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HARMONIC NATURAL LIFTS OF METRICS ON TANGENT BUNDLES

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Abstract. Let $(M,\ g)$ be a Riemannian manifold, g^s and g^h be respectively the corresponding Sasaki metric and the horizontal lift on the tangent bundle TM. In this paper we study the harmonicity of g-natural metrics with respect to g^s and then with respect to g^h. We compute the tension fields in terms of the Ricci curvature and we describe the harmonicity conditions accordind to whether the basis metric g is Einstein or not. In the case of a Ricci flat manifold (M, g) we obtain that the horizontal lift g^h is an harmonic metric w.r.t. the Sasaki metric g^s, and conversely that g^s is harmonic w.r.t. g^h on TM. Futhermore for a connected non-Einstein manifold (M, g), we prove that there exists no nontrivial Riemannian g-natural metric on TM that is harmonic w.r.t the horizontal lift g^h.

1. Introduction

In [1] the authors introduced g-natural metrics on the tangent bundle TM of a Riemannian manifold (M, g) as metrics on TM which come from g through first order natural operators defined between the natural bundle of Riemannian metrics on M and the natural bundle of $(0, 2)$-tensors fields on the tangent bundles. Classical well-known metrics like Sasaki metric (cf. [13] , [8]), horizontal lift (cf. [4]) or Cheeger-Gromoll metrics (cf. [5] , [12]) are examples of natural metrics on the tangent bundle. By associating the notion of F-tensors fields they got a characterization of g-natural metrics on TM in terms of the basis metric g and of some functions defined on the set of positive real numbers, and obtained necessary and sufficient conditions for g-natural metrics to be either nondegenerate or Riemannian (see [11] for more details on natural operators and F-tensors fields).

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Harmonic metrics arised from an interesting application of harmonic maps and have been introduced by the authors in [6]. Let $\phi : (M,\ g) \longrightarrow (N,\ h)$ be immersion between two Riemannian manifolds (M, g) and (N, h). If ϕ is an harmonic map, then $\phi^* h$ is a Riemannian metric on M such that the identity map $Id_M : (M,\ g) \longrightarrow (M,\ \phi^* h)$ is an harmonic map [9]. Thus, for a given Riemannian manifold (M, g), it became natural and interesting to seek for pseudo-Riemannian metrics $\hat{g}$ on M for which the identity $Id_M : (M,\ g) \longrightarrow (M,\ \hat{g})$ is an harmonic map. Such metrics are said to be harmonic w.r.t.the given metric g. The authors in [6], who introduced formally the notion, obtained an intrinsic characterization of harmonic metrics and used it to extend the definition of harmonicity to symmetric $(0, 2)$)-tensors.
In this paper we consider the Tangent bundle TM of a Riemannian manifold (M, g) and we study the harmonicity of nondegenerate g-natural metrics firstly w.r.t. the Sasaki metric g^s, and then w.r.t. g^h, the horizontal lift of g.
In the next section we give some basics and known results on g-natural metrics, and then on harmonic metrics. In Section 3, we study harmonicity on TM of nondegenerate g-natural metrics with respect to the Sasaki metric g^s, and the Section 4 is devoted to the harmonicity on TM of nondegenerate g-natural metrics with respect to horizontal lift g^h of g.

2. Preliminaries

2.1. **g-natural metrics on tangent bundles.** Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection of g. The tangent space of TM at a point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to ∇ :

$$T_{(x,u)}TM = H_{(x,u)}M \oplus V_{(x,u)}M\ .$$

A system of local coordinates $(U\,;\, x_i,\ i = 1, \cdots, m)$ in M induces on TM a system of local coordinates $(\pi^{-1}(U)\,;\, x_i, u^i,\ i = 1, \cdots, m)$.
Let $X = \sum_{i=1}^m X^i \frac{\partial}{\partial x_i}$ be the local expression in U of a vector field X on M. Then, the horizontal lift X^h and the vertical lift X^v of X are given, with respect to the induced coordinates, by :

$$X^h = \sum_i X^i \frac{\partial}{\partial x_i} - \sum_{i,j,k} \Gamma^i_{jk} u^j X^k \frac{\partial}{\partial u^i} \quad \text{and} \tag{1}$$

$$X^v = \sum_i X^i \frac{\partial}{\partial u^i}, \tag{2}$$

where the Γ^i_{jk} are the Christoffel's symbols defined by g.
Next, we introduce some notations which will be used to describe vectors obtained from lifted vectors by basic operations on TM.
Let T be a tensor field of type $(1, s)$ on M. If $X_1, X_2, \cdots, X_{s-1} \in T_x M$, then $h\{T(X_1, \cdots, u, \cdots, X_{s-1})\}$ and $v\{T(X_1, \cdots, u, \cdots, X_{s-1})\})$ are horizontal and vertical vectors repectively at the point (x, u) which are defined by:

$$h\{T(X_1, \cdots, u, \cdots, X_{s-1})\} = \sum u^\lambda \left(T(X_1, \cdots, \frac{\partial}{\partial x_{\lambda\,|x}}, \cdots, X_{s-1}) \right)^h$$

$$v\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum u^\lambda \left(T(X_1,\cdots,\frac{\partial}{\partial x_{\lambda\,|x}},\cdots,X_{s-1})\right)^v .$$

