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A SPECIAL CONIC ASSOCIATED WITH THE REULEAUX NEGATIVE PEDAL CURVE

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Abstract. The Negative Pedal Curve of the Reuleaux Triangle w.r. to a pedal point M located on its boundary consists of two elliptic arcs and a point P_0. Intriguingly, the conic passing through the four arc endpoints and by P_0 has one focus at M. We provide a synthetic proof for this fact using Poncelet's porism, polar duality and inversive techniques. Additional interesting properties of the Reuleaux negative pedal w.r. to pedal point M are also included.

1. Introduction

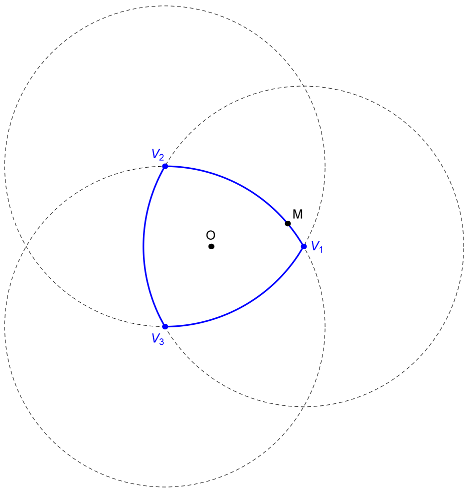

Figure 1. The sides of the Reuleaux Triangle $\mathcal{R}$ are three circular arcs of circles centered at each Reuleaux vertex V_i, $i = 1, 2, 3$. of an equilateral triangle.

The Reuleaux triangle $\mathcal{R}$ is the convex curve formed by the arcs of three circles of equal radii r centered on the vertices V_1, V_2, V_3 of an equilateral triangle and that mutually intercepts in these vertices; see Figure 1. This triangle is mostly known due to its constant width property [4].

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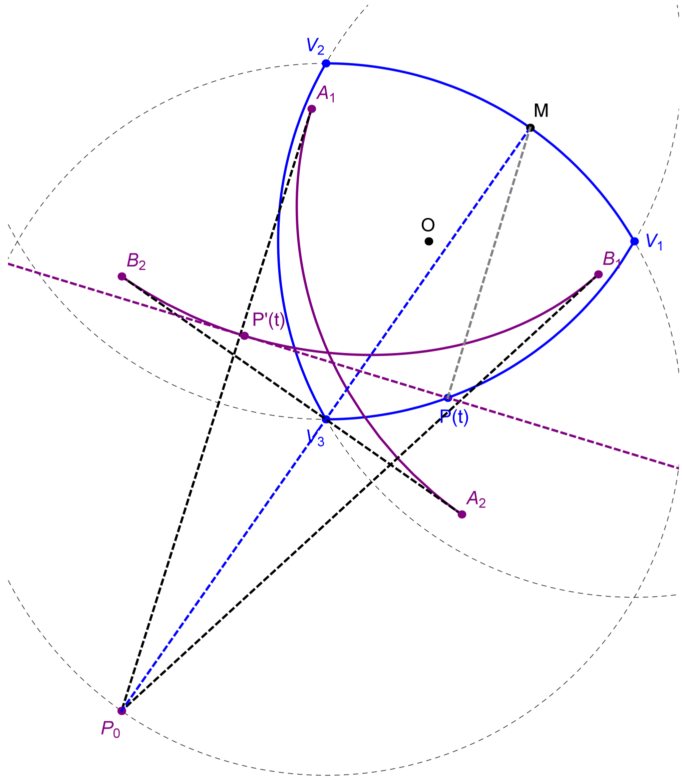

FIGURE 2. The negative pedal curve $\mathcal{N}$ of the Reuleaux Triangle $\mathcal{R}$ w.r. to a point M on its boundary consist on an point P_0 (the antipedal of M through V_3) and two elliptic arcs A_1A_2 and B_1B_2 (green and blue). A sample P is shown illustrating instantaneous tangency to $\mathcal{N}$ at P'.

Here, we study some properties of the negative pedal curve $\mathcal{N}$ of $\mathcal{R}$ w.r. to a pedal point M lying on one of its sides. This curve is the envelope of lines passing through points P on $\mathcal{R}$ and perpendicular to PM [3, Negative Pedal Curve]; see Figure 2. Trivially, the negative pedal curve of arc V_1V_2 is a point which we call P_0. We show that the negative pedal curves of the other two sides are elliptic arcs with a common focus on M and whose major axis measures $2r$; see Prop. 3.1.

