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RABINOWITZ CIRCLES

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Abstract In a recent paper Rabinowitz Conics assosiated with points in a triangle have been defined. We show that two of these Rabinowitz Conics are in fact circles. These are associated with the Bickart points of a triangle. The two circles are congruent and their centers are foci of the Steiner inellipse.

1. Introduction

Let P be a point in the plane of $\triangle ABC$. From each vertex draw three congruent segments, one to P, the other ones in the same direction as the other segments connecting P with a vertex. We get six new points, which lie on a conic, as proven in a recent paper [2] by *Rabinowitz et al.* See figure 1. One of the authors, *Ercole Suppa*, coined the name *Rabinowitz* $P-$*conic*, a naming we will follow. In this paper we will show that if P is one of the Bickart points (foci of the Steiner circumellipse) of $\triangle ABC$, the P-Rabinowitz conic is a circle. It is centered at the complement of P, one of the foci of the Steiner inellipse. The two Rabinowitz circles are congruent.

2. Rabinowitz Conics

Let P be a point in the plane of $\triangle ABC$. Let A_B be the point such that $\overrightarrow{AA_B}$ and $\overrightarrow{BP}$ have the same direction, while $AP = AA_B$. Similarly define A_C, B_A, B_C, C_A, and C_B. *Rabinowitz et al.* [2] have proven the following theorem.

Theorem 2.1. *The points A_B, A_C, B_A, B_C, C_A, and C_B lie on a conic.*

Proof. Notice that $\triangle PA_BA_C$, $\triangle B_APB_C$, and $\triangle C_AC_BP$ are homothetic, so that $B_CC_B \parallel A_BA_C$, $A_CC_A \parallel B_CB_A$, and $A_BB_A \parallel C_AC_B$. Hence Pascal's hexagon theorem can be applied. $\square$

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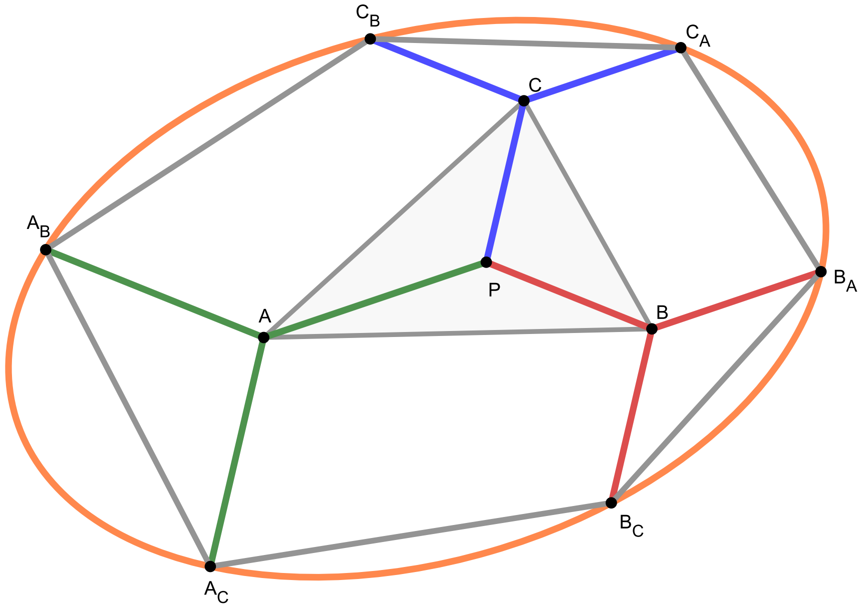

FIGURE 1. The P-Rabinowitz conic

In an open question in [2] it was wondered if there exist Rabinowitz $P-$conics that are circles. As an example the authors noted that in the particular case of an equilateral triangle with P as its center, indeed the Rabinowitz conic is a circle. In private communication *Stanley Rabinowitz* conjectured, based on numerical calculation in a 6-9-13 triangle, that in scalene triangles there exist two points yielding a circle and that the resulting circles are congruent. We will show that this conjecture is correct. If P is one of the Bickart points (foci of the Steiner circumellipse), the Rabinowitz $P-$conic is indeed a *Rabinowitz circle* and these two circles are congruent.

