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RELATION BETWEEN SIDES OF A TRIANGLE WHEN $\alpha = r\beta$

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Abstract. Let ABC be a triangle with sides a, b and c, and angles α, β and γ, satisfying $\alpha = r\beta$, $r \in \mathbb{R}^+$. We show that the relation between the sides of ΔABC is given by the continued fraction of r and successive transformations $(a, b, c) \to (a \star b, bc, ac)$, where the operator $\star$ will be specified later on. We also apply these results to commensurable triangles.

1. Introduction

It is well known that if in a ΔABC the relation $\alpha = 2\beta$ is satisfied, then the relation between the sides of the triangle is given by [2]

$$a^2 = b(b + c).$$

A natural question is: what is the relation between the sides of the triangle when $\alpha = r\beta$, $r \in \mathbb{R}^+$? Richard Parris [1] states that it is clear that there exists a polynomial describing the relation between the sides of the triangle for $r \in \mathbb{Q}^+$, however he does not give a formalism. Indeed, he poses the question if there is an explicit formula for the aforementioned polynomial. In this article we seek to answer to these questions.

2. Preliminaries

Let's suppose without loss of generality that $r \geq 1$, and therefore $\alpha \geq \beta$ and $a \geq b$. Let D be a point on BC such that $\angle CAD = \angle ABC = \beta$ (see Figure 1).

We observe that ΔDAC is similar to ΔABC with ratio $AC : BC = b : a$, and therefore $CD = AC \cdot AC/BC = b^2/a$ and $AD = AB \cdot AC/BC = bc/a$. From this, $BD = BC - CD = a - b^2/a = (a^2 - b^2)/a$.

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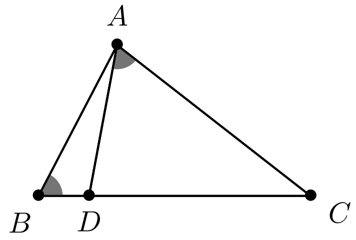

FIGURE 1. Let D be a point on BC such that $\angle CAD = \angle ABC = \beta$.

Let's introduce the operator $\star$ such that

$$a \star b = a^2 - b^2.$$

In sum, ΔABD has sides $BD = (a \star b)/a$, $AD = bc/a$ and $AB = c$. We apply a dilation of ratio $a : 1$ to show that

$$\Delta ABD \sim (a \star b, bc, ac).$$

The corresponding angles of ΔABD are $\angle DAB = \angle CAB - \angle CAD = \alpha - \beta$, $\angle ABD = \beta$ and $\angle BDA = \angle DCA + \angle CAD = \gamma + \beta$.

Let's define the transformation of sides of ΔABC by

$$\mathcal{S} : (a, b, c) \to (a \star b, bc, ac)$$

and the transformation of angles of ΔABC by

$$\mathcal{A} : (\alpha, \beta, \gamma) \to (\alpha - \beta, \beta, \gamma + \beta).$$

We say that $\mathcal{S}$ and $\mathcal{A}$ are similar because so are the resulting triangles.

3. CASE $r \in \mathbb{N}^+$

Let's treat the case $\alpha = n\beta$, with $n \in \mathbb{N}^+$. This means that $\gamma = \pi - (n+1)\beta$. If we apply n times the transformations $\mathcal{S}$ and $\mathcal{A}$ to the sides and angles of ΔABC, which we call $\mathcal{S}_n$ and $\mathcal{A}_n$, then the resulting triangle has sides $\mathcal{S}_n(a, b, c)$ and angles $\mathcal{A}_n(\alpha, \beta, \gamma)$. We observe that

$$\mathcal{A}_n(\alpha, \beta, \gamma) = (\alpha - n\beta, \beta, \gamma + n\beta).$$

But for $\alpha = n\beta$ and $\gamma = \pi - (n+1)\beta$, we get

$$\mathcal{A}_n(\alpha, \beta, \gamma) = (0, \beta, \pi - \beta).$$

The first angle of the resulting triangle is zero, and therefore, the first side of the resulting triangle must also be zero. In other words the first term of $\mathcal{S}_n(a, b, c)$ must be zero. Let's introduce the transformation $\mathcal{F}$ such that $\mathcal{F}(a, b, c) = a$, then the relation between the sides of the triangle when $\alpha = n\beta$, with $n \in \mathbb{N}^+$ is given by:

$$\mathcal{F}(\mathcal{S}_n(a, b, c)) = 0.$$

Example. For $n = 2$.

