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A SURVEY ON CONICS IN HISTORICAL CONTEXT: an overview of definitions and their relationships.

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Abstract. Conics are undoubtedly one of the most studied objects in geometry. Throughout history different definitions have been given, depending on the context in which a conic is seen. Indeed, conics can be defined in many ways: as conic sections in three-dimensional space, as loci of points in the euclidean plane, as algebraic curves of the second degree, as geometrical configurations in (desarguesian or non-desarguesian) projective planes, … In this paper we give an overview of all these definitions and their interrelationships (without proofs), starting with conic sections in ancient Greece and ending with ovals in modern times.

1. Introduction

There is a vast amount of literature on conic sections, the majority of which is referring to the groundbreaking work of Apollonius. Some influential books were e.g. [49], [13], [51], [6], [55], [59] and [26]. This paper is an attempt to bring together the variety of existing definitions for the notion of a conic and to give a comparative overview of all known results concerning their connections, scattered in the literature. The proofs of the relationships can be found in the references and are not repeated in this article. In the first section we consider the oldest definition for a conic, due to Apollonius, as the section of a plane with a cone. Thereafter we look at conics as loci of points in the euclidean plane and as algebraic curves of the second degree going back to Descartes and some of his contemporaries. Next we enlighten the connection between conic sections and conics as discovered by Dandelin. Then we focus on two important projective definitions for conics in pappian planes, attributed to Steiner and Von Staudt. Closely connected with the notion of a conic is that of an oval. We formulate the famous theorem of Segre on ovals in finite planes over Galois fields of odd order and we also mention a nice characterization theorem due to Buekenhout which has its origin in a theorem of Pascal concerning inscribed hexagons in a conic.

Keywords and phrases: Conic section, Conic, Oval, Euclidean plane, Projective plane, History of Mathematics

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Both the definitions of Steiner and Von Staudt can be generalized to non-pappian projective planes. This topic is discussed in another section with several interesting results obtained by Strambach and with an algebraic description of conics in non-desarguesian planes due to Krüger. Finally we look at conics and ovals in topological planes where a theorem of Buchanan on ovals in the complex plane surprisingly interfaces with Segre's theorem for ovals in finite desarguesian planes.

2. Apollonius' definition: conics as conic sections

Conics or conic sections are geometrical objects with a long history, dating back to the ancient Greeks. It is generally accepted that conics appeared for the first time in the work of Menaichmos, who lived in the fourth century BC. He discovered the curves as a by–product of his search for a solution to the Delian problem of doubling the cube. One of the most prominent geometers of the Greek tradition was Apollonius from Perga ($\pm 262-190$ BC). He was born in the city of Bergama in Turkey, in ancient times known as Pergamon and an important center for hellenistic culture. Apollonius' *Conica*, a work consisting of eight books, seven of which are preserved, was of great importance and its influence on later work was enormous[2].

The *Conica* was the first systematic study of conics in the style of the famous *Elementa* of Euclides. The oldest preserved manuscript referring to books I to IV is "Vaticanus graecus 206" dating from the 12th century and stored in the Biblioteca Apostolica Vaticana. The content of books V to VII is only known from Arabic translations, e.g. the ones kept in the Bodleian Library of the University of Oxford under the name "MS Marsh 667".

Apollonius defined conics as *conic sections*, i.e. as the intersection curves of a (right or oblique) double–napped cone with a plane not passing through the top of the cone. The type of the conic section (ellipse, parabola or hyperbola) depends on the inclination angle of the plane. So the setting of Apollonius' definition of a plane conic is in fact a spatial one. One needs the three-dimensional euclidean space (see figure 1).

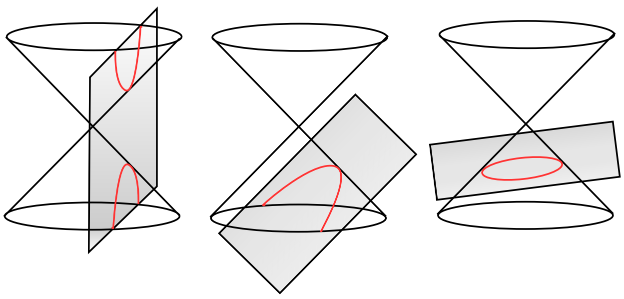

Figure 1. Conic sections by Apollonius: intersection curves of a (double) cone and a plane.

[2]In the Renaissance period there was a renewed interest for the ancient Greeks. Several translations in Latin of books I to IV were made, the first of which in 1537 by Giovanni Battista Memmo and another in 1548 by Francesco Maurolico. The most important translation of the 16th century is due to Federico Commandino. His *Apollonii Pergaei Conicorum libri quattuor* had a leading position till the 18th century.