3. Rabinowitz Circles

The foci of the Steiner inellipse are labelled P_{118} and U_{118} in *Clark Kimberling*'s Bicentric Pairs of Points [1], while the Bickart points are P_{116} and U_{116}. Labelling is so that P_{118} is the complement of U_{116} and U_{118} is the complement of P_{116}. We will use these labels.

Theorem 3.1. *The Rabinowitz U_{116}-conic is a circle with center P_{118}. The Rabinowitz P_{116}-conic is a circle with center U_{118}.*

Proof. We consider the case of the Rabinowitz U_{116}-conic, the Rabinowitz P_{116}-conic is treated likewise.

To avoid naming problems later in this paper, we will rename the points from theorem 2.1 for the Rabinowitz U_{116}-conic as $A_1 = A_B$, $A_2 = A_C$, $B_1 = B_C$, $B_2 = B_A$, $C_1 = C_A$, and $C_2 = C_B$. Let $A'B'C'$ be the medial triangle. See figure 2. For the Rabinowitz P_{116}-conic we will use subscripts 3 and 4 in stead of 1 and 2 respectively. See figure 4.

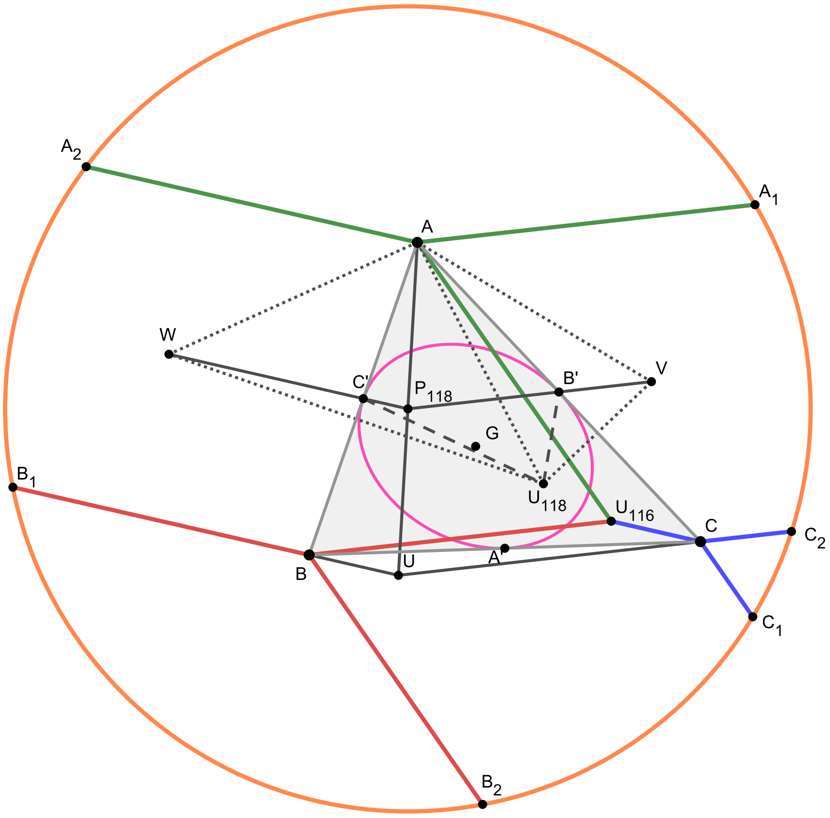

FIGURE 2. Rabinowitz U_{116}-conic is a circle with center P_{118}

First note that if we reflect U_{118} through AC and AB to V and W respectively, then

$$P_{118}V = P_{118}B' + B'U_{118} = P_{118}C' + C'U_{118} = P_{118}W.$$

So V and W are equidistant to P_{118}. Clearly the reflections let also

$$AV = AU_{118} = AW,$$

so V and W are equidistant to A as well. This makes $AVP_{118}W$ a kite and thus AP_{118} bisects $\angle VP_{118}W$. By parallelism this means that AP_{118} bisects $\angle A_1AA_2$. From this

$$\triangle AP_{118}A_1 \cong \triangle AP_{118}A_2$$

and hence

$$P_{118}A_1 = P_{118}A_2.$$

In the same way: $P_{118}B_1 = P_{118}B_2$ and $P_{118}C_1 = P_{118}C_2$.