We apply the transformation $\mathcal{S}$ twice and we get

$$\mathcal{S}_2(a, b, c) = \mathcal{S}(a \star b, bc, ac) = (a \star b \star bc, abc^2, ac(a \star b))$$

And therefore, the relation between the sides of the triangle is given by

$$\mathcal{F}(\mathcal{S}_2(a,b,c)) = a \star b \star bc = 0.$$

This relation can readily be simplified to $a^2 = b(b+c)$.

Table 1 gives the relations between the sides of the triangle when $\alpha = n\beta$ for $n = 1,2,3,4,5$.

n	Expression $= 0$
1	$a \star b$
2	$a \star b \star bc$
3	$a \star b \star bc \star abc^2$
4	$a \star b \star bc \star abc^2 \star a^2bc^3(a \star b)$
5	$a \star b \star bc \star abc^2 \star a^2bc^3(a \star b) \star a^3bc^4(a \star b)^2(a \star b \star bc)$

Table 1. Relations between sides of a triangle for different values of n.

So far, we have used an identical reasoning to the one by Richard Parris to get the relations between the sides of a triangle for $r \in \mathbb{N}^+$, however we have introduced the operator $\star$ and the transformations $\mathcal{S}$ and $\mathcal{A}$, which simplify significantly obtaining the aforementioned relations. In the next sections we will use the concept of continued fraction, which is not used by Richard Parris either.

4. Case $r \in \mathbb{Q}^+$

Let's treat the case $\alpha = r\beta$, with $r \in \mathbb{Q}^+$. Let $[r]$ denote the integer part of r. It is readily seen that the transformation of the angles $\mathcal{A}$ can be performed at most $[r]$ times and that the resulting triangle has the following angles:

$$\mathcal{A}_{[r]}(\alpha,\beta,\gamma) = ((r-[r])\beta, \beta, \pi - (r-[r]+1)\beta).$$

Let now $r = [a_0, a_1, a_2, ..., a_N]$ be the continued fraction of r. This sequence is finite because $r \in \mathbb{Q}^+ \subset \mathbb{Q}$. By definition of the continued fraction, $[r] = a_0$. So, the resulting triangle has angles:

$$\mathcal{A}_{a_0}(\alpha,\beta,\gamma) = ((r-a_0)\beta, \beta, \pi - (r-a_0+1)\beta).$$

The relation between the second and first angle of this triangle is

$$\frac{\beta}{(r-a_0)\beta} = \frac{1}{r-a_0} = \frac{1}{[0, a_1, a_2, ..., a_N]} = [a_1, a_2, ..., a_N].$$

Let now introduce the transformation $\mathcal{P}$ which permutes the first term with the second one, i.e. $\mathcal{P}(a,b,c) = (b,a,c)$ (remark that a permutation of angles is similar to a permutation of sides). Now, the relation between the first and the second angle of the triangle of angles $\mathcal{P} \circ \mathcal{A}_{a_0}$ is $[a_1, a_2, ..., a_N]$. We can iterate and see that the relation between the first and the second angle of the triangle of angles $\mathcal{P} \circ \mathcal{A}_{a_1} \circ \mathcal{P} \circ \mathcal{A}_{a_0}$ is $[a_2, \ldots, a_N]$ and so on, until we get that the relation between the angles is zero. Formally, let $\mathcal{A}_r$ be defined by,

$$\mathcal{A}_r = \mathcal{A}_{a_N} \circ \mathcal{P} \circ \mathcal{A}_{a_{N-1}} \circ \cdots \circ \mathcal{P} \circ \mathcal{A}_{a_0}.$$

Then $\mathcal{F}(\mathcal{A}_r(\alpha,\beta,\gamma)) = 0$, and therefore the first side of the resulting triangle must be zero. The relation between the sides of the triangle is given by,

$$\mathcal{F}(\mathcal{S}_r(a,b,c)) = 0,$$

where $\mathcal{S}_r$ is similarly constructed to $\mathcal{A}_r$.