3. Conics as loci in the euclidean plane and conics as algebraic curves

In the (real) euclidean plane conics are defined as loci of points. This definition can be found in any textbook on euclidean geometry. An ellipse is the locus of points for which the sum of the distances to two given points has a constant value. So it is the set of points P for which $d(P,F)+d(P,F')=2a$ with $2a>0$ a fixed constant. An hyperbola is the locus of points for which the absolute value of the difference of the distances to two given points is a constant: $|d(P,F)-d(P,F')|=2a$. The points F and F' are called "foci" and the line joining them, assuming $F\neq F'$, is the "major axis" (a circle is a special case of an ellipse for $F=F'$). The name "focus" has been introduced by Kepler. In *Ad Vitellionem Paralipomena, quibus Astronomiae pars Optica Traditur* (1604) he studies optical problems but in the fourth chapter general properties of conics are discussed and there the word focus appears for the first time. A parabola is the locus of points for which the distance to a given point F is equal to the distance to a given line ℓ (with F not on ℓ). The point F and the line ℓ are called "focus" and "directrix" respectively. A parabola can also be interpreted as the limiting case of a one–parameter family of ellipses $E\lambda(F,F',2a\lambda)$ with F a fixed point and F' going to infinity along the major axis while $2a\lambda-d(F,F')$ remains fixed. The three separate definitions are called the "*classical definitions of a conic as point loci* " (or the "*two foci definitions*" (for ellipse and hyperbola) (figure 2).

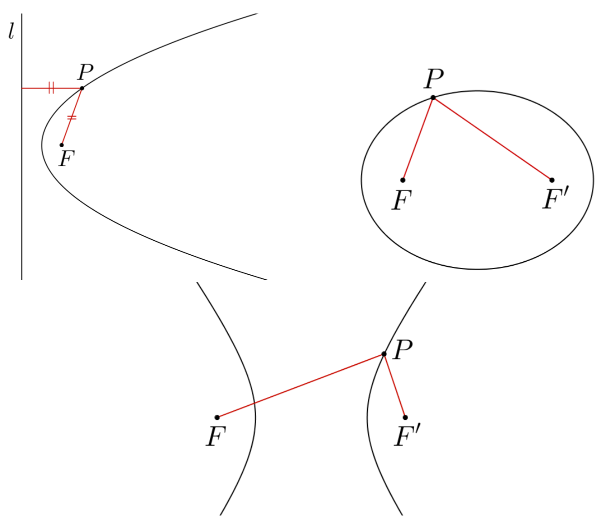

Figure 2. Classical definition of conics as point loci in the euclidean plane.

By a suitable choice of a coordinate system in the plane, a conic defined as point locus, can be represented algebraically by a standard equation of the second degree with real coëfficiënts. For a parabola the standard equation is $y^2 = 2px$ (with p the distance from F to ℓ). For the ellipse and the hyperbola the standard equations are $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a^2-c^2=b^2$ and

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $c^2 - a^2 = b^2$ respectively (where $2c$ is the distance between the foci F and F'). Conversely, any algebraic equation of one of these three forms, represents a conic. The proofs for this are elementary.

The interpretation of conics in the euclidean plane as *algebraic curves of the second degree* goes back to the pioneering work of PIERRE DE FERMAT (1607–1665): *Ad locos planos et solidos Isagoge* from 1629 and published posthumously in 1679. Also the "new method" of DESCARTES (1596–1650) (referring to the use of coordinates in geometry, now called "analytic geometry") which first appeared in *La Géométrie* has contributed to the study of conics by means of their algebraic equations. Descartes' method was further disseminated by the Dutch mathematician Frans VAN SCHOOTEN (1615–1660). In a second latin translation of Descartes' work under the title *Geometria a Renato Des Cartes* (1659–1661), an appendix written by Johan DE WITT (1625–1672) was added. In the appendix, *Elementa Curvarum Linearum, liber primus et liber secundus*, conics are first defined as loci of points and the geometric theory found in the books of Apollonius was developed, in the second part they are characterized by means of second degree equations in two indeterminates. A similar approach is found in *Tractatus De Sectionibus Conicis* (1655) by the British mathematician and contemporary John WALLIS (1616–1703).

A common definition for the three types of conics as loci of points is also possible if one introduces a positive constant ε, called excentricity. A conic is now defined as the locus of points P for which the distance $d(P, F)$ to a given point F equals $\varepsilon \cdot d(P, \ell)$ with ℓ a given line, not through P (for $\varepsilon = 1$ we recognize the definition of a parabola). This *focus-directrix definition* of a conic can be used as an alternative for the three separate classical definitions (two foci definitions) formulated above (figure 3).

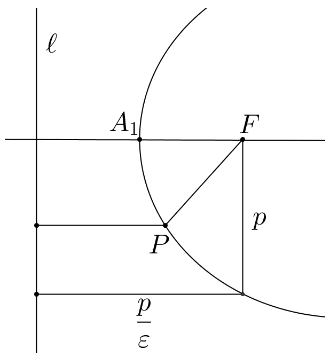

FIGURE 3. Focus–directrix definition of a conic.

If one chooses a coordinate sytem with the x–axis through F and perpendicular to ℓ and with the y–axis coinciding with ℓ, then an equation of the form $(1-\varepsilon^2)x^2 - \frac{2p}{\varepsilon}x + \frac{p^2}{\varepsilon^2} = 0$ appears. The parameter p in this formula satisfies $p = \varepsilon \cdot d(F, \ell)$. Applying a coordinate transformation $\begin{cases} x = x' + \frac{p}{\varepsilon(1+\varepsilon)} \\ y = y' \end{cases}$ finally reduces the equation to $y^2 = 2px - (1 - \varepsilon^2)x^2$ (the accents by the undeterminates are omitted). According to the value of the excentricity ε one can distinguish three types of conics. For $\varepsilon < 1$ we call them of elliptic type, for $\varepsilon = 1$ of parabolic type and for $\varepsilon > 1$ of hyperbolic type.