Let V_3 be the center of the circular arc where pedal point M lies, and let V_1, V_2 be the endpoints of said arc. Let arc A_1A_2 (resp. B_1B_2) be the negative pedal image of the Reuleaux side V_1V_3 (resp. V_2V_3) where point A_1 is the image of V_1, and B_1 the image of V_2. The endpoints of $\mathcal{N}$ whose preimage is V_3 are respectively A_2 when V_3 is regarded as a point of side V_1V_3, and B_2 when V_3 is regarded as a point of side V_2V_3 of the Reuleaux triangle.

Our main result (Theorem 2.1, Section 2) is an intriguing property of the conic $\mathcal{C}^*$ – called here the *endpoint conic* – that passes through the endpoints A_1, A_2, B_1, B_2 of the negative pedal curve $\mathcal{N}$, and through P_0: that one of its foci is precisely the pedal point M; see Figure 3. We also give a full geometric description of its axes, directrix and vertices, and a criterion for identifying its type, according to the location of the pedal point M.

In Section 3 we prove other properties of the Reuleaux triangle and its negative pedal curve, involving tangencies, collinearities and homotheties.

The proofs combine elementary techniques with inversive arguments and polar reciprocity. A review of polar reciprocity and other concepts, including the description of the negative pedal curve as a locus of points, as well as an alternative description of it as an envelope of lines is postponed to the Appendix.

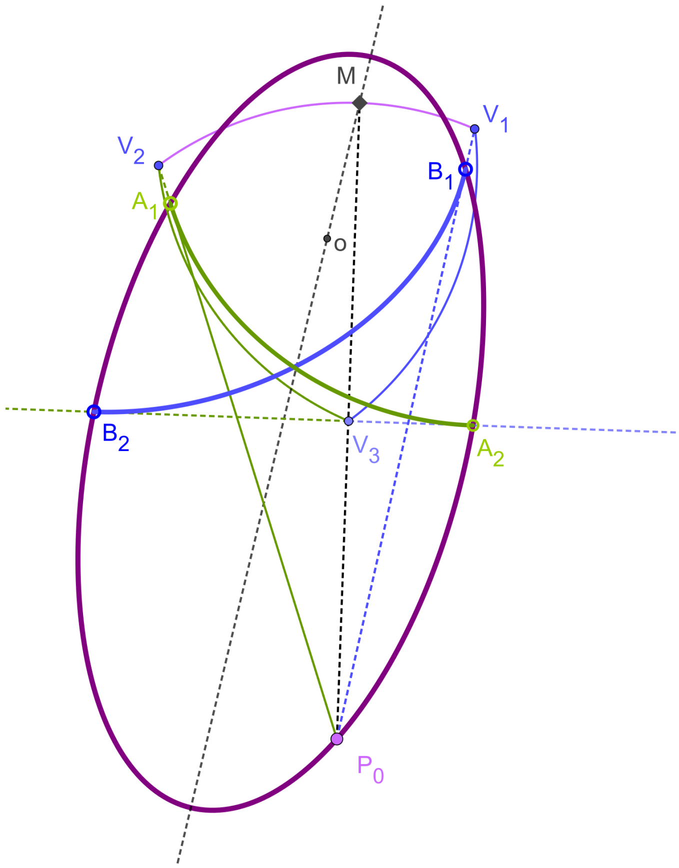

FIGURE 3. The sides of the Reuleaux $\mathcal{R}$ are three circular arcs centered at the vertices V_1, V_2, V_3 of an equilateral triangle. Its negative pedal curve $\mathcal{N}$ w.r. to a pedal point M on its boundary consists of a point P_0 (the antipode of M through V_3) and of two elliptic arcs A_1A_2 and B_1B_2 (green and blue). The endpoint conic $\mathcal{C}^*$ (purple) passes through P_0 and the four endpoints of the two elliptic arcs of $\mathcal{N}$. It has a focus on M and its focal axis passes through the center of the Reuleaux triangle.

2. Main Result: The Endpoint Conic

Referring to Figure 3.