Proof. Let $r_k = [a_0, ..., a_k]$, we will prove by induction that for $\alpha = r\beta = [a_0, ..., a_N]\beta$ and for $k \in \{1, ..., N-1\}$, then $\mathcal{A}_{r_k}(\alpha,\beta,\gamma) = (\alpha_{r_k}, \beta_{r_k}, \gamma_{r_k})$, where

$$\begin{aligned}\alpha_{r_k} &= \beta \prod_{i=1}^{k+1} ([a_i, ..., a_N])^{-1}\\ \beta_{r_k} &= \beta \prod_{i=1}^{k} ([a_i, ..., a_N])^{-1}.\end{aligned}$$

Let's prove for $k = 1$. We first have that $(\alpha,\beta,\gamma) = ([a_0, ..., a_N]\beta, \beta, \gamma)$. We apply a_0 times the transformation of angles $\mathcal{A}$ and we get:

$$\begin{aligned}\alpha_{r_0} &= [a_0, ..., a_N]\beta - a_0\beta\\ &= [0, a_1, ..., a_N]\beta\\ &= \frac{\beta}{[a_1, ..., a_N]}.\end{aligned}$$

And $\beta_{r_0} = \beta$. We apply once the permutation $\mathcal{P}$ and a_1 times the transformation of angles $\mathcal{A}$ and we get:

$$\begin{aligned}\alpha_{r_1} &= \beta - a_1 \frac{\beta}{[a_1, ..., a_N]}\\ &= \frac{[0, a_2, ..., a_N]}{[a_1, ..., a_N]}\beta\\ &= \frac{\beta}{[a_1, ..., a_N][a_2, ..., a_N]}\\ &= \beta \prod_{i=1}^{2} ([a_i, ..., a_N])^{-1}\end{aligned}$$

and

$$\begin{aligned}\beta_{r_1} &= \alpha_{r_0}\\ &= \frac{\beta}{[a_1, ..., a_N]}\\ &= \beta \prod_{i=1}^{1} ([a_i, ..., a_N])^{-1},\end{aligned}$$

which finishes the proof for $k = 1$.

We will assume true this result for $k \in \{1, ..., N-2\}$ and will prove that it is also true for $k+1$. Indeed, we apply the permutation $\mathcal{P}$ and a_{k+1} times the transformation $\mathcal{A}$ to $\mathcal{A}_{r_k}$ and we get $\mathcal{A}_{r_{k+1}}$, where

$$\begin{aligned}\alpha_{r_{k+1}} &= \beta\prod_{i=1}^{k}([a_i,...,a_N])^{-1} - a_{k+1}\beta\prod_{i=1}^{k+1}([a_i,...,a_N])^{-1} \\ &= \beta\prod_{i=1}^{k}([a_i,...,a_N])^{-1}\frac{[a_{k+1},...,a_N]-a_{k+1}}{[a_{k+1},...,a_N]} \\ &= \beta\prod_{i=1}^{k}([a_i,...,a_N])^{-1}\frac{1}{[a_{k+1},...,a_n][a_{k+2},...,a_N]} \\ &= \beta\prod_{i=1}^{k+2}([a_i,...,a_N])^{-1}.\end{aligned}$$

The demonstration for $\beta_{r_{k+1}}$ is straightforward:

$$\begin{aligned}\beta_{r_{k+1}} &= \alpha_{r_k} \\ &= \beta\prod_{i=1}^{k+1}([a_i,...,a_N])^{-1}.\end{aligned}$$

Using the previous reasoning to $k = N-1$, we apply the transformations $\mathcal{P}$ and $\mathcal{A}_{a_N}$ to the angles $\mathcal{A}_{r_{N-1}}$ and the fact that for $k = N-1$, $[a_{k+1},...,a_N]-a_{k+1} = 0$, to get that $\mathcal{F}(\mathcal{A}_r) = 0$. □