One can prove in an elementary way that a conic defined by the focus-directrix definition is also a conic classicaly defined as locus of points. Moreover a conic of parabolic, elliptic, hyperbolic type is indeed a parabola, ellipse, hyperbola in the sense of the classical definition and the converse is also true.

By the way we also mention that analogues of the different definitions given above still make sense in non-euclidean metric geometries (elliptic and hyperbolic planes), see e.g. [52], [54], [25], [20], [21] [18], [27], [35], [33] and [11]. We do not mention all results here, but there are several remarkable differences with the euclidean case. For example in hyperbolic planes at least six types of conics (or even more according to the used definition) can occur instead of three and the different definitions (as conic section, as point locus in the plane and as algebraic curve) are no longer equivalent!

4. Conics are conic sections: Hamilton, Dandelin and Morton

Apollonius' definition of a conic section in euclidean 3–space has at first glance nothing to do with the three classical definitions or with the equivalent focus–directrix definition of a conic as point loci in the plane. The first one who noticed a connection between both concepts was the Irish mathematician Hugh Hamilton (1729–1805), Bishop of Ossory, in *De Sectionibus Conicis: Tractatus Geometricus* (1758) translated into English as "A Geometrical Treatise of the Conic Sections" in 1773.

> "He was the first to deduce the properties of the conic section from the properties of the cone, by demonstrations which were general, unencumbered by lemmas, and proceeding in a more natural and perspicuous order" [58].

But usually reference is made to two Belgian mathematicians and friends with each other: Adolphe Quetelet (1796–1874) and Pierre Dandelin (1794–1847). Quetelet wrote his doctoral thesis on conics *De quibusdam locis geometricis nec non de curva focali* and Dandelin described a simple geometric construction for finding the foci of a conic section. If a conic section is defined as the intersection curve of a plane π with a cone $\mathcal{K}$ then the point of contact of the plane π with a sphere touching $\mathcal{K}$ and π at the same time and with center lying on the axis of the cone, is a focus of the conic section. As a consequence it follows that conic sections in the sense of Apollonius are the same objects as conics classicaly defined as loci of points in the plane (two foci definition). Dandelin proved this result in [16] in 1822 using what are now called "Dandelin spheres". This theorem is also known under the name "Théorème de Dandelin–Quetelet" or "Théorème belge sur la section conique".

> "Dans ces derniers temps, MM. Quetelet et Dandelin, en considerant les coniques dans le solide, sont parvenus à de forts beaux résultats nouveaux, dont le suivant offre, je crois, la première construction qu'on ait donnée des foyers des coniques dans le cône ..." [12]

Independently of Dandelin the British mathematician Pierce MORTON (1803–1859) described in *On the focus of a conic section* (1819) [36] a similar construction and in addition he also could interpret the directrix and the excentricity. More precisely he showed that the directrix is the secant of the plane π with the plane containing the tangent circle of the touching sphere and that the excentricity $\varepsilon = \frac{\cos\beta}{\cos\alpha}$ with α the half top angle of the cone and β the angle of the plane π with the axis of the cone. It was the first complete proof of the equivalence of Apollonius' spatial definition of conic section and the focus–directrix definition in the plane.

> "A complete determination of the foci and directrices of the sections of the cone by means of the focal spheres was at length proposed by Pierce Morton" [55]

5. Two projective definitions: Von Staudt and Steiner

The development of projective geometry with Girard DESARGUES (1591–1661) as one of its pioneers, shed a new light on conics. In 1639 Desargues wrote *Brouillon Project d'une atteinte aux événements des rencontres du cône avec un plan*, which can be translated as "draft for a study on the intersections of a cone with a plane". It contains a coherent theory of conics using properties which remain invariant under projection. In this work Desargues presents a theory of conic sections in a way that was fundamentally new in his time. For the first time "points and lines at infinity" were introduced, a radical innovation. The culmination point of projective geometry is reached in the 19th century with Jean-Victor PONCELET, Michel CHASLES, Jakob STEINER and Karl VON STAUDT.

The Swiss mathematician Jakob STEINER (1796–1863) was very important for synthetic geometry. In *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander* (1832) and in the posthumously published *Vorlesungen über synthetische Geometrie* (1867) he lays the foundation of projective geometry and sheds new light on conic sections. In the original formulation of the definition the underlying projective plane was the real projective plane, but this can be generalized without any problem to projective planes over any field $\mathbb{F}$.
His definition sounds as follows: let α be a projective but not perspective mapping between two pencils in a projective plane PG(2,$\mathbb{F}$) (a pencil is the set of all lines through a given point). Then the intersection points of corresponding lines form a (non–degenerate) *Steiner conic* [50], [51].
In this definition a projective mapping between pencils is used. One could also take a projective map between point rows, resulting into dual, tangential or line conics (instead of punctual or point conics). Tangential Steiner conics have been considered by Chasles [13] independently from Steiner.

Next we come to another definition for a conic in a pappian projective plane, which is due to German mathematician VON STAUDT (1798–1867). His definition which first appeared in *Geometrie der Lage* (1847) [56] is based on the concept of a polarity. It was originally only for the real projective plane, but it remains valid in pappian projective planes PG(2,$\mathbb{F}$) over an arbitrary field $\mathbb{F}$ with characteristic not equal 2.

