E\} = \mu_0 \cap \varepsilon_0$ *are real if and only if, S considered as a point of the plane, is outside the closure of the Steiner ellipse of ABC. This is equivalent to the condition* $\Sigma = s_1 s_2 + s_2 s_3 + s_3 s_1 \leq 0$.

Proof. The standard method ([14, p.127]) to find the intersection points is to solve the line equation w.r.t. to one variable, $y_3 = -s_3(y_1/s_1 + y_2/s_2)$ say, and replace in the equation of the conic μ_0. The discriminant of the resulting homogeneous quadratic equation is found equal to $d = -4(s_2 s_3 + s_1 s_3 + s_1 s_2)/(s_1 s_2)^2$. This is non-negative precisely when the stated condition holds. □

Figure 12 shows a case of a conic κ_1' for which lemma 2 is satisfied. The pencil of conics $\{t\mu_0 - \varepsilon_0^2\}$ in this case has all its members passing through the real points $\{D,E\}$, tangent there to the lines $\{DG, EG\}$. Line ε_0 is the tripolar of S w.r.t. ABC. Figure 13 shows the limiting case of the preceding one in which S lies *on* the Steiner ellipse. Then the conic μ_0

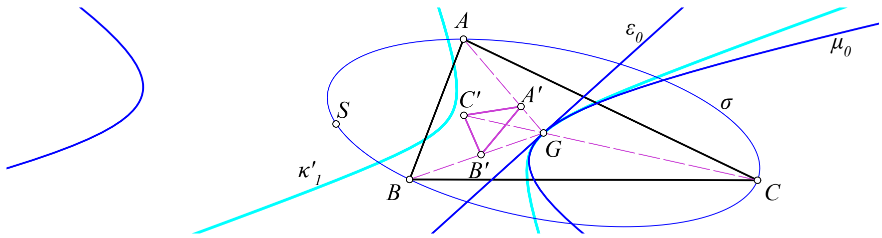

Figure 13: A case with S *on* the Steiner ellipse

is a hyperbola through G and the line ε_0 is tangent there to μ_0. All conics of the pencil $\mathcal{D}$ are also tangent to ε_0 at G.

Our discussion so far, shows also an opposite behavior when all $\{s_i > 0\}$ and S lies in the inner domain of the triangle. For the new configuration theorem 4 translates to the following property.

Lemma 3. *With the notation adopted so far, if all $\{s_i > 0\}$, implying that $S(s_1, s_2, s_3)$, as a point of the plane, lies inside the triangle ABC, then the conic $\kappa'_1 = \Sigma_1\mu_0 - \varepsilon_0^2$ is virtual.*

If all $\{s_i < 0\}$, then $S(s_1, s_2, s_3)$ considered as a point of the plane, is again inside the triangle. Nevertheless, as we saw in the example of section 5, in this case we have real, as well as virtual conics. Next lemma generalizes a property of this example.

Lemma 4. *Assume that all $\{s_i < 0\}$. Then, if the coefficient $\Sigma_1 > 0 \Leftrightarrow \Sigma < -\Pi$ the conic κ'_1 is virtual, whereas if $\Sigma_1 < 0 \Leftrightarrow \Sigma > -\Pi$, the conic is real. For $\Sigma_1 = 0$ the conic κ'_1 is degenerate coinciding with the double line ε^2.*

Proof. The first condition, with $\Sigma_1 > 0$, implies trivially the corresponding claim, since then the whole expression in (12) is negative for every point Y of the plane. In the case of the second condition $\Sigma_1 < 0$, we find a real point on the median of the triangle $G + tC = (1, 1, 1 + t)$ satisfying the equation of the conic κ'_1, thereby proving that the conic is real. In fact, evaluating the expression in (12) for $G + tC$ we come to a quadratic equation in t, whose discriminant d is equal to

$$d = -4\Sigma_1(s_1 + s_2)(\Sigma_1\Pi - \Sigma)/\Pi^2 = -4\Sigma_1(s_1 + s_2)/\Pi .$$

Under the present hypothesis all the factors of this product are negative implying that $d > 0$, i.e. there is a real t and consequently a real point $G + tC$ satisfying the equation of the conic. The last statement is obvious. □

In view of the three preceding lemmata, the remaining cases of interest are those corresponding to S defining points of the plane lying in the domains inside the Steiner ellipse and outside the triangle as in figure 14, which corresponds to the conditions:

$$s_1s_2 + s_2s_3 + s_1s_2 > 0, \quad s_1 > 0, \quad s_2 > 0, \quad s_3 < 0, \tag{14}$$

analogous conditions holding for the other two domains shown.