Theorem 2.1. *The conic $\mathcal{C}^*$ which contains P_0 and the endpoints A_1, A_2, B_1, B_2 of the negative pedal curve of the Reuleaux triangle $\mathcal{R}$ with respect to a pedal point M located on a side of $\mathcal{R}$, has one focus on M and its axes pass through the circumcenter of $\triangle V_1V_2V_3$.*

The proof will require some additional steps which steadily use an inversive approach and polar duality.

The main idea is to use polar reciprocity: in order to prove that the focus of the C^* is M, we show that there exists a circle $\mathcal{C}$ to which the five polars of the endpoints A_0, A_1, B_0, B_1 and P_0 are tangent.

Recall that whenever we perform a polar transform (or a polar duality) of a conic, w.r. to an inversion circle (see Figure 4), points on the conic transform into their polars w.r. to the inversion circle, and those polars become tangents to the dual curve of the (initial) conic [2, Article 306].

This indirect and inversive approach is appropriate as the polars of A_1, A_2, B_1, B_2, and P_0 can be readily analyzed.

Classic facts about polar reciprocity guarantee that the dual (curve) of a conic is a circle iff the center of the inversion circle (which, in our case, is M) is the focus of the endpoint conic; see Proposition A.2.

The reader not familiar with the topic may find useful the details in Appendix and the references therein.

Referring to Figure 4.

Lemma 2.1. *Let M be a point on the side V_1V_2 of the Reuleaux triangle and $\mathcal{I}$ the inversion circle. Let V_1', V_2', V_3' be the inverses of points V_1, V_2, V_3 and let arcs $V_1'V_3'$ and $V_2'V_3'$ be, respectively, the inverses of arcs V_1V_3 and V_2V_3 of the Reuleaux triangle. Then:*

(1) *The polars of A_1 and B_1 are the tangents at V_1' and V_2' to circular arcs $V_1'V_3'$ and $V_2'V_3'$. The poles of the tangents at V_1' and V_2' to arcs $V_1'V_3'$ and $V_2'V_3'$, are the points A_1, B_1.*

(2) *The polars of points A_2, B_2 are the tangents in V_3' to arcs $V_1'V_3'$ and $V_2'V_3'$, respectively. The poles of the tangents at V_3' to arcs $V_1'V_3'$ and $V_2'V_3'$, are the points A_2, B_2.*

(3) *The inverse of arc V_1V_2 is the line $V_1'V_2'$ excluding segment $[V_1'\ V_2']$ and the polar of P_0 is the line $V_1'V_2'$.*

Proof. Inversion w.r. to a circle maps circles to either circles or lines: thus, the inverses of arcs V_1V_3 and V_2V_3 are the two circular arcs $V_1'V_3'$ and $V_2'V_3'$, respectively. On the other hand, since arc V_1V_2 passes through the inversion center, its image is the union of two half-lines.

All other statements derive from the description of the negative pedal curve as the dual of its inverse, as shown in Proposition A.3.

Lemma 2.2. *Using the notation in Lemma 2.1:*

(1) *the angles at V_1', V_2', and V_3' between arcs $V_1'V_2'$, $V_1'V_3'$, and $V_1'V_2'$ respectively (the inverses of the sides of the Reuleaux triangle), are* $120°$.

(2) *$\triangle V_1'O'V_2'$ determined by the tangents at V_1', V_2' to said arcs and the line $V_1'V_2'$ is equilateral.*

Proof. These statements derive from the fact that inversion is preserves angles between curves.

Given the above results, Theorem 2.1 is equivalent to the following Lemma; see Figure 4:

Lemma 2.3. *The five polars of endpoints A_1, A_2, B_1, B_2, and P_0 are tangent to circle* $\mathbf{c}$*, the exinscribed circle in $\triangle V_1'O'V_2'$, which is externally-tangent to side $V_1'V_2'$.*