Example. For $r = [1,2] = 1 + 1/2 = 3/2$.
We apply once ($a_0 = 1$) the transformation $\mathcal{S}$ and the transformation $\mathcal{P}$

$$(\mathcal{P}\circ\mathcal{S}_1)(a,b,c) = \mathcal{P}(a\star b, bc, ac) = (bc, a\star b, ac).$$

Then, we apply $a_1 = 2$ more times the transformation $\mathcal{S}$

$$(\mathcal{S}_2\circ\mathcal{P}\circ\mathcal{S}_1)(a,b,c) = (bc\star(a\star b)\star ac(a\star b), ..., ...).$$

And therefore, the relation between the sides of the triangle when $\alpha = 3\beta/2$ is given by

$$\mathcal{F}(\mathcal{S}_{3/2})(a,b,c) = \mathcal{F}(\mathcal{S}_2\circ\mathcal{P}\circ\mathcal{S}_1)(a,b,c) = a\star b\star bc\star ac(a\star b) = 0.$$

Let's explain this example in order to understand the foundation of the reasoning of this section. If in a ΔABC $\alpha = 3\beta/2$, then the transformation of the angles $\mathcal{A}$ can be performed only once ($a_0 = [3/2] =1$). The resulting triangle has angles $\mathcal{A}(3\beta/2,\beta,\gamma) = (\beta/2,\beta,\gamma+\beta)$ and sides $\mathcal{S}(a,b,c) = (a\star b, bc, ac)$. If we permute the first side (and angle) with the second one, then the resulting triangle has angles $\alpha' = \beta, \beta' = \beta/2$ and $\gamma' = \gamma+\beta$, and sides $(bc, a\star b, ac)$. In this triangle the relation $\alpha' = 2\beta'$ holds, and then we can apply the relation between the sides of a triangle for $r = a_1 = 2$ and sides $(bc, a\star b, ac)$, or in other terms, $\mathcal{F}(\mathcal{S}_2)(bc, a\star b, ac) = 0$.

Table 2 gives relations between sides of a triangle for different r.

CF(r)	r	Expression = 0
$[1,2]$	$3/2$	$a \star b \star bc \star ac(a \star b)$
$[1,1,2]$	$5/3$	$a \star b \star bc \star ac(a \star b) \star abc^2(a \star b \star bc)$
$[1,1,3]$	$7/4$	$a \star b \star bc \star ac(a \star b) \star abc^2(a \star b \star bc) \star a^3b^2c^5(a \star b)(a \star b \star bc)$
$[1,3]$	$4/3$	$a \star b \star bc \star ac(a \star b) \star a^2bc^3(a \star b)$
$[1,4]$	$5/4$	$a \star b \star bc \star ac(a \star b) \star a^2bc^3(a \star b) \star a^3b^2c^5(a \star b)(a \star b \star bc)$
$[2,2]$	$5/2$	$a \star b \star bc \star abc^2 \star ac(a \star b)(a \star b \star bc)$
$[2,3]$	$7/3$	$a \star b \star bc \star abc^2 \star ac(a \star b)(a \star b \star bc) \star a^3bc^4(a \star b)^2(a \star b \star bc)$
$[2,1,2]$	$8/3$	$a \star b \star bc \star abc^2 \star ac(a \star b)(a \star b \star bc) \star a^2bc^3(a \star b)(a \star b \star bc \star abc^2)$
$[3,2]$	$7/2$	$a \star b \star bc \star abc^2 \star a^2bc^3(a \star b) \star ac(a \star b)(a \star b \star bc)(a \star b \star bc \star abc^2)$

Table 2. Relations between sides of a triangle for different values of r.

5. Case $r \in \mathbb{I}^+$

Let's treat the case $\alpha = r\beta$, with $r \in \mathbb{I}^+$. It is clear by construction that the relation $\mathcal{F}(\mathcal{A}_r(\alpha,\beta,\gamma)) = 0$ for $r \in \mathbb{Q}^+$, also holds for $r \in \mathbb{I}^+$. Indeed,

$$\mathcal{F}(\mathcal{A}_r(\alpha,\beta,\gamma)) = \beta \prod_{i=1}^{\infty} ([a_i, ...])^{-1},$$

which tends to zero. Therefore $\mathcal{F}(\mathcal{S}_r(a,b,c)) = 0$ for $r \in \mathbb{I}^+$. However, the continued fraction of r is infinite and so is $\mathcal{F}(\mathcal{S}_r(a,b,c))$. To what does it converge?