In particular, if T is the identity tensor of type $(1,1)$, then we obtain the geodesic flow vector field at (x,u), $\xi_{(x,u)} = \sum_\lambda u^\lambda \left(\frac{\partial}{\partial x_\lambda}\right)^h_{(x,u)}$, and the canonical vertical vector at (x,u), $\mathcal{U}_{(x,u)} = \sum_\lambda u^\lambda \left(\frac{\partial}{\partial x_\lambda}\right)^v_{(x,u)}$.

Also $h\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ and $v\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ are defined by similar way.

Let us introduce the notations

$$h\{T(X_1,\cdots,X_s)\} =: T(X_1,\cdots,X_s)^h \tag{3}$$

and

$$v\{T(X_1,\cdots,X_s)\} =: T(X_1,\cdots,X_s)^v . \tag{4}$$

Thus $h\{X\} = X^h$ and $v\{X\} = X^v$, for each vector field X on M.

From the preceding quantities, one can define vector fields on TU in the following way: If $u = \sum_i u^i \left(\frac{\partial}{\partial x_i}\right)_x$ is a given point in TU and $X_1,\cdots,X_{s-1}$ are vector fields on U, then we denote by

$$h\{T(X_1,\cdots,u,\cdots,X_{s-1})\} \quad \text{(respectively} \quad v\{T(X_1,\cdots,u,\cdots,X_{s-1})\})$$

the horizontal (respectively vertical) vector field on TU defined by

$$h\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum_\lambda u^\lambda T(X_1,\cdots,\frac{\partial}{\partial x_\lambda},\cdots,X_{s-1})^h$$

$$(\text{ resp. } \quad v\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum_\lambda u^\lambda T(X_1,\cdots,\frac{\partial}{\partial x_\lambda},\cdots,X_{s-1})^v\ ).$$

Moreover, for vector fields $X_1,\cdots,X_{s-t}$ on U, where $s\,,\,t \in \mathbb{N}^*\,(s>t)$, the vector fields $h\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ and $v\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ on TU, are defined by similar way.

Now, for $(r,s) \in \mathbb{N}^2$, we denote by $\pi_M : TM \to M$ the natural projection and F the natural bundle defined by

$$\begin{aligned} FM &= \pi_M^*(\underbrace{T^*\otimes\cdots\otimes T^*}_{r \text{ times}} \otimes \underbrace{T\otimes\cdots\otimes T}_{s \text{ times}})M \to M, \\ Ff(X_x,S_x) &= (Tf.X_x, (T^*\otimes\cdots\otimes T^*\otimes T\otimes\cdots\otimes T)f.S_x) \end{aligned} \tag{5}$$

for $x \in M$, $X_x \in T_xM$, $S \in (T^*\otimes\cdots\otimes T^*\otimes T\otimes\cdots\otimes T)M$ and any local diffeomorphism f of M.

We call the sections of the canonical projection $FM \to M$ F-tensor fields of type (r,s). So, if we denote the product of fibered manifolds by $\oplus$, then the F-tensor fields are mappings

$A : TM \oplus \underbrace{TM\oplus\cdots\oplus TM}_{s \text{ times}} \to \bigsqcup_{x\in M} \otimes^r T_xM$ which are linear in the last s summands and such that $\pi_2 \circ A = \pi_1$, where π_1 and π_2 are respectively the natural projections of the source and target fiber bundles of A. For $r = 0$ and

$s = 2$, we obtain the classical notion of F-metrics. So, F-metrics are mappings $TM \oplus TM \oplus TM \to \mathbb{R}$ which are linear in the second and the third arguments.
Moreover let us fix $(x, u) \in TM$ and a system of normal coordinates
$S := (U\,;\, x_i\,, i = 1, \cdots, m)$ of (M, g) centred at x. Then we can define on U the vector field $\mathbf{U} := \sum_i u^i \frac{\partial}{\partial x_i}$, where $(u^1, \cdots, u^m)$ are the coordinates of $u \in T_xM$ with respect to its basis $(\frac{\partial}{\partial x_i}_{|x}; \, i = 1, \cdots, m)$.
Let P be an F-tensor field of type (r, s) on M. Then on U, we can define an (r, s)-tensor field P_u^S (or P_u if there is no risk of confusion) associated to u and S by