A polarity π of a projective plane is an incidence preserving but not type preserving (points are mapped onto lines and lines onto points) automorphism of order two. If the image $\pi(P)$ of a point P under a polarity π is a line that goes through P, then P is called an absolute point of π. A polarity does not necessarily have absolute points. If π has no absolute points, we call it an elliptic polarity, in the other case we use the term hyperbolic polarity. If char $\mathbb{F} \neq 2$ then a hyperbolic polarity of PG(2,$\mathbb{F}$) has more than one absolute point. If $\mathbb{F}$ is infinite, there are an infinite number of absolute points. The action of a polarity of a pappian projective plane on the points and the lines can be descibed algebraically as follows: if the line $[u, v, w]$ is the image of the point (x, y, z), then

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \cdot \begin{bmatrix} x^\theta \\ y^\theta \\ z^\theta \end{bmatrix}$$

and the image of the line $[u, v, w]$ is the point (x, y, z) with

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^\theta & v^\theta & w^\theta \end{bmatrix} \cdot A^{-1}$$

in which θ is a field automorphism of $\mathbb{F}$, $\theta^2 = 1$ and A is a non–singular 3×3–matrix over $\mathbb{F}$ with $(A^T)^\theta = A$.
The polarities of PG(2,$\mathbb{F}$) can be classified in three types:
orthogonal polarities: if $\theta = 1$ and $A^T = A$ and char $\mathbb{F} \neq 2$
pseudo–polarities: if char $\mathbb{F} = 2$ and $\theta = 1$ and $A^T = A$ and not all of the diagonal elements of A are zero.
hermitian or *unitary* polarities: if $\theta \neq 1$, $\theta^2 = 1$ and $(A^T)^\theta = A$.

Now we are ready to give Von Staudt's definition of a conic: a *Von Staudt conic* in a pappian projective plane PG(2,$\mathbb{F}$) with $\mathbb{F}$ a field (char $\mathbb{F} \neq 2$), is the set of absolute points of a hyperbolic orthogonal polarity of the plane. By this definition a Von Staudt conic is never empty.

From the matrix representation of a polarity it is clear that any Von Staudt conic is also an irreducible algebraic conic, i.e.the set of points with homogeneous coordinates (x, y, z) which are a solution of the quadratic equation $F(x, y, z) = 0 \Leftrightarrow a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz = 0$ with not alle coefficients $a_{ij} \in \mathbb{F}$ equal to zero. If $F(x, y, z)$ can not be factorized in linear factors over $\mathbb{F}$ we say that the algebraic conic is irreducible. Conversely, any irreducible algebraic conic over a field with characteristic $\neq 2$ is equivalent with a Von Staudt conic.
Indeed, the conic $a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz = 0$ defines the matrix $A = \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} \end{bmatrix}$. Since A is symmetric and non–singular, it defines a hyperbolic orthogonal polarity and the set of absolute points (hence the Von Staudt conic) is the given algebraic conic.
Remark that in the original definition of Von Staudt the term polarity was not accompanied by the specification "orthogonal" because it was only for the real projective plane and orthogonal polarities are the only type that can occur in such a plane.

Any Von Staudt conic is also a Steiner conic. If one considers a variable point of a Von Staudt conic that is connected with two fixed points of that conic, one obtains a pair of lines which are corresponding lines in a projectivity between the pencils through the fixed points.
Also the converse theorem is valid (though not trivial): any Steiner conic in a projective plane over a field with characteristic $\neq 2$ is also a Von Staudt conic. In pappian planes (with char $\neq 2$) both definitions are thus equivalent (for a proof see e.g. [17], [15] or [19]).
Also note that Von Staudt's conics are self-dual objects (by the correspondence between absolute points and absolute lines).

In contrast to conics in the euclidean plane the distinction between the three types of conics is lost in a projective setting. All conics are projectively equivalent. So it is no longer possible to distinguish parabolas, ellipses and hyperbolas. If one considers the affine plane over the reals (which is in fact the real euclidean plane stripped of metric concepts) non–degenerate conics defined algebraically by a second degree equation $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$ can be classified by looking at the discriminant $\delta = C^2 - 4AB$. It determines the number of intersection points of the conic with the line at infinity. If $\delta = 0$ the conic is of parabolic type (one intersection point), while for $\delta < 0$ it is of elliptic type (no intersection points) and for $\delta > 0$ of hyperbolic type (two intersection points). This classification remains (partially) valid in the affine plane AG(2,$\mathbb{F}$) over an arbitrary field. If $\delta = 0$ the conic is of parabolic type, if $\delta \neq 0$ it is non–parabolic (the difference between elliptic and hyperbolic is not always possible). E.g. if $\mathbb{F}$ is the complex field, then a non–parabolic conic in AG(2,$\mathbb{C}$) has always two intersection points with the line at infinity. If $\mathbb{F}$ is a finite field GF(q), then there is for $\delta \neq 0$ a hyperbolic and an elliptic type according to δ is or is not a square in GF(q).

6. Conics and ovals in finite projective planes: Segre's theorem

The Italian mathematician Beniamino Segre (1903–1977) was one of the founders of finite geometry and of projective geometry over a finite field in particular (today also known as Galois geometry). One of the most famous theorems in that branch of geometry deals with conics and ovals.