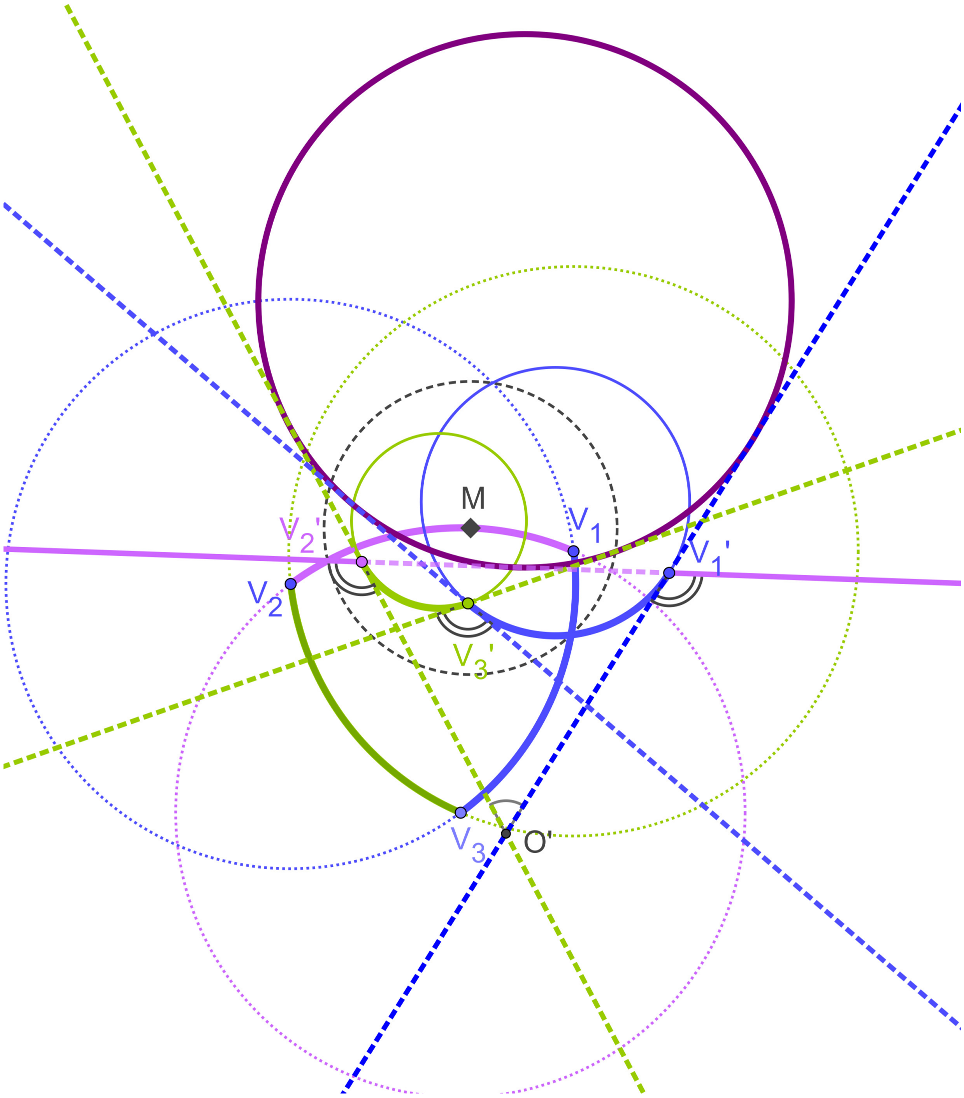

Figure 4. The Reuleaux triangle $\mathcal{R}$ and its inverse: (1) arcs $V_1'V_3'$ (blue) and $V_2'V_3'$ (green) are the inverses of sides V_1V_3 and V_2V_3 of $\mathcal{R}$, while line $V_1'V_2'$ except for segment $[V_1'V_2']$ itself (violet) is the image of arc V_1V_2; (2) the polars of A_1 and B_1 are the tangents in V_1' and V_2' to arcs $V_1'V_3'$ and $V_2'V_3'$; (3) the polars of A_2 and B_2 are the tangents in V_3' to arcs $V_1'V_3'$ and $V_2'V_3'$; (1)v) line $V_1'V_2'$ is the polar of P_0. (v) all five polars above are tangent to the exinscribed circle $\mathbf{c}$ of $\triangle V_1'O'V_2'$ (purple). The angle between circles $\mathbf{c_0}$ (green) and C_0 (blue) in V_3' is 120° iff V_3' is on the circle $\mathbf{C}$ concentric with $\mathbf{c}$ and passing through V_1' and V_2' (not shown); in this case, the tangents at V_3' (dashed green and blue) to circles $\mathbf{c_0}$ and $\mathbf{C_0}$, are also tangent to the exinscribed circle $\mathbf{c}$.