$$P_u(X_1, \cdots, X_s) := P(\mathbf{U}_z; X_1, \cdots, X_s)\,, \tag{6}$$

for all $(X_1, \cdots, X_s) \in T_zM$, $\forall z \in U$.
On the other hand, if we fix $x \in M$ and s vectors $X_1, \cdots, X_s$ in T_xM, then we can define a C^∞-mapping $P_{(X_1,\cdots,X_s)} : T_xM \to \otimes^r T_xM$, associated to $(X_1, \cdots, X_s)$ by

$$P_{(X_1,\cdots,X_s)}(u) := P(u;\, X_1, \cdots, X_s)\,, \tag{7}$$

for all $u \in T_xM$.
Let $s > t$ be two non-negative integers, T be a $(1, s)$-tensor field on M and P^T be an F-tensor field of type $(1, t)$ of the form

$$P^T(u; X_1, \cdots, X_t) = T(X_1, \cdots, u, \cdots, u, \cdots, X_t), \tag{8}$$

for all $(u; X_1, \cdots, X_t) \in TM \oplus \cdots \oplus TM$, i.e., u appears $s - t$ times at positions $i_1, \cdots, i_{s-t}$ in the expression of T. Then

- P_u^T is a $(1, t)$-tensor field on a neighborhood U of x in M, for all $u \in T_xM$;
- $P^T_{(X_1,\cdots,X_t)}$ is a C^∞-mapping $T_xM \to T_xM$, for all $X_1, \cdots, X_t$ in T_xM.

We have the following definition :

Definition 2.1. *Let (M, g) be a Riemannian manifold. A g-natural metric on the tangent bundle of M is a metric on TM which is the image of g by a first order natural operator defined from the natural bundle of Riemannian metrics $S^2_+T^*$ on M into the natural bundle of $(0, 2)$-tensor fields $(S^2T^*)T$ on the tangent bundles (cf.* [1] *,* [2]*).*

Tangent bundles equipped with g-natural metrics are called g-natural tangent bundles.

The following result gives the classical expression of g-natural metrics:

Proposition 2.2. [1] *Let (M, g) be a Riemannian manifold and G a g-natural metric on TM. There exists six smooth functions $\alpha_i,\ \beta_i : \mathbb{R}^+ \to \mathbb{R},\ i = 1, 2, 3,$ such that* *for any $x \in M$ and all vectors $u,\ X,\ Y \in T_xM$, we have*

$$\begin{cases} G_{(x,u)}\left(X^h, Y^h\right) &= (\alpha_1+\alpha_3)(t)g_x(X,Y) \\ & \quad +(\beta_1+\beta_3)(t)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}\left(X^h, Y^v\right) &= \alpha_2(t)g_x(X,Y)+\beta_2(t)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}\left(X^v, Y^h\right) &= \alpha_2(t)g_x(X,Y)+\beta_2(t)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}\left(X^v, Y^v\right) &= \alpha_1(t)g_x(X,Y)+\beta_1(t)g_x(X,u)g_x(Y,u), \end{cases}$$

where $t = g_x(u,u)$, X^h and X^v are respectively the horizontal lift and the vertical lift of the vector $X \in T_xM$ at the point $(x,u) \in TM$.

Notations 2.3.

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$, $i = 1,2,3$,
- $\alpha(t) = \alpha_1(t)(\alpha_1+\alpha_3)(t) - \alpha_2^2(t)$,
- $\phi(t) = \phi_1(t)(\phi_1+\phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$.

For a g-natural metric to be nondegenerate or Riemannian, there are some conditions to be satisfied by the functions α_i and β_i of Proposition 2.1. It holds:

Proposition 2.4. [1] *A g-natural metric G on the tangent bundle of a Riemannian manifold $(M,\ g)$ is :*

(i) *nondegenerate if and only if the functions α_i, β_i, $i = 1,2,3$ defining G are such that*

$$\alpha(t)\phi(t) \neq 0 \tag{9}$$

for all $t \in \mathbb{R}^+$.

(ii) *Riemannian if and only if the functions α_i, β_i, $i = 1,2,3$ defining G, satisfy the inequalities*

$$\begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases} \tag{10}$$

for all $t \in \mathbb{R}^+$.
For $\dim M = 1$, this system reduces to $\alpha_1(t) > 0$ and $\alpha(t) > 0$, for all $t \in \mathbb{R}^+$.

Before giving the formulas relating both Levi-Civita connexions ∇ of (M,g) and $\bar{\nabla}$ of (TM,G) let us introduce the following notations:

Notations 2.5. *For a Riemannian manifold (M,g), we set :*

$$\begin{array}{ll} T^1(u;X_x,Y_x) = R(X_x,u)Y_x, & T^2(u;X_x,Y_x) = R(Y_x,u)X_x\,, \\ T^3(u;X_x,Y_x) = g(R(X_x,u)Y_x,u)u, & T^4(u;X_x,Y_x) = g(X_x,u)Y_x, \\ T^5(u;X_x,Y_x) = g(Y_x,u)X_x, & T^6(u;X_x,Y_x) = g(X_x,Y_x)u, \\ T^7(u;X_x,Y_x) = g(X_x,u)g(Y_x,u)u. & \end{array} \tag{11}$$

where $(x,u) \in TM$, $X_x, Y_x \in T_xM$ and R is the Riemannian curvature of g.