Lemma 5. *The conic $\mu_0 : \sum y_i^2/s_i = 0$ appearing in equation* (12) *is always a hyperbola for S satisfying the conditions* (14)*.*

Proof. We use again the standard method of lemma 2 to show that the intersections of the conic with the line at infinity $y_1 + y_2 + y_3 = 0$ consists of real points. We may assume that S is normalized, i.e. $\sum s_i = 1$ and substitute in the conic equation $y_3 = -y_1 - y_2$. The discriminant of the resulting homogeneous quadratic equation is positive $d = -4(s_1 + s_2 + s_3)/(s_1s_2s_3) = -4/(s_1s_2s_3) > 0$. □

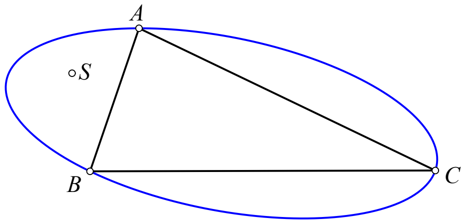

Figure 14: Domains of interest for S

Remark 3. By a general and trivial property of a pencil of conics, the common points to two members are common to all members of the pencil. Thus, by lemma 2 and for S satisfying (14) all members $\{\kappa'_t = t\mu_0 - \varepsilon_0^2, t \in \mathbb{R}\}$ of the pencil generated by $\{\mu_0, \varepsilon_0^2\}$ do not intersect pairwise in real points and in particular all $\{\kappa'_t, t \neq 0\}$ and ε_0 do not intersect in real points.

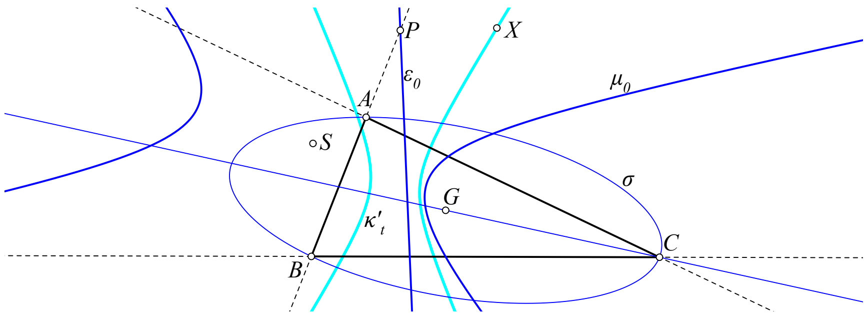

Figure 15: Case of existence of a real conic κ'_1

Figure 15 shows a case of existence of a real conic κ'_t. Since the various conics $\{\kappa'_t\}$ cover the plane, we are looking after those $\{t\}$ which define real conics. Next lemma lists some properties useful for this task, suggested by this figure. Their proofs result from general facts valid for bitangent pencils or/and the simple matrix representation of these conics, which is a linear combination of the matrices representing the conics μ_0 and ε_0^2 :

$$\kappa'_t = t\mu_0 - \varepsilon_0^2 : \quad t \cdot \begin{pmatrix} \frac{1}{s_1} & 0 & 0 \\ 0 & \frac{1}{s_2} & 0 \\ 0 & 0 & \frac{1}{s_3} \end{pmatrix} - \begin{pmatrix} \frac{1}{s_1^2} & \frac{1}{s_1 s_2} & \frac{1}{s_1 s_3} \\ \frac{1}{s_2 s_1} & \frac{1}{s_2^2} & \frac{1}{s_2 s_3} \\ \frac{1}{s_3 s_1} & \frac{1}{s_2 s_2} & \frac{1}{s_3^2} \end{pmatrix}. \tag{15}$$