A prominent property of conics is the fact that a line can intersect a (non–degenerate) conic in at most two points. This fact is the basis of the definition of an oval. An *oval* in a (finite or infinite) projective plane is a set $\mathcal{O}$ of points such that any line of the plane is incident with either $0, 1$ or 2 points of $\mathcal{O}$ and for each point of $\mathcal{O}$ there exists a unique line (called *tangent line*) which is incident with that point and with no other points of $\mathcal{O}$. In a finite projective plane of order q, an oval can equivalently be defined as a set of $q + 1$ points no three of which are collinear.

Any irreducible (non–empty) conic in a pappian projective plane PG(2,$\mathbb{F}$) with $\mathbb{F}$ an arbitrary field, is an oval. In particular any irreducible conic in PG(2,q) is an oval.
If $\mathbb{F}$ is infinite, it is not hard to find examples of ovals which are not conics,

e.g. in PG(2,$\mathbb{R}$) the point set $\mathcal{O} = \{(x, y, z) \,|\, y^4 = xz\}$ is an oval but not a conic.
In the finite case, the situation is more intrigating.

First let q be even. By a theorem of Qvist [39] all tangent lines of an oval intersect in a common point, the nucleus. Adding this point to the oval gives rise to an *hyperoval*. If one starts in particular with a conic $\mathcal{K}$, the hyperoval $\mathcal{H}$ which arises by adding the nucleus N is called a *regular hyperoval* or *hyperconic*. Starting from such a hyperconic $\mathcal{H} = \mathcal{K} \cup \{N\}$ and omitting a point P, distinct from N gives an oval which is also called a *pointed conic*. If $q = 2^h$ with $h > 2$ one obtains an oval which is not a conic, since it has $q \geq 5$ points in common with $\mathcal{K}$ (two distinct conics can not have more than four points in common). A classification of (hyper)ovals of PG(2,q), q even, is not known yet and it seems to be very difficult. For small values of q Segre has proved the following: if $q = 2, 4, 8$ all hyperovals are regular and if $q = 2$ or 4 they are conics; for $q = 8$ they are either conics or pointed conics.

Next, let us consider the case of q odd. In 1949 the Finnish mathematicians Järnefelt and Kustaanheimo formulated the conjecture that any oval in a finite projective plane PG(2,q) with q odd must be a conic [28]. In a review, Marshall Hall Jr. said that "The reviewer finds this conjecture implausible." But in 1955 the conjecture was proved by Beniamino Segre [47]. Hall was again a reviewer of the paper of Segre, where he then said "The fact that this conjecture seemed implausible to the reviewer seems to have been at least a partial incentive to the author to undertake this work. It would be very gratifying if further expressions of doubt were as fruitful." The remarkable theorem had indeed a great influence on the development of finite geometry. Two mention only two examples in which the statement plays a crucial role: in circle geometry any finite ovoidal Laguerre plane of odd order must be miquelian and in the theory of generalized quadrangles the quadrangle of Tits $T_2(\mathcal{O})$ is always classical for q odd. Both results follow from Segre's theorem.
The original proof of Segre is partially of algebraic and partially of geometric nature. The so-called lemma of tangents (any inscribed triangle of an oval lies in perspective with the triangle formed by the tangent lines in the points) forms a crucial element in the proof.

7. Pascalian ovals and a theorem of Buekenhout

Blaise Pascal (1623–1662) which is best known for his contribution to probability theory, also left an indelible mark in connection with conic sections. In 1640, at the age of sixteen, he writes *Essay pour les coniques*. This one page pamphlet contains a famous theorem named after him and which is also known as the "hexagrammum mysticum theorem": if six arbitrary points are chosen on a conic and joined by lines in any order to form a hexagon, then the three pairs of opposite sides of the hexagon meet at three points which lie on a straight line. Only two copies of this essay have been preserved, one in the Bibliothèque Nationale de Paris and the other as a part of a set of manuscripts by Leibniz in the Königlichen Öffentlichen Bibliothek of Hannover in Germany. The Essay can be viewed as an announcement for a comprehensive work on conics *Conicorum opus completum* which was written by Pascal after 1640 but was never published and the manuscript of which is lost.

It is intrigating that an age-old theorem turned out to be interesting in modern geometry. In his doctoral thesis entitled "*Étude intrinsèque des ovales*" Francis BUEKENHOUT studies among other things ovals in projective planes. He formulates the property of Pascal for an oval as follows. Let $\mathcal{O}$ be an oval in a projective plane (finite or infinite). An inscribed hexagon of $\mathcal{O}$ is a cyclic ordered set of six (not necessarily distinct) points $A_0, A_1, \ldots, A_5$ of $\mathcal{O}$ such that all lines A_iA_{i+1}, $i = 0, \ldots, 5$ are distinct and with the assumptions that A_i and A_{i+3} are distinct and that the line A_iA_{i+1} is a tangent line to $\mathcal{O}$ if $A_i = A_{i+1}$. Such hexagon has the property of Pascal if the three intersection points $A_iA_{i+1} \cap A_{i+3}A_{i+4}$ are collinear. An oval is called pascalian if for all inscribed hexagons the property of Pascal is valid.