Let G be a g-natural metric on TM defined by the functions α_i, β_i of Proposition 2.2. It holds:

Proposition 2.6. [7] *Let $(x,u) \in TM$ and $X, Y \in \mathfrak{X}(M)$, we have*

$$
\begin{aligned}
(12)\quad \left(\bar{\nabla}_{X^h} Y^h\right)_{(x,u)} &= (\nabla_X Y)^h_{(x,u)} + h\{A(u; X_x, Y_x)\} + v\{B(u; X_x, Y_x)\},\\
(13)\quad \left(\bar{\nabla}_{X^h} Y^v\right)_{(x,u)} &= (\nabla_X Y)^v_{(x,u)} + h\{C(u; X_x, Y_x)\} + v\{D(u; X_x, Y_x)\},\\
(14)\quad \left(\bar{\nabla}_{X^v} Y^h\right)_{(x,u)} &= h\{C(u; Y_x, X_x)\} + v\{D(u; Y_x, X_x)\},\\
(15)\quad \left(\bar{\nabla}_{X^v} Y^v\right)_{(x,u)} &= h\{E(u; X_x, Y_x)\} + v\{F(u; X_x, Y_x)\},
\end{aligned}
$$

where $P(u; X_x, Y_x) = \sum_{i=1}^{7} f_i^P(|u|^2) T^i(u; X_x, Y_x)$, for $P = A, B, C, D, E, F$, and the functions f_i^P defined in terms of α_i, β_i) (see [7] *for their exact expressions).*

2.2. **Harmonic maps, harmonic metrics.**

Definition 2.7. *Let $\varphi : (M,\ g) \longrightarrow (N,\ h)$ be a C^2 map between two Riemannian manifolds (M, g) and (N, h) with compact support. The energy density of φ, denoted by $e(\varphi)$ is defined by:*

$$e(\varphi) = \frac{|d\varphi|^2}{2},$$

where $|d\varphi|$ is the Hilbert-Schmidt norm of $d\phi$ induced by the metrics g and h on $T^*(M) \otimes \varphi^{-1}TN$ that is defined by:

$$|d\varphi|^2 = tr_g(\varphi^* h).$$

In local coordinates, $e(\varphi) = \frac{g^{ij}}{2} h_{\alpha\beta} \partial_i \varphi^\alpha \partial_j \varphi^\beta$. The Dirichlet energy of φ, over M is defined by

$$E(\varphi) = \int_M e(\varphi) dv_g,$$

where dv_g is the volume measure induced by g.
The map φ is said to be harmonic, if it is a critical point of the energy functionnal E.
In the case where the map φ has a noncompact support, the map φ is said to be harmonic if its restriction to any compact subset of M is harmonic.
The Euler-Lagrange equations with respect to the energy functional E obtained by the first variation formula give rise to the following characterization: the map $\varphi : (M,\ g) \longrightarrow (N,\ h)$ is harmonic if and only if its tension field $\tau(\phi)$ vanishes identically, where $\tau(\phi)$ is the contraction w.r.t. g of the second fundamental form $\nabla^{\varphi^{-1}TN} d\varphi$ of φ defined by

$$
\begin{aligned}
\nabla^{\varphi^{-1}TN} d\varphi(X,Y) &= (\nabla_X^{\varphi^{-1}TN} d\varphi)(Y)\\
&= \nabla^N_{d\varphi(X)} d\varphi(Y) - d\varphi(\nabla^M_X Y),\ \forall X, Y \in \Gamma(TM),
\end{aligned}
$$

whith ∇^M and ∇^N the Levi-Civita connections on $(M,\ g)$ and $(N,\ h)$ respectively.
In local coordinates $(x^i)_{i=1}^m$ at $x \in M$ and $(u^\alpha)_{\alpha=1}^n$ at $f(x) \in N$, the Euler-Lagrange equations are given by the system:

$$-\Delta_g f^\alpha + g^{ij} \bar{\Gamma}^\alpha_{\gamma\beta} \frac{\partial f^\gamma}{\partial x^j} \frac{\partial f^\beta}{\partial x^i} = 0 \text{ for all } \alpha = 1, ..., n\ ,$$