This lemma will be equivalent to the assertion that the focus of C^* coincides with pedal point M when we show that the two tangents at V_3' to arcs $V_1'V_3'$ and $V_2'V_3'$ respectively are also tangent to the excircle of $\triangle V_1'O'V_2'$. Referring to Figure 4, this can be restated as follows:

Lemma 2.4. *Let $\triangle V_1'O'V_2'$ be equilateral and let O be the center of its exinscribed circle* $\mathbf{c}$*, externally-tangent to side $[V_1'V_2']$. Let* $\mathbf{C}$ *be another circle, concentric with* $\mathbf{c}$*, that passes through V_1' (and V_2'). Finally, let* $\mathbf{c_0}$ *and* $\mathbf{C_0}$

be the two circles tangent to the sides $[OV_1']$ and $[OV_2']$ of the triangle, at V_1' and V_2', respectively. Then:

(1) *$\mathbf{c_0}$ and $\mathbf{C_0}$ intersect at an angle of 120° iff the three circles $\mathbf{c_0}$, $\mathbf{C_0}$, and $\mathbf{C}$ pass through one common point; let V_3' be that point.*
(2) *if the condition above is fulfilled, then the two tangents at the common point V_3' to circles $\mathbf{c_0}$ and $\mathbf{C_0}$ are also tangent to the exinscribed circle $\mathbf{c}$.*

While assertion (1) is automatic, once we identify circles $\mathbf{c_0}$ and $\mathbf{C_0}$ as the inverses of the circles which define the Reuleaux triangle, assertion (2) is not obvious and will require additional steps.

Though our construction is in general asymmetric, there are two regular hexagons associated with it, used in the results below.

Lemma 2.5. *Let $[A_0A_1\ldots A_5]$ be a regular hexagon with inscribed circle $\mathbf{c}$ and circumcircle $\mathbf{C}$. Let P_0 be a point on arc A_0A_1 of $\mathbf{C}$ and let P_0P_1, $P_1P_2, \ldots, P_5P_6$ be the tangents from $P_0, P_1, \ldots, P_5$ to $\mathbf{c}$.*

Let $\mathbf{c_0}$ be the circle tangent to side $[A_0A_5]$ of the hexagon at A_0 whose center is inside the hexagon; let P_0 be its second intersection point with $\mathbf{C}$. Then:

(1) *Points P_6 and P_0 coincide and the hexagon $[P_0P_1 \ldots P_5]$ is regular and congruent with $[A_0A_1 \ldots A_5]$. Both hexagons share the same incircle and circumcircle.*
(2) *Let P_0P_1 be the tangent from P_0 to $\mathbf{c}$; then it tangents (in P_0) the circle $\mathbf{c_0}$, as well.*

Referring to Figure 5:

Proof. 1) When we perform the construction of tangent lines $P_0P_1, \ldots, P_5P_6$, the process ends in six steps and $P_0 = P_6$, thanks to Poncelet's porism, since $\mathbf{c}$ and $\mathbf{C}$ are the incircle and the circumcircle of a hexagon. Note the latter is regular since its inscribed and circumscribed circles are concentric.

2) Let T_0 be the intersection of the perpendicular bisector of segment $[A_0P_0]$ with line A_0A_5. We shall prove that T_0P_0 is tangent at P_0 to circle $\mathbf{c_0}$ and that T_0, P_0, P_1 are collinear.

Let c_0 be the center of circle $\mathbf{c_0}$; since T_0 is a point on the perpendicular bisector of $[A_0P_0]$, and since A_0 and P_0 are the two intersections of circles $\mathbf{c_0}$ and $\mathbf{c}$, then O, c_0 and T_0 are collinear.

Next, $\triangle T_0A_0c_0 = \triangle T_0P_0c_0$, as they have respectively-congruent sides, hence:

$$\angle T_0P_0c_0 = \angle T_0A_0c_0 = 90^\circ$$

which proves that line T_0P_0 is tangent at P_0 to circle $\mathbf{c_0}$.

Furthermore, $\triangle T_0A_0O = \triangle T_0P_0O$ as they have respectively-congruent sides, hence:

$$\angle T_0P_0O = \angle T_0A_0O$$

By hypothesis, T_0, A_0, and A_5 are collinear, and $\triangle A_0A_5O$ is equilateral, hence the external angle $\angle T_0A_0O = 120^\circ$; hence $\angle T_0P_0O = 120^\circ$ as well.

Since by (1) $\triangle P_0OP_1$ is equilateral, then $\angle OP_0P_1 = 60°$ and $\angle T_0P_0P_1 = 180°$, proving that points T_0, P_0, P_1 are collinear.