Lemma 6. *With the notation and conventions adopted so far and for S satisfying the conditions* (14) *the following properties hold (see Figure 15).*

1. *The polar of the centroid G of the triangle ABC w.r.t. to every conic* $\kappa'_t = t\mu_0 - \varepsilon_0^2$ *is the line* ε_0.
2. *The pol of the median CG of the triangle ABC w.r.t. every member conic* κ'_t *of the pencil is the intersection point* $P = AB \cap \varepsilon_0$.
3. *Point P is an outer point w.r.t. to every real member conic of the pencil* $\kappa'_t \neq \varepsilon_0$ *and every real conic* κ'_t *of the pencil intersects the line GC.*
4. *Point S is the center of* μ_0 *and the centers of all the conics* κ'_t *lie on line SG.*
5. *The tangents of all real conics* κ_t *at their intersection points with line SG are parallel to* ε_0, *which therefore represents the conjugate direction to the one determined by SG w.r.t. every conic* κ'_t.

Proof. nr-1: Applying the matrix representing κ'_t to $G = (1,1,1)$ we find the coefficients of the polar of G and the equation of the polar:

$$t\varepsilon_0(X) - \varepsilon_0(G)\varepsilon_0(X) = \left(t - \sum \frac{1}{s_i}\right)\varepsilon_0(X) = 0.$$

nr-2: By a trivial property of pencils of conics we know that all polars p_Q of points Q lying on line GC pass through the same point P necessarily lying on ε_0. Since ABC is self-conjugate w.r.t. ABC, line AB is the polar of C w.r.t. μ_0, hence P is the intersection $P = BC \cap \varepsilon_0$ as claimed.

nr-3: Since P lies on ε_0 which is not intersected by any real member of the pencil $\kappa'_t \neq \varepsilon_0$, it lies outside of these conics. Hence the polar of P w.r.t. such a conic, which is GC intersects the conic.

nr-4,5: Denoting the conic and its matrix representation by the same symbol, we see that $S^t\mu_0 = (1,1,1)$, which are the coefficients of the line at infinity. This shows that S is the pol w.r.t. μ_0 of the line at infinity, i.e. it is the center of the conic. For the general case, consider the point at infinity $P_0 = (s_1(s_2 - s_3), s_2(s_3 - s_1), s_3(s_1 - s_2))$ of the line ε_0. The remaining claims result immediately if we show, that $\{S, G\}$ are contained in the polar of P_0 w.r.t. every conic κ'_t, which is easily verified. This implies, that line SG is the polar of P_0 for every real conic κ'_t intersecting the line SG. But since P_0 is on ε_0, it is outside every such real conic and consequently every such conic intersects SG, at points at which their tangents are parallel to ε_0, hence the center of the conic is on SG and the direction of SG is conjugate to that of ε_0 w.r.t. every real conic κ_t. □

Lemma 7. *Assume that a point $S \in \mathbb{R}^3$ satisfies the conditions* (14)*. Then, the corresponding conic κ'_1 is real if and only if the condition $\Sigma < -\Pi$ is valid. In the case $-S$ satisfies the conditions* (14) *the conic κ'_1 is real.*

Proof. Since every real conic κ'_λ intersects the median GC of the triangle, it suffices to show that the stated condition is equivalent to the existence of a point on GC satisfying the equation of κ'_1. Consider a point $P_t = G + tC = (1,1,1+t)$ of the median GC of the triangle ABC and the real conic $\kappa'_\lambda = \lambda\mu_0 - \varepsilon_0^2$ passing through that point. Proceeding as in lemma 4 we find a quadratic equation in t with discriminant:

$$d = -4\lambda(s_1 + s_2)(\lambda\Pi - \Sigma)/\Pi^2 . \tag{16}$$

Thus, t is real if and only if d is positive. Assume first that S satisfies the condition 14. Then the condition $d > 0$ is equivalent to $\lambda(\lambda\Pi - \Sigma) < 0$, which setting $\lambda = \Sigma_1$ leads to the stated inequality.