6. Commensurable triangles

Now we will focus on commensurable triangles with one angle equal to an arbitrary rational number times another angle. We will use the transformation $\mathcal{S}$ in order to get similar results found in [1].

Let's consider an isosceles triangle with sides (m,n,n) with $m,n \in \mathbb{N}^+$ and angles (α,β,β). The triangle resulting from the transformation $\mathcal{S}$ has angles $\mathcal{A}(\alpha,\beta,\beta) = (\alpha-\beta,\beta,2\beta)$ and therefore it is a triangle where one of its angles is twice another. The sides of this triangle are $\mathcal{S}(m,n,n) = (m \star n, n^2, mn)$. We can generalise and see that the triangle of sides $\mathcal{S}^*_r(m,n,n)$ has an angle r times another. The operator $\mathcal{S}^*_r$ is similarly constructed to $\mathcal{S}_r$, but instead of the permutation $\mathcal{P}$, we use the permutation $\mathcal{P}^* : (a,b,c) \to (a,c,b)$, and we make the operation the other way around. In other terms,

$$\mathcal{S}^*_r = \mathcal{S}_{a_0} \circ \mathcal{P}^* \circ \mathcal{S}_{a_1} \circ \cdots \circ \mathcal{P}^* \circ \mathcal{S}_{a_N-1}$$

and similar for $\mathcal{A}^*_r$. The values of m and n have to be such that the transformation $\mathcal{A}^*_r$ is well defined, i.e. $\gamma = \pi - (r+1)\beta > 0$, which implies $0 < \beta < \pi/(r+1)$, but $\cos(\beta) = (m/2)/n$, and therefore

$$\cos\left(\frac{\pi}{r+1}\right) < \frac{m}{2n} < 1.$$

Proof. The proof for $r \in \mathbb{N}^+$ is straightforward. Let $r_k = [a_{N-k}, ..., a_N]$, we will prove by induction that for $k \in \{1, ..., N\}$, then $\mathcal{A}^*_{r_k}(\alpha,\beta,\beta) = (\alpha^*_{r_k}, \beta^*_{r_k}, \gamma^*_{r_k})$, where

$$\beta^*_{r_k} = \beta \prod_{i=N-k+1}^{N} [a_i, ..., a_N]$$

$$\gamma^*_{r_k} = \beta \prod_{i=N-k}^{N} [a_i, ..., a_N].$$

Let's prove for $k = 1$. We first have that the angles of the isosceles triangle are (α, β, β) and we apply $a_N - 1$ times the transformation of angles $\mathcal{A}$ and we get:

$$\begin{aligned}\gamma_{r_0} &= \beta + (a_N - 1)\beta \\ &= a_N \beta\end{aligned}$$

and $\beta_{r_0} = \beta$. We apply once the permutation $\mathcal{P}$ and a_{N-1} times the transformation of angles $\mathcal{A}$ and we get:

$$\begin{aligned}\gamma_{r_1} &= \beta + a_{N-1}a_N\beta \\ &= [a_{N-1}, a_N]a_N\beta \\ &= \beta \prod_{i=N-1}^{N} [a_i, ..., a_N]\end{aligned}$$

and

$$\begin{aligned}\beta_{r_1} &= \gamma_{r_0} \\ &= a_N\beta \\ &= \beta \prod_{i=N}^{N} [a_i, ..., a_N],\end{aligned}$$

which finishes the proof for $k = 1$.