where Δ_g is the Laplace-operator on (M, g) and $\bar{\Gamma}^{\alpha}_{\gamma\beta}$ the Christoffel symbols of (N, h). Refer to [9] for more basic informations on harmonic maps.
Let (M, g) be an m-dimensional Riemannian manifold. It is easy to check that the identity map $Id : (M, g) \longrightarrow (M, g)$ is harmonic. However if we consider another metric h on M, then the identity map $Id : (M, g) \longrightarrow (M, h)$ is not any more automatically harmonic. A metric h on M is said to be harmonic w.r.t. g if the identity map $Id : (M, g) \longrightarrow (M, h)$ is harmonic [6].
In a local coordinate system $(x^i)_{i=1}^m$ on M, the metric h is harmonic w.r.t. g if and only if:

$$g^{ij}\left(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}\right) = 0 \ , \ k = 1, ..., m \ ,$$

where Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$ are the Christoffel symbols w.r.t. g and h respectively.
Equivalently the metric h is harmonic w.r.t. g if and only if

$$d(tr_g h) + 2\delta h = 0 \ ,$$

where d and δ are the differential and the codifferential operators defined on (M, g) respectively. From this characterization a symmetric $(0, 2)$-tensor T on (M, g) is said to be harmonic with respect to g if it satisfies the equation

$$d(tr_g T) + 2\delta T = 0 \ .$$

Some interesting results have been obtained on harmonic symmetric $(0, 2)$-tensors by the authors in [6] and also in [14].

3. Harmonicity of g-natural metrics w.r.t. g^s.

Let (M, g) be Riemannian manifold. The nondegenerate g-natural metric G is harmonic with respect to Sasaki metric g^s on the tangent TM induced by g if the identity map $Id : (TM,\ g^s) \longrightarrow (TM,\ G)$ is an harmonic map.

By direct computation we have the following result:

Proposition 3.1. *Let (M, g) be an n-dimensional Riemannian manifold, G a nondegenerate g-natural metric on TM and g^s the Sasaki metric induced by g on TM. The tension field $\tau(Id)$ of the identity map $Id : (TM,\ g^s) \longrightarrow (TM,\ G)$ is given at a point $(x, u) \in TM$ by:*

$$\tau(Id)_{(x,u)} = \ \tau^h_{(x,u)} + \ \tau^v_{(x,u)} \in T_{(x,u)}TM, \tag{16}$$

where the horizontal part $\tau^h_{(x,u)}$ and the vertical part $\tau^v_{(x,u)}$ of $\tau(Id)_{(x,u)}$ are given by:

$$\begin{aligned} \tau^h_{(x,u)} \ &= \ [-(f^A_1 + f^A_2)R_{ic}(u) - f^A_3 g(R_{ic}(u), u)u]^h_{(x,u)} \\ &+[(f^A_4 + f^A_5 + nf^A_6 + g(u, u)f^A_7) \\ &+(f^E_4 + f^E_5 + nf^E_6 + g(u, u)f^E_7)]u^h_{(x,u)} \end{aligned} \tag{17}$$

and

$$\begin{aligned}\tau^v_{(x,u)} &= [-(f_1^B+f_2^B)R_{ic}(u) - f_3^B g(R_{ic}(u),u)u]^v_{(x,u)}\\ &+[(f_4^B+f_5^B+nf_6^B+g(u,u)f_7^B)\\ &+(f_4^F+f_5^F+nf_6^F+g(u,u)f_7^F)]u^v_{(x,u)},\end{aligned} \tag{18}$$

with the functions f_i^P, $i=1,\cdots,7$; $P=A,B,E,F$ *defined by Proposition 2.6 and* $R_{ic}(u) = -g^{ij}R(u,\partial_i)\partial_j$ *in local coordinates, where* R *is the Riemannian curvature of* $(M,\ g)$.

Thus the identity map $Id : (TM,\ g^s) \longrightarrow (TM,\ G)$ is harmonic if and only if:

$$\tau^h \equiv 0 \ \text{ and } \ \tau^v \equiv 0\ .$$

Remark 3.2. *We can observe that:*

(1) $R_{ic}(X) = -g^{ij}R(X,\partial_i)\partial_j$ *in local coordinates such that the ricci curvature* r_{ic} *is the* $(0,2)-tensor$ *with* $r_{ic}(X,Y) = g^{ij}g(R(X,\partial_i)Y,\partial_j) = g(R_{ic}(X),Y)$, $\forall x\in M$ *and* $\forall X,Y\in T_xM$.
(2) *the horizontal part and the vertical part of the tension field* $\tau(Id)$ *are independent of the functions* f_1^E, f_2^E, f_3^E, f_1^F, f_2^F *and* f_3^F.
(3) *the horizontal part* τ^h *and the vertical part* τ^v *of the tension field* $\tau(Id)$ *depend on the Ricci curvature* r_{ic}*; which suggests the idea to deal with the harmonicity equations according to whether* r_{ic} *is proportional to the metric* g *or not.*

In the case of non-Einstein manifolds we have:

Theorem 3.3. *Let* (M,g) *be a connected non-Einstein Riemannian manifold,* G *a nondegenerate* g*-natural metric on* TM *defined by the functions* α_i, β_i $i=1,2,3$*; in Proposition 2.2.*
The g*-natural metric* G *is harmonic with respect to the Sasaki metric* g^s *induced by* g *on* TM *if and only if*

$$\begin{cases}\alpha_2\equiv 0,\\ (f_4^A+f_5^A+nf_6^A+tf_7^A)+(f_4^E+f_5^E+nf_6^E+tf_7^E)=0, \quad \forall t\in\mathbb{R}_+,\\ (f_4^B+f_5^B+nf_6^B+tf_7^B)+(f_4^F+f_5^F+nf_6^F+tf_7^F)=0;\end{cases}$$

where the functions f_i^P, $i=4,\cdots,7$, $P=A,B,E,F$ *are evaluated at* t.