Assume now that $-S$ satisfies the condition 14. The only difference to the preceding case is in the sign of the $\{s_i\}$, which now are $\{s_1 < 0, s_2 < 0, s_3 > 0\}$, implying $\Pi > 0$. Since the conics of the pencil are the same, the real ones are determined again by the condition $d > 0$ in equation (16). This is now equivalent to $\lambda(\lambda\Pi - \Sigma) > 0$, which setting $\lambda = \Sigma_1$ leads to the inequality $\Pi + \Sigma > 0$, which is true. □

Next theorem recapitulates the results proved in the preceding lemmata.

Theorem 5. *Let $T(t_1, t_2, t_3) \in \mathbb{R}^3$ be a generic vector of parameters and $O(o_1, o_2, o_3)$ be a point of the plane not lying on the side-lines of the triangle $\tau = ABC$ expressed in absolute barycentrics. Let also $\{s_1 = t_1/o_1, s_2 = t_2/o_2, s_3 = t_3/o_3\}$, $\Pi = s_1s_2s_3$ and $\Sigma = s_1s_2 + s_2s_3 + s_3s_4$. Then the perspectivity of τ to the triangle $\tau' = A'B'C'$ with vertices $A' = O + t_1A, B' = O + t_2B$, $C' = O + t_3C$ is realized by the polarity w.r.t. a real conic represented by equation* (11) *if and only if, S interpreted as a point of the projective plane, satisfies one of the following conditions:*

1. *S lies in the closure of the outer domain of the Steiner ellipse of ABC, i.e. it satisfies $\Sigma = s_1s_2 + s_2s_3 + s_3s_1 \leq 0$.*

2. *S lies inside the Steiner ellipse of ABC and outside the inner domain of the triangle satisfying* $0 < \Sigma < -\Pi$.
3. *S lies inside the Steiner ellipse of ABC and outside the inner domain of the triangle satisfying* $\Pi > 0$.
4. *S lies inside the triangle having all* $\{s_i < 0\}$ *and satisfying* $\Sigma > -\Pi > 0$.

Bibliography

[1] Askwith, E.H., *A course of pure Geometry*, Cambridge University Press, Cambridge, 1930.

[2] Baker, H., *Principles of Geometry, vols I?VI*, Cambridge university press, Cambridge, 1930.

[3] Berger, M., *Geometry vols I, II*, Springer Verlag, Heidelberg, 1987.

[4] Boeger, R., *Geometrie der Lage*, Goeschensche Verlagshandlung, Leipzig, 1900.

[5] Coxeter, H., *Projective Geometry*, Springer, New York, 1987.

[6] Emch, A., *An introduction to projective Geoemtry and its applications*, John Wiley, London, 1905.

[7] Gibson, C., *Elementary Euclidean Geometry*, Cambridge University Press, Cambridge, 2003.

[8] Loney, S., *The elements of coordinate geometry I, II*, AITBS publishers, Delhi, 1991.

[9] Pamfilos, P., *Barycentric coordinates*, http://users.math.uoc.gr/~pamfilos/eGallery /problems/Barycentrics.pdf, 2019.

[10] Pamfilos, P., *Conic Homographies and Bitangent Pencils*, Forum Geometricorum, **9(2009)**, 229-257, 2009.

[11] Salmon, G., *A treatise on Conic Sections*, Longmans Green and Co., London, 1917.

[12] Strang, G., *Linear Algebra and its Applications*, Thomson Learning 3rd Edition, London, 1988.

[13] Veblen, P. and Young, J., *Projective Geometry vol. I, II*, Ginn and Company, New York, 1910.

[14] Yiu, P., *Introduction to the Geometry of the Triangle*, http://math.fau.edu/Yiu/Geome try.pdf, 2013.

DEPARTMENT OF MATHEMATICS
AND APPLIED MATHEMATICS
UNIVERSITY OF CRETE
HERAKLION, 70013 GR
E-mail address: pamfilos@uoc.gr