We will assume true this result for k and will prove that it is also true for $k + 1$. Indeed, we apply the permutation $\mathcal{P}^*$ and a_{N-k-1} times the transformation $\mathcal{A}$ and we get $\mathcal{A}^*_{r_{k+1}}(\alpha, \beta, \beta) = (\alpha^*_{r_{k+1}}, \beta^*_{r_{k+1}}, \gamma^*_{r_{k+1}})$, where

$$\begin{aligned}\gamma^*_{r_{k+1}} &= \beta \prod_{i=N-k+1}^{N} [a_i, ..., a_N] + a_{N-k-1}\beta \prod_{i=N-k}^{N} [a_i, ..., a_N] \\ &= \beta \prod_{i=N-k}^{N} [a_i, ..., a_N]\left(a_{N-k-1} + \frac{1}{[a_{N-k}, ..., a_N]}\right) \\ &= \beta \prod_{i=N-k}^{N} [a_i, ..., a_N][a_{N-k-1}, ..., a_N] \\ &= \beta \prod_{i=N-k-1}^{N} [a_i, ..., a_N].\end{aligned}$$

The demonstration for $\beta^*_{r_{k+1}}$ is straightforward:

$$\begin{aligned}\beta^*_{r_{k+1}} &= \gamma^*_{r_k} \\ &= \beta \prod_{i=N-k}^{N} [a_i, ..., a_N].\end{aligned}$$

It is readily seen that for $k = N$ the ratio between the third and the second angles of $\mathcal{A}_r^*(\alpha, \beta, \beta)$ is

$$\gamma^*_{r_N} : \beta^*_{r_N} = \beta \prod_{i=0}^{N} [a_i, ..., a_N] : \beta \prod_{i=1}^{N} [a_i, ..., a_N]$$

which is precisely $r = [a_0, ..., a_N]$. □

Table 3 gives the sides of a triangle for which $\gamma = r\beta$ for $r = 1, 2, 3, 4$ (for simplified and explicit expressions refer to [1]).

r	a	b	c
1	m	n	n
2	$m \star n$	n^2	mn
3	$m \star n \star n^2$	mn^3	$mn(m \star n)$
4	$m \star n \star n^2 \star mn^3$	$m^2n^4(m \star n)$	$mn(m \star n)(m \star n \star n^2)$

TABLE 3. Commensurable triangles such that $\gamma = r\beta$ for $r = 1, 2, 3, 4$.

Example. Let's make an example for $r = [1, 3] = 4/3$.

We start with an isosceles triangle of sides (m, n, n) and angles (α, β, β). We apply $a_1 - 1 = 2$ times the transformation $\mathcal{S}$ and once the permutation $\mathcal{P}^*$. The resulting triangle has angles $(\mathcal{P}^* \circ \mathcal{A}_2)(\alpha, \beta, \beta) = (\alpha - 2\beta, 3\beta, \beta)$ and sides $(\mathcal{P}^* \circ \mathcal{S}_2)(m, n, n) = (m \star n \star n^2, mn(m \star n), mn^3)$.

We apply once again $(a_0 = 1)$ the transformation $\mathcal{S}$, and we get a triangle with angles $\mathcal{A}(\alpha - 2\beta, 3\beta, \beta) = (\alpha - 5\beta, 3\beta, 4\beta)$ and therefore the ratio between its angles is $\gamma' : \beta' = 4 : 3$. The sides of the resulting triangle are $\mathcal{S}(m \star n \star n^2, mn(m \star n), mn^3)$, i.e.

$$(m \star n \star n^2 \star mn(m \star n), m^2n^4(m \star n), mn^3(m \star n \star n^2)).$$

Table 4 gives the sides of a triangle for which $\gamma = r\beta$ for different $r \in \mathbb{Q}^+$.

CF(r)	r	a	b	c
$[1, 2]$	$3/2$	$m \star n \star mn$	mn^3	$n^2(m \star n)$
$[1, 3]$	$4/3$	$m \star n \star n^2 \star mn(m \star n)$	$m^2n^4(m \star n)$	$mn^3(m \star n \star n^2)$
$[1, 1, 2]$	$5/3$	$m \star n \star mn \star n^2(m \star n)$	$mn^5(m \star n)$	$mn^3(m \star n \star mn)$
$[2, 2]$	$5/2$	$m \star n \star mn \star mn^3$	$mn^5(m \star n)$	$n^2(m \star n)(m \star n \star mn)$

TABLE 4. Commensurable triangles such that $\gamma = r\beta$ for different values of r.

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