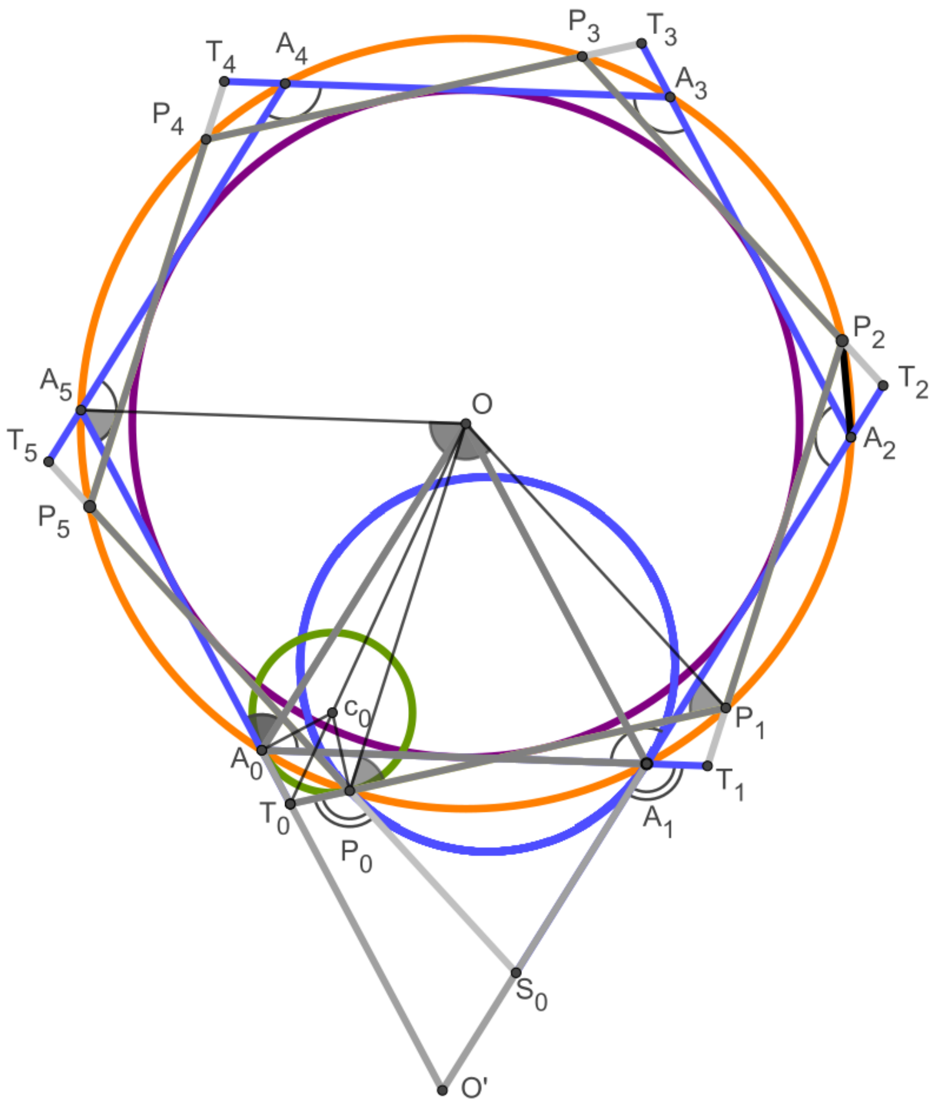

FIGURE 5. The angle between circles $\mathbf{c_0}$ (green circle) and $\mathbf{C_0}$ (blue circle) in P_0 is 120° iff P_0 is on the circumcircle of the regular hexagon $[A_0A_1 \ldots A_5]$ (orange circle).

We now reformulate Lemma 2.4 as follows:

Lemma 2.6. *Let $[A_0A_1 \ldots A_5]$ be a regular hexagon whose incircle is $\mathbf{c}$ and circumcircle is $\mathbf{C}$. Let $\mathbf{c_0}$ be the circle tangent to side $[A_0A_5]$ of the hexagon at A_0 and let $\mathbf{C_0}$ be the circle tangent to side $[A_1A_2]$ at A_1. Then the angle between circles $\mathbf{c_0}$ and $\mathbf{C_0}$ is 120° iff the three circles $\mathbf{c_0}, \mathbf{C_0}$, and $\mathbf{C}$ have one common point.*

Proof. " $\Leftarrow$ " First assume circles $\mathbf{c_0}$ and $\mathbf{C_0}$ intersect at a point P_0 on circumcircle $\mathbf{C}$. Referring to Figure 5, if P_0 is on arc A_0A_1 of $\mathbf{C}$ then by Lemma 2.5 P_0P_1 is a common tangent to circles $\mathbf{c_0}$ and $\mathbf{c}$. In particular, P_0P_1 is the tangent at P_0 to circle $\mathbf{c_0}$.