Now consider U, the reflection of U_{116} through A'. We see that $BUCU_{116}$ is a parallelogram. We can see that

$$UB_1 = UC_2 = BU_{116} + CU_{116}.$$

Also

$$BU \parallel CU_{116}$$

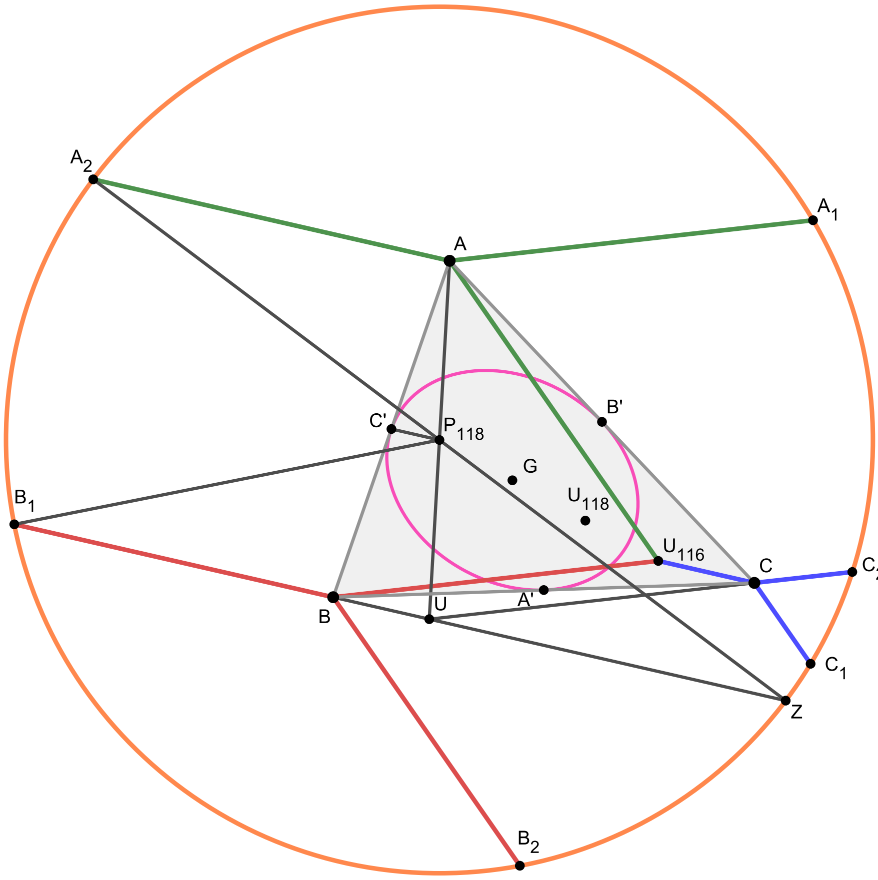

FIGURE 3. $P_{118}C'$ bisects $\angle A_2P_{118}B_1$

and

$$BU = CU_{116}.$$

As $C'P_{118}$ is the complement of CU_{116}, we have

$$C'P_{118} \parallel CU_{116}$$

and

$$C'P_{118} = \frac{1}{2}CU_{116}.$$

The parallels combine to

$$BU \parallel C'P_{118}$$

and the equalities combine to

$$BU = 2 \cdot C'P_{118}.$$

So $C'P_{118}$ is midparallel of $\triangle ABU$ and U is the reflection of A through P_{118}. This means that the line $AU = AP_{118}$ bisects $\angle B_1UC_2$ as well. And thus

$$\triangle P_{118}B_1U \cong \triangle P_{118}C_2U$$

and hence

$$P_{118}B_1 = P_{118}C_2.$$

In the same way $P_{118}C_1 = P_{118}A_2$ and $P_{118}A_1 = P_{118}B_2$.