Proof :
Suppose that G is harmonic with respect to the Sasaki metric g^s induced by g. Since $(M,\ g)$ is a connected non-Einstein manifold and R_{ic} is diagonalizable, there exits $x\in M$ such that R_{ic} has two different real eigenvalues (Λ_1 and Λ_2) on T_xM. Let e_1 and e_2 be eigenvectors w.r.t Λ_1 and Λ_2 respectively.
Then $R_{ic}(e_1+e_2) = \Lambda_1 e_1 + \Lambda_2 e_2$ is not parallel to e_1+e_2.
So $\tau^v(e_1+e_2)=0$ gives $-(f_1^B+f_2^B)(g(e_1+e_2,\ e_1+e_2)) = -\frac{\alpha_2^2(g(e_1+e_2,\ e_1+e_2))}{\alpha(g(e_1+e_2,\ e_1+e_2))} = 0$ and $(f_4^B+f_5^B+nf_6^B+tf_7^B)(g(e_1+e_2,\ e_1+e_2))+(f_4^F+f_5^F+nf_6^F+tf_7^F)(g(e_1+e_2,\ e_1+e_2))=0$.
Futhermore $\forall l\in\mathbb{R}_+^*$, $\tau^v(\sqrt{l}(e_1+e_2),\ \sqrt{l}(e_1+e_2))=0$,
then $\alpha_2(lg(e_1+e_2,\ e_1+e_2))=0$ $\forall l\in\mathbb{R}_+^*$, so $\alpha_2\equiv 0$.

By the similar way, $(f_4^B(t) + f_5^B(t) + nf_6^B(t) + tf_7^B(t)) + (f_4^F(t) + f_5^F(t) + nf_6^F(t) + tf_7^F()t) = 0 \ \forall t \in \mathbb{R}_+$.
Elsewhere, $\tau^h \equiv 0$ gives $(f_4^A + f_5^A + nf_6^A + tf_7^A) + (f_4^E + f_5^E + nf_6^E + tf_7^E) \equiv 0$, since from $\alpha_2 \equiv 0$ we have $(f_1^A + f_2^A) \equiv 0$ and $f_3^A \equiv 0$.

Conversely, if $\begin{cases} \alpha_2 \equiv 0, \\ (f_4^A + f_5^A + nf_6^A + tf_7^A) + (f_4^E + f_5^E + nf_6^E + tf_7^E) \equiv 0, \quad \forall t \in \mathbb{R}_+, \\ (f_4^B + f_5^B + nf_6^B + tf_7^B) + (f_4^F + f_5^F + nf_6^F + tf_7^F) \equiv 0; \end{cases}$
holds then $\tau^h \equiv 0$ and $\tau^v \equiv 0$. This completes the proof.

□

In the following we characterize harmonic g-natural metrics with respect to Sasaki metric g^s when (M, g) is an Einstein manifold.

Theorem 3.4. *Let $(M,\ g)$ be an Einstein manifold, and G a nondegenerate g-natural metric on TM.*
The nondegenerate g-natural metric G is harmonic with respect to the Sasaki metric g^s induced by g if and only if

$$\begin{cases} -\Lambda[(f_1^A + f_2^A)(t) + t(f_3^A)(t)] + (f_4^A + f_5^A + nf_6^A)(t) + \\ tf_7^A(t) + (f_4^E + f_5^E + nf_6^E)(t) + tf_7^E(t)) = 0, \\ \\ -\Lambda[(f_1^B + f_2^B)(t) + t(f_3^B)(t)] + (f_4^B + f_5^B + nf_6^B)(t) + \\ tf_7^B(t) + (f_4^F + f_5^F + nf_6^F)(t) + tf_7^F(t) = 0; \end{cases}$$

$\forall t \in \mathbb{R}_+$, *where Λ is the constant of the Ricci curvature; i.e.:*
$R_{ic}(u) = \Lambda u$, $\forall u \in TM$, *and $n = \dim M$.*