Next, let $P_0P'_5$ be the tangent from P_0 to the incircle $\mathbf{c}$ (distinct from P_0P_1). Similarly, let $P'_5P'_4$, $P'_4P'_3$, $\ldots P'_1P'_0$ be the tangents from $P'_5, P'_4, \ldots, P'_1 \in \mathbf{C}$ to the incircle $\mathbf{c}$.

Then, as above, points P'_0 and P_0 coincide, and hexagon $[P'_0P'_1 \ldots P'_5]$ is regular; since hexagons $[P_0P_1 \ldots P_5]$ and $[P'_0P'_1 \ldots P'_5]$ have one common point (P_0), are regular, and are both inscribed in $\mathbf{C}$, they must coincide.

Once again, Lemma 2.5 guarantees that P_0P_5 is a common tangent to circles $\mathbf{C_0}$ and $\mathbf{c}$. In particular, line P_0P_5 is the tangent at P_0 to circle $\mathbf{C_0}$.

Since hexagon $[P_0P_1 \ldots P_5]$ is regular, $\angle P_1P_0P_5 = 120°$. This guarantees that the angle between circles $\mathbf{c_0}$ and $\mathbf{C_0}$, which is the angle between their tangents at P_0, is also $120°$.

" $\Rightarrow$ " By hypothesis, circles $\mathbf{c_0}$ and $\mathbf{C_0}$ intersect at an angle of $120°$. We shall prove that, necessarily, point P_0 must be on the circumcircle $\mathbf{C}$.

Call P'_0 the intersection point between circle $\mathbf{c_0}$ and arc A_0A_1 of circle $\mathbf{C}$. We shall prove that P_0 and P'_0 coincide.

Let $\mathbf{C'_0}$ be the circle tangent at A_1 to line A_1A_2 that passes through P'_0. Then, by the first part of this proof, circles $\mathbf{c_0}$ and $\mathbf{C'_0}$ intersect at an angle of $120°$. So circles $\mathbf{C_0}$ and $\mathbf{C'_0}$ would be two circles, both tangent at A_1 to line A_1A_2, which intersect circle $\mathbf{c_0}$ at the same angle. Hence circles $\mathbf{C_0}$ and $\mathbf{C'_0}$ must coincide, as do points P_0 and P'_0.

Finally we can prove Theorem 2.1. Refer to Figure 6.

Proof. The above lemmas prove that the focus of $\mathcal{C}^*$ coincides with M. We end the proof by showing that the axis of $\mathcal{C}^*$ passes through G, the circumcenter of $\triangle V_1V_2V_3$. Equivalently, we prove that the directrix of $\mathcal{C}^*$ is perpendicular to the line that joins points M and G. We shall use an inversive argument.

As shown in Proposition A.1 in the Appendix, the directrix of a conic whose polar-dual is some circle, is precisely the polar of the center of that circle (w.r. to the inversion circle). In other words, the directrix of the $\mathcal{C}^*$ is the polar of O.

Recall V'_1, V'_2 and V'_3 are, respectively, the inverses of V_1, V_2, V_3 w.r. to the inversion circle centered in M. Since M, O, and G are, respectively, the center of the inversion circle, the circumcenter of $\triangle V'_1V'_2V'_3$, and the circumcenter of $\triangle V_1V_2V_3$ M, O, G must be collinear.

In turn, this implies that the polar of O (that is perpendicular to OM), is perpendicular to GM, as well.

Thus, the axis of $\mathcal{C}^*$ and line MO will either be parallel or coincide. Since M is the focus of $\mathcal{C}^*$, is major axis is line MO and G is on this line.