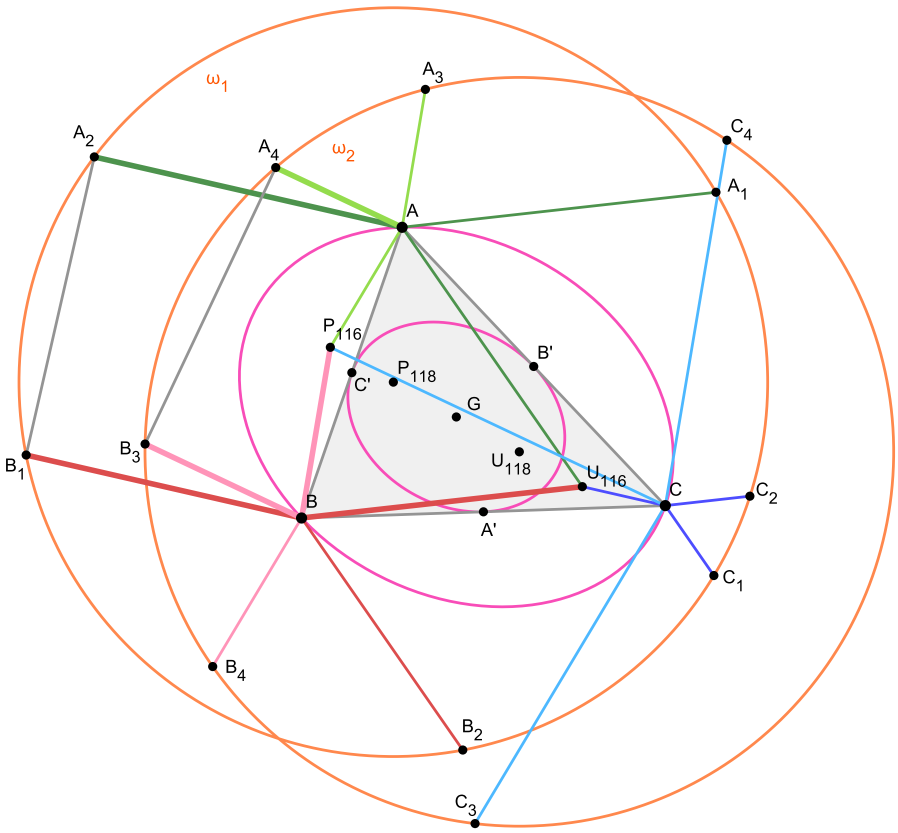

FIGURE 4. $B_1A_2 = B_3A_4$

We conclude that $A_1A_2B_1B_2C_1C_2$ is circumscribed by a circle with center P_{118} as desired. □

We will write ω_1 for the Rabinowitz U_{116}-circle and ω_2 for the Rabinowitz P_{116}-circle.

Remark 3.1. *Note that the line AU contains a diameter of ω_1, and bisects both $\angle A_1AA_2$ and $\angle B_1UC_2$. Hence A_1C_2 and B_1A_2 are congruent as they are symmetric about AU. Similarly C_1B_2 is congruent to these as well. And of course a similar statement can be made for ω_2.*

Remark 3.2. *Note that if we reflect A_2 through P_{118} to Z, that Z will lie on line C_2U. This shows that $\angle A_2P_{118}B_1 = 2\cdot\angle A_2ZB_1 = 2\cdot\angle A_2P_{118}C'$. So $P_{118}C'$ bisects $\angle A_2P_{118}B_1$. See figure 3. Similarly $P_{118}A'$ bisects $\angle B_2P_{118}A_1$ and $P_{118}B'$ bisects $\angle C_2P_{118}A_1$.*

Theorem 3.2. *Rabinowitz circles ω_1 and ω_2 are congruent.*

Proof. Considering figure 4 we see that that

$$B_1B + BB_3 = U_{116}B + BP_{116} = U_{116}A + AP_{116} = A_2A + AA_4.$$

Remark that the tangent at C to the Steiner circumellipse is parallel to AB, hence according to the reflection property of ellipse foci $\angle U_{116}CP_{116}$ is bisected by the $C-$altitude of $\triangle ABC$. By parallelism $\angle B_1BB_3$ and $\angle A_2AA_4$