Proof:
Since (M, g) is an Einstein manifold with the real Λ such that $R_{ic}(u) = \Lambda u$, $\forall u \in TM$, we obtain

$$\begin{aligned} \tau^h(x,u) &= -\Lambda[(f_1^A + f_2^A)(t) + t(f_3^A)(t)]u + [(f_4^A + f_5^A + nf_6^A)(t) + \\ & \quad tf_7^A(t) + (f_4^E + f_5^E + nf_6^E)(t) + tf_7^E(t)]u, \\ \tau^v(x,u) &= -\Lambda[(f_1^B + f_2^B)(t) + t(f_3^B)(t)]u + [(f_4^B + f_5^B + nf_6^B)(t) + \\ & \quad tf_7^B(t) + (f_4^F + f_5^F + nf_6^F)(t) + tf_7^F(t)]u; \end{aligned}$$

$\forall u \in TM$, with $t = g(u, u)$.
This completes the proof. □

In particular for Ricci flat manifolds, we get:

Corollary 3.5. *Let $(M,\ g)$ be a Ricci flat manifold. Then the horizontal lift g^h of g is an harmonic metric with respect to Sasaki metric g^s on TM.*

Proof:
For the horizontal lift of g, we have $\alpha_2 = 1$, $\alpha_1 = \alpha_3 = 0$, $\beta_1 = \beta_2 = \beta_3 = 0$ then $(f_4^A + f_5^A + nf_6^A)(t) + tf_7^A(t) + (f_4^E + f_5^E + nf_6^E)(t) + tf_7^E(t)) = 0$ and

$(f_4^B+f_5^B+nf_6^B)(t)+tf_7^B(t)+(f_4^F+f_5^F+nf_6^F)(t)+tf_7^F(t)=0\ \forall\, t\in\mathbb{R}_+$. Futhermore $\Lambda=0$.

This completes the proof.

□

4. Harmonicity of g-natural metrics w.r.t. g^h.

Let (M,g) be Riemannian manifold. The nondegenerate g-natural metric G is harmonic with respect to the horizontal lift g^h of g on the tangent TM if the identity map $Id:(TM,\ g^h)\longrightarrow(TM,\ G)$ is an harmonic map.

By direct computation we have the following result:

Proposition 4.1. *Let (M,g) be an n-dimensional Riemannian manifold, G be a nondegenerate g-natural metric on TM and g^h be the horizontal lift of g on TM. The tension field $\tilde{\tau}(Id)$ of the identity map $Id:(TM,\ g^h)\longrightarrow(TM,\ G)$ is given at a point $(x,u)\in TM$ by :*

$$\tilde{\tau}(Id)_{(x,u)}=\tilde{\tau}^h_{(x,u)}+\tilde{\tau}^v_{(x,u)}\in T_{(x,u)}TM, \tag{19}$$

where the horizontal part $\tilde{\tau}^h_{(x,u)}$ and the vertical part $\tilde{\tau}^v_{(x,u)}$ of $\tilde{\tau}(Id)_{(x,u)}$ are given by:

$$\begin{aligned}\tilde{\tau}^h_{(x,u)} &= [-2f_2^C R_{ic}(u)-2f_3^C g(R_{ic}(u),u)u\\ &\quad +2(f_4^C+f_5^C+nf_6^C+g(u,u)f_7^C)u]^h_{(x,u)}\end{aligned} \tag{20}$$

and

$$\begin{aligned}\tilde{\tau}^v_{(x,u)} &= [-2f_2^D R_{ic}(u)-2f_3^D g(R_{ic}(u),u)u\\ &\quad +2(f_4^D+f_5^D+nf_6^D+g(u,u)f_7^D)u]^v_{(x,u)},\end{aligned} \tag{21}$$

with f_i^P, $i=2,\cdots,7$; $P=C,D$ defined by Proposition 2.6 and $R_{ic}(u)=-g^{ij}R(u,\partial_i)\partial_j$ in local coordinates where R is the Riemannian curvature of (M,g).

Thus the identity map $Id:(TM,\ g^h)\longrightarrow(TM,\ G)$ is harmonic if and only if:

$$\tilde{\tau}^h\equiv 0\ \text{ and }\ \tilde{\tau}^v\equiv 0\ .$$

Remark 4.2. *We can also observe that:*

1. *the horizontal part and the vertical part of the tension field $\tilde{\tau}(Id)$ are independent of the functions f_1^C and f_1^D.*
2. *the horizontal part $\tilde{\tau}^h$ and the vertical part $\tilde{\tau}^v$ of the tension field $\tilde{\tau}(Id)$ depend on the Ricci curvature r_{ic}; which suggests the idea to deal with the harmonicity equations according to whether r_{ic} is proportional to the metric g or not.*

In the case of non-Einstein manifolds we have:

Theorem 4.3. *Let (M,g) be a connected non-Einstein Riemannian manifold, G a nondegenerate g-natural metric on TM defined by the functions α_i, β_i $i=1,2,3$; in Proposition 2.2.*

The g-natural metric G is harmonic with respect to the horizontal lift g^h of g on TM if and only if

$$\begin{cases} \alpha_1 \equiv 0, \\ f_4^C + f_5^C + nf_6^C + tf_7^C = 0, \quad \forall t \in \mathbb{R}_+, \\ f_4^D + f_5^D + nf_6^D + tf_7^D = 0; \end{cases}$$

where the functions f_i^P, $i = 4, \cdots, 7$, $P = C, D$ are evaluated at t.