By Pascal's theorem any conic in a pappian plane is of course also a pascalian oval. In [10] the following interesting characterization theorem is proved: if in a projective plane there exists a pascalian oval $\mathcal{O}$, then the plane must be pappian and $\mathcal{O}$ is a conic. The proof makes use of involutions and is based on a theorem of Tits about transitive permutation groups. A simpler and shorter proof, using coordinates, was given in [2] and in [41].

Buekenhout's theorem is a nice illustration of the fact that a rather simple geometric property (the existence of a pascalian oval in the plane) can have very strong consequences for that plane (it must be pappian). In Buekenhout's approach the theorem of Pascal is expressed entirely within the language of permutation groups and it was the starting point of further research about so-called *abstract ovals*. For a recent survey on that subject, see [5].

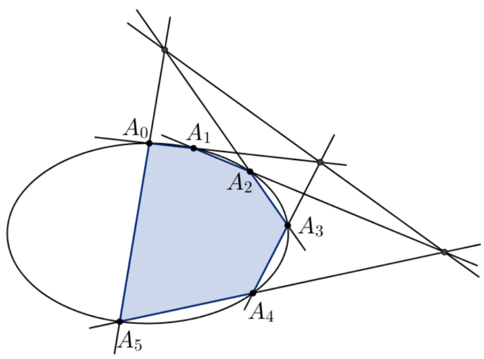

FIGURE 4. Oval with the Pascal property

8. IDIOSYNCRASY IN NON–CLASSICAL PROJECTIVE PLANES AND SOME MORE DEFINITIONS: KRÜGER CONICS AND OSTROM CONICS

Since the publication of the famous book *Grundlagen der Geometrie* by David HILBERT (1862–1943) the development of non–desarguesian geometry has grown steadily. It is well known that the properties valid in non–desarguesian planes can differ substantially from those in pappian planes over a field. Let us first recall some definitions. Given a projective plane $\mathcal{P}$, a point P and a line ℓ in $\mathcal{P}$, we say that $\mathcal{P}$ is (P, ℓ)–desarguesian if for each pair of triangles ABC and $A'B'C'$ which are perspective from P and for which two pairs of corresponding side lines intersect each other on ℓ, also the intersection point of the third pair of corresponding side lines is on ℓ. If $\mathcal{P}$ is (P, ℓ)–desarguesian for any choice of P and ℓ, we call $\mathcal{P}$ a desarguesian plane. Any desarguesian plane is isomorphic to a PG(2,$\mathbb{F}$) with $\mathbb{F}$ a skewfield or a field (in the second case the plane is called pappian and additionally the axiom of Pappus is valid). If the above formulated property (also called Desargues configuration) is not valid for at least one point–line pair, we call the plane non–desarguesian. A particular class of non–desarguesian planes are the Moufang planes. They are (P, ℓ)–desarguesian only for any incident point–line pair.

Both the definitions of Von Staudt and Steiner for a conic in a pappian plane can be generalized for desarguesian (but non–pappian) and for non-desarguesian planes. A very lucid paper in which a comparison between both definitions is discussed in detail by Strambach in [53]. A Von Staudt conic in an arbitrary projective plane is defined as the set of absolute points of a hyperbolic orthogonal polarity. Hyperbolic means that the set of absolute points is not empty. Since we have no longer a matrix representation for polarities in non–desarguesian planes, a different definition for "orthogonal" is required: a polarity is orthogonal if the set of absolute points forms an oval in the plane. By this definition Von Staudt conics are special ovals (with the points being absolute points of a polarity), we could say that conics are *polar ovals*.

A projective plane does not always possess polarities. In a finite projective plane any polarity has absolute points, but in an infinite non–desarguesian plane there may exist polarities with a finite number of absolute points or even without absolute points. As a consequence the existence of Von Staudt conics in non–desarguesian planes is not always garantueed. On the other side there are several examples known of special non–desarguesian planes with polar ovals, e.g. the Figueroa planes [14], [7], the Coulter-Matthews planes [30] and the Albert planes [1]. It has also been proved that a translation plane $\mathcal{P}$ admits a Von Staudt conic if and only if $\mathcal{P}$ can be coordinatized by a commutative semifield with char $\neq 2$, see [22] and [29]. The first example of a Von Staudt conic in a finite projective plane over a semifield is given in [57] while in [23] examples in finite and infinite planes are given.

Not any oval is a Von Staudt conic and in an infinite plane there are always examples of ovals that are not conics. Strambach has proved namely the following theorem: if in a projective plane each oval is a Von Staudt conic, i.e. each oval is a polar oval, then the plane must be finite [53]. In the Hughes–planes of order q^2 an example of an oval is constructed by extending a conic in the Baer subplane PG(2,q) of the Hughes plane. This oval defines a polarity but the set of absolute points does not coincide with the oval. Hence it is not a Von Staudt conic, see [43]. Ovals in special classes of projective planes were also studied in e.g. [31], [32] and [44].

The definition of Steiner also can be extended for non-pappian planes. A Steiner conic is the locus of intersection points of corresponding lines of a projective mapping α (but not a perspectivity) between two pencils (through the points S and T). In pappian planes $\alpha(ST) \neq ST$ because α is not a perspecitivity, but in non-pappian planes it may happen that $\alpha(ST) = ST$ even if α is not a perspectivity. The Steiner conics corresponding to an α for which $\alpha(ST) = ST$ are called degenerate (see e.g. [40]). We now restrict ourselves to the non–degenerate case.