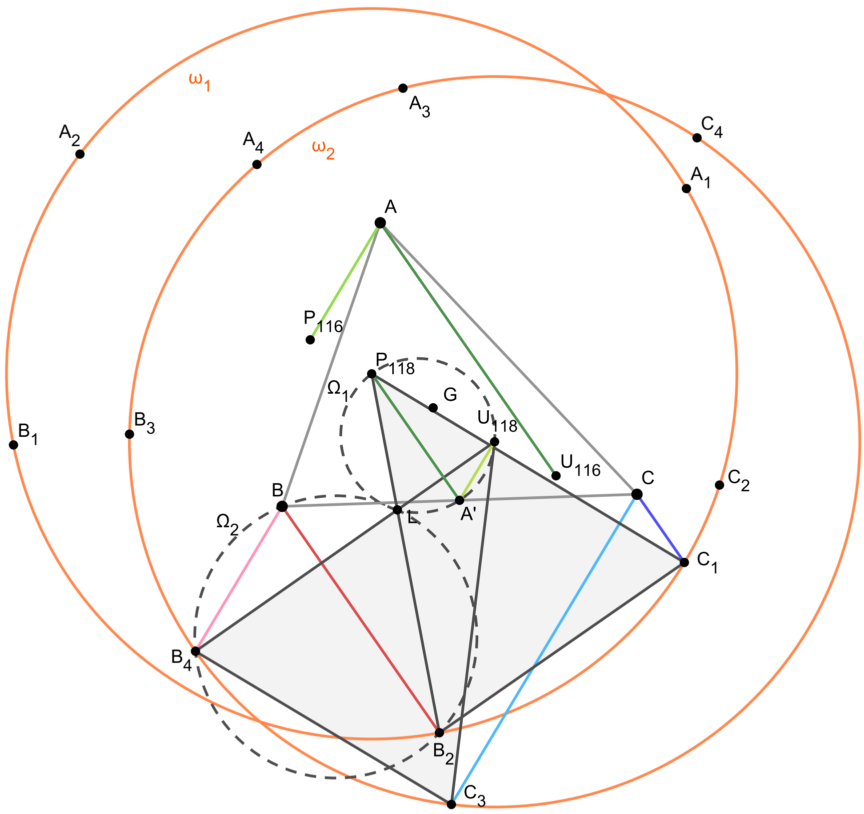

FIGURE 5. $\triangle B_4C_3U_{118} \cong \triangle B_2C_1P_{118}$

are bisected by lines perpendicular to AB as well, and these angles are thus bisected externally by AB itself. From these remarks we see that if we reflect A_4 and B_3 through AB, the images together with B_1 and A_2 form a rectangle. So $B_3A_4 = B_1A_2$. Now this together with remark 3.1 shows that

$$A_1C_2 = B_1A_2 = C_1B_2 = A_3C_4 = B_3A_4 = C_3B_4.$$

Now consider $\Omega_1 = (P_{118}U_{118}A')$. Let $L = \Omega_1 \cap B_2P_{118}$. As $\angle U_{118}LP_{118} = \angle U_{118}A'P_{118} = \angle U_{116}AP_{116} = \angle B_4BB_2$ we see that $U_{118}L$ meets $\Omega_2 = (BLB_2)$ again in B_4. From this we see that the

$$\angle B_4U_{118}, B_2P_{118} = \angle P_{118}A, U_{118}A.$$

Similarly

$$\angle C_3U_{118}, C_1P_{118} = \angle P_{118}A, U_{118}A.$$

We see that $\triangle B_4C_3U_{118}$ and $\triangle B_2C_1P_{118}$ are isosceles triangles, with congruent base sides and corresponding legs that make equal angles to each other. We conclude that $\triangle B_4C_3U_{118} \cong \triangle B_2C_1P_{118}$ and hence that ω_1 and ω_2 are congruent, as the legs of $\triangle B_4C_3U_{118}$ and $\triangle B_2C_1P_{118}$ are their radii. □

We finish with noting that Ω_1, Ω_2 and $\Omega_3 = (C_1CC_3)$ concur in the point where the minor axis of the Steiner inellipse meets BC. We leave details to the reader.

References

[1] Kimberling, C.H., *Bicentric Pairs of Points*, `https://faculty.evansville.edu/ck6/encyclopedia/BicentricPairs.html`.

[2] Rabinowitz, S., Suppa, E., Altıntaş, A., and van Lamoen, F.M., *Rabinowitz Conics Associated with a Triangle*, International Journal of Computer Discovered Mathematics, **5(2020)** 1–12.

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