Proof :
Suppose that G is harmonic with respect to the horizontal lift g^h. As in the proof of the Theorem 3.3, there exists $x \in M$ and $u_0 \in T_xM$ such that $R_{ic}(u_0)$ and u_o are not parallel. So $\tilde{\tau}_h(u_o) = 0$ gives $-f_2^C(g(u_0,\ u_0)) = \frac{\alpha_1^2(g(u_0,\ u_0))}{2\alpha(g(u_0,\ u_0))} = 0$ and $(f_4^C + f_5^C + nf_6^C + tf_7^C)(g(u_0,\ u_0)) = 0$.

Then $\alpha_1 \equiv 0$ and $(f_4^C + f_5^C + nf_6^C + tf_7^C)(t) = 0,\ \forall t \in \mathbb{R}_+$.
Elsewhere, $\tilde{\tau}^v \equiv 0$ gives $f_4^D + f_5^D + nf_6^D + tf_7^D \equiv 0$, since from $\alpha_1 \equiv 0$ we have $f_2^D \equiv 0$ and $f_3^D \equiv 0$.

Conversely, if $\begin{cases} \alpha_1 \equiv 0, \\ f_4^C + f_5^C + nf_6^C + tf_7^C = 0, \quad \forall t \in \mathbb{R}_+, \\ f_4^D + f_5^D + nf_6^D + tf_7^D = 0; \end{cases}$

holds then $\tilde{\tau}^h \equiv 0$ and $\tilde{\tau}^v \equiv 0$. This completes the proof.

□

In particular for Riemannian g-natural metrics, we get :

Corollary 4.4. *Let (M, g) be a connected non-Einstein Riemannian manifold. Then any Riemannian g-natural metric is not harmonic with respect to the horizontal lift g^h of g on TM.*

Proof:
If a Riemannian g-natural metric is harmonic with respect to the horizontal lift g^h when $(M,\ g)$ is a connected non-Einstein manifold, then by the Theorem 4.3 $\alpha_1 \equiv 0$. This is absurd by Proposition 2.4.

□

Futhermore, we establish the following result to characterize harmonic g-natural metrics with respect to the horizontal lift g^h when (M, g) is an Einstein manifold.

Theorem 4.5. *Let $(M,\ g)$ be an Einstein manifold, and G a nondegenerate g-natural metric on TM.*

The nondegenerate g-natural metric G is harmonic with respect to the horizontal lift g^h of g if and only if

$$\begin{cases} -\Lambda[f_2^C(t) + tf_3^C(t)] + (f_4^C + f_5^C + nf_6^C)(t) + tf_7^C(t) = 0, \\ \\ -\Lambda[f_2^D(t) + tf_3^D(t)] + (f_4^D + f_5^D + nf_6^D)(t) + tf_7^D(t) = 0; \end{cases}$$

$\forall t \in \mathbb{R}_+$, where Λ is the constant of the Ricci curvature; i.e.: $R_{ic}(u) = \Lambda u$, $\forall u \in TM$, and $n = \dim M$.

Proof:
Since (M, g) is an Einstein manifold with the real Λ such that $R_{ic}(u) = \Lambda u$,

$\forall u \in TM$, we obtain

$$\begin{cases} \tilde{\tau}^h(u) = -\Lambda[f_2^C(t) + tf_3^C(t)]u + [(f_4^C + f_5^C + nf_6^C)(t) + tf_7^C(t)]u, \\ \\ \tilde{\tau}^v(u) = -\Lambda[f_2^D(t) + tf_3^D(t)]u + [(f_4^D + f_5^D + nf_6^D)(t) + tf_7^D(t)]u; \end{cases}$$

$\forall u \in TM$ with $t = g(u, u)$.

This completes the proof. □

In particular for Ricci flat manifolds, we get:

Corollary 4.6. *Let $(M,\ g)$ be a Ricci flat manifold. Then the Sasaki metric g^s induced by g is an harmonic metric with respect to the horizontal lift g^h of g on TM.*

Proof:
For the Sasaki metric induced by g, we have $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, $\beta_1 = \beta_2 = \beta_3 = 0$ then $(f_4^C + f_5^C + nf_6^C)(t) + tf_7^C(t) = 0$ and $(f_4^D + f_5^D + nf_6^D)(t) + tf_7^D(t) = 0\ \forall\ t \in \mathbb{R}_+$. Futhermore $\Lambda = 0$.

This completes the proof.

□

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