The deviant behaviour of Steiner conics in non-pappian planes is immediately clear in desarguesian non-pappian planes PG(2,$\mathbb{K}$) with $\mathbb{K}$ a skewfield. It is known that any Steiner conic is such a plane can be seen, after a suitable coordinatization, as the pointset $\{(1, X, Y) \mid XY = 1\} \cup \{(0, 1, 0)\} \cup \{(0, 0, 1)\}$. If $P(1, x_0, y_0)$ is a point of the conic, then all lines through P, except for one, has exactly one other point in common with the conic if and only if x_0 belongs to the center of $\mathbb{K}$. Such a point P is called regular. For a non–regular point Q the number of intersection points of a line through Q with the conic depends on the number of solutions of an equation of the kind $xa - ax = b$. If one considers for example the plane PG(2,$\mathbb{H}$) over the skewfield of real quaternions $\mathbb{H} = \{a + bi + cj + dk \mid i^2 = j^2 = k^2 = ijk = -1\}$, then $Q(1, i, -i)$ is non–regular and the line $Y = -X$ through Q intersects the conic in at least three other points $(1, i, -i)$, $(1, j, -j)$ and $(1, k, -k)$, even in an infinite amount of points of the form $(1, q, -q)$ with $q = bi + cj + dk$ and $b^2 + c^2 + d^2 = 1$. This illustrates the strange fact that a conic in a desarguesian, non–pappian plane, is not always an oval. There are also examples known of Steiner conics which are not ovals (hence not Von Staudt conics) in finite non-desarguesian projective planes of order bigger than 4. After all these observations it is natural to ask under which conditions the class of Steiner conics and the class of Von Staudt conics in a plane are identical. This question was answered in [53] and partially in [4]. Both concepts coincide only if the plane is pappian over a field of characteristic distinct from two. Moreover it is true that a projective plane is pappian if any Steiner conic is an oval in the plane.

Segre's theorem (see section 6) can not further be generalized for non-desarguesian finite planes of odd order. Indeed, if the class of ovals coincides with the class of Steiner conics then the plane is the pappian plane PG(2,q) with q odd, see [53]. We will need this result in the next section.

For Steiner conics in desarguesian planes there always is a standard algebraic equation. In order to find an algebraic description for conics in non-desarguesian planes, Krüger introduced an alternative definition in [34]. Given four points A, B, C and D, no three of which are collinear, he defines a Krüger conic in an arbitrary projective plane as the locus of intersection points of corresponding lines for a projectivity α between the pencils with base points A and B which is the composition of the perspectivities $[A, BC, D]$ and $[D, AC, B]$ (see figure 5). In this we used the notation $[S, \ell, T]$ to indicate a perspectivity with axis ℓ from the pencil of lines through S to the pencil of lines through T.

Each Krüger conic is a Steiner conic, but not conversely. The two concepts are the same only if the plane is desarguesian (not necessarily pappian) [53]. Making a specific choice for the coordinatization of the plane, a Krüger conic can be seen as the point set $\{(x, y) \mid x \circ y = 1\} \cup \{(\infty), (0)\}$ with $x \circ y = T(x, y, 0)$ the multiplication associated with the coordinatising planar ternary ring (R, T). Another choice of the quadrangle for coordinatization leads to the equivalent description as the set $\{(x, y) \mid y = x^2\} \cup \{(\infty)\}$ with $x^2 = x \circ x$. Krüger investigated those conics especially in Moufang planes (coordinatized by an alternative division ring).
A Krüger conic is not always a Von Staudt conic. For example $\{(x, y) \mid y = x^2\} \cup \{(\infty)\}$ in a Moufang plane over the real octonions $\mathbb{O}$ is not an oval as there are lines in the plane that intersect the conic in infinitely many points [2].

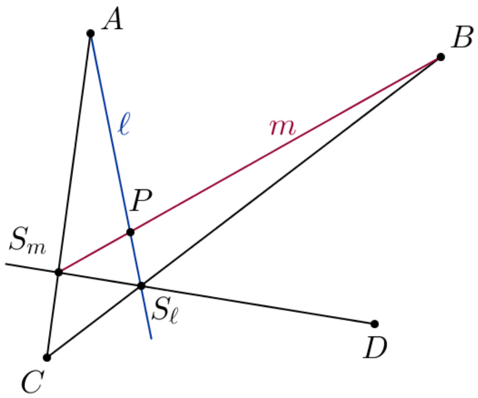

Figure 5. Definition of a Krüger conic

In the literature still another definition of conics in non–pappian projective planes was introduced by Ostrom [37], based on a generalization of harmonic sets. An Ostrom conic is a special kind of oval. Examples are known in infinite non–pappian planes, but it is not known whether there exist examples in finite non–desarguesian planes. Garner in [24] conjectures that a finite projective plane admitting an Ostrom conic must be pappian (and than the concept coincides with that of a Steiner conic and a Von Staudt conic). He also gives an explicit example of a Von Staudt conic which is not an Ostrom conic in a finite plane of order 27 over an Albert twisted semifield. In a pappian plane any conic is always an Ostrom conic. In [42] an example is given of an oval which is not a Von Staudt conic (hence not an Ostrom conic) in a projective plane of order 9 over a Dickson nearfield.

9. Ovals and conics in topological planes

When one connects geometry with topology a new field of research arises: "topological geometry". The study of topological projective planes as part of this, was initiated by Kolmogorov, but interest increased in particular since 1950 by publications of Salzmann [45], [46] and Skornyakov [48]. A topological projective plane is a projective plane in which both the point and the line set are endowed with a non-trivial topology such that "connecting distinct points" and "intersecting distinct lines" are continuous operations. A well–known important classification theorem states that the only compact, connected topological Moufang planes are the desarguesian topological planes PG(2,$\mathbb{K}$) with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and the non–desarguesian topological plane PG(2,$\mathbb{O}$) over the alternative octonion division ring. These are the so–called classical topological planes and they have topological dimension 2,4,8 and 16 respectively. The lines in the topological projective plane over the complex numbers are closed subsets homeomorphic to 2–spheres. Since any conic in the complex plane is an oval and since the stereographic projection from a point on a conic upon a line in the plane is an homeomorphism we see that conics in PG(2,$\mathbb{C}$) are also homeomorphic to 2–spheres. Hence they are closed ovals. It is remarkable that also the converse is true. Thomas Buchanan, an American-born mathematician who lived in Germany and a student of Strambach, proved that any closed oval in the topological projective plane over the complex numbers is a conic. The proof of this rather surprising theorem which can be found in [8] makes use of topological tools as well as of results from complex functions theory (e.g. the theorem of Casorati–Weierstrass on holomorphic functions).

Now let $\mathcal{O}$ be an oval in a topological projective plane and denote by $\mathcal{L}_{\mathcal{O}}$ the set of lines intersecting $\mathcal{O}$ in at least one point. The map

$$\mathcal{O} \times \mathcal{O} \to \mathcal{L}_{\mathcal{O}} : (P, Q) \mapsto \begin{cases} PQ & \text{if } P \neq Q \\ t_Q & \text{if } P = Q \end{cases}$$

induces a bijection $\psi_{\mathcal{O}} : \mathcal{O} * \mathcal{O} \to \mathcal{L}_{\mathcal{O}}$ between the (symmetrized) cartesian product $\mathcal{O} * \mathcal{O}$ and $\mathcal{L}_{\mathcal{O}}$ which is continuous in all points (P, Q) of $\mathcal{O} * \mathcal{O}$ met $P \neq Q$. We put $\varphi_{\mathcal{O}} = \psi_{\mathcal{O}}^{-1} : \mathcal{L}_{\mathcal{O}} \to \mathcal{O} * \mathcal{O} : \ell \mapsto \ell \cap \mathcal{O}$
An oval in a topological projective plane is by definition a topological oval if the map $\varphi_{\mathcal{O}}$ is continuous. A topological oval in a compact, connected projective plane is compact and hence also closed. Conversely it was proved that each closed oval in a compact connected projective plane of finite dimension is a topological oval. Do there exist topological ovals in any compact plane? The answer is negative. There do not exist topological ovals in compact projective planes with topological dimension larger than four [9].

In line with Buchanan's theorem that closed ovals (in particular topological ovals) in PG(2,$\mathbb{C}$) are conics, the question arises whether the complex plane is the only compact plane (with a non–discrete topology) in which topological ovals and conics are the same objects. To answer this question one needs some more topology and algebra (such as completeness, discrete valuation, non–archimedean local field, Hensel's lemma). With these auxiliary tools the following theorem was proved in [53]: in a compact projective plane $\mathcal{P}$ the class of topological ovals is the same as the class of Steiner conics if and only if $\mathcal{P}$ is the complex plane. In this theorem it is assumed tacitly that the topological plane is non–discrete (so the plane must be infinite). If one considers a finite projective plane endowed with the discrete topology, then one obtains a compact, totally disconnected projective plane.
The result mentioned in section 8 that ovals and Steiner conics are identical objects only in the finite pappian plane PG(2,q) with q odd can be reformulated in topological terms: in a finite compact projective plane of odd order

the class of topological ovals coincides with the class of Steiner conics if and only if the plane is a pappian plane over a Galois field of odd order. Combining this with the preceeding theorem yields another remarkable theorem which combines Segre's theorem and the theorem of Buchanan: in a compact projective plane $\mathcal{P}$ (possibly with a discrete topology in case the plane is finite) the class of topological ovals is the same as the class of Steiner conics if and only if $\mathcal{P}$ is either the complex plane PG(2,$\mathbb{C}$) or a finite projective plane PG(2,q) over the Galois field GF(q) with q odd.

10. Summary

Throughout history, there has been a wide variety of definitions regarding the term "conic section". In this paper we have provided an overview focusing on some milestones. The theorem of Dandelin unifies the spatial definition of Apollonius' conic sections and the focus–directrix definition of conics in the euclidean plane, the theorem of Segre and the theorem of Buekenhout puts ovals and conics in a common framework while the theorem of Buchanan adds a topological aspect. Other results under consideration deal with the projective definitions by Steiner and Von Staudt for pappian planes as well as their generalization to non–desarguesian projective planes. Extensions of the different definitions to non-euclidean metric geometries are only touched sideways.

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