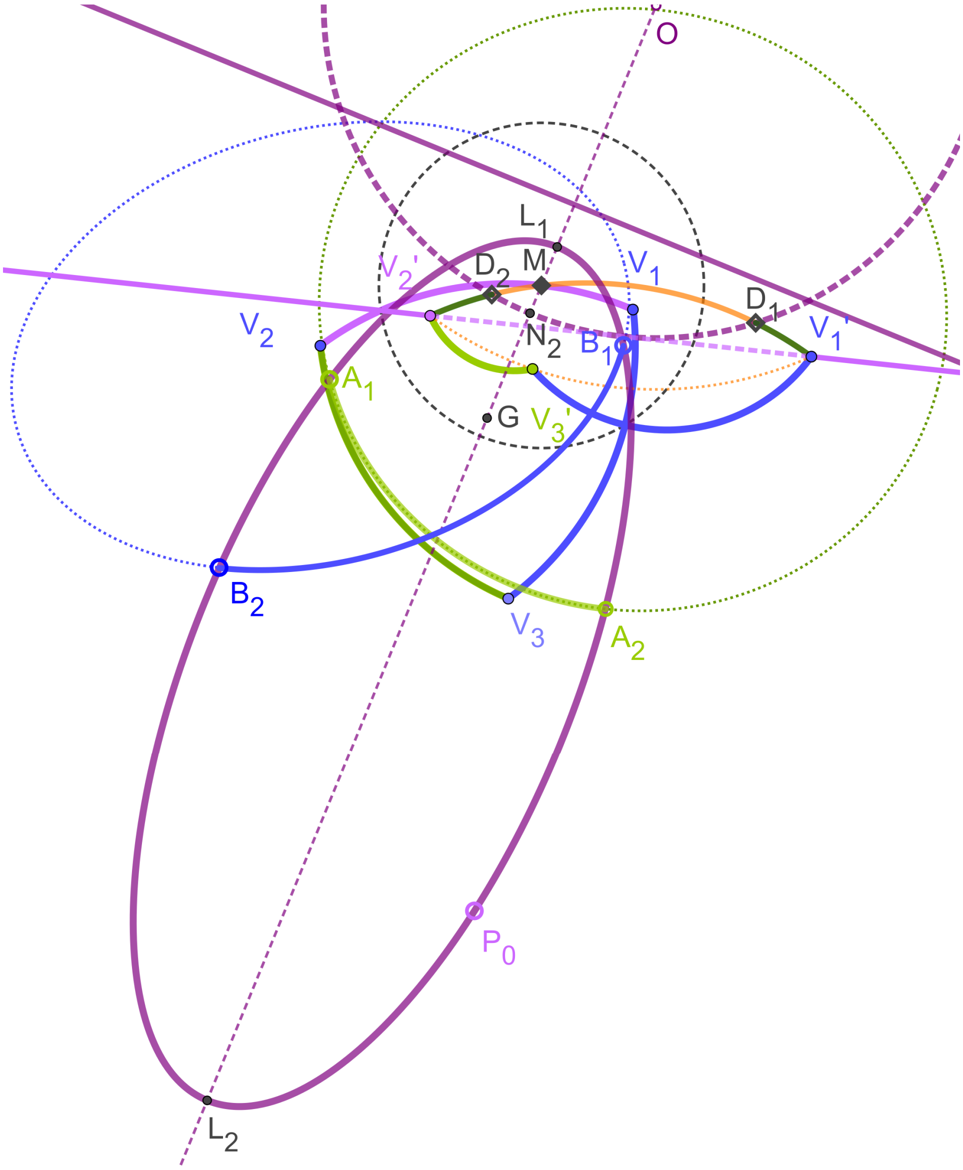

FIGURE 6. The endpoint conic $\mathcal{C}^*$ (purple) is the dual of circle $\mathbf{c}$ (dashed purple) w.r. to the inversion circle $\mathcal{I}$ centered on M (dashed black); its directrix is the polar of O, the circumcenter of $\mathbf{c}$; its major axis passes through G, and its vertices, L_1 and L_2 are the inverses of antipodal points N_1 and N_2, the diameter of $\mathbf{c}$ passing through M (point N_1, the antipode of N_2 w.r. to O, not shown).

The above results reveal that the endpoint conic is in fact the polar-dual of a special circle, which depends on the vertices of the Reuleaux triangle and on the location of pedal point M; see Proposition A.1.
Using the notation in Lemma 2.3 and referring to Figure 6:

Corollary 2.1. *The endpoint conic $\mathcal{C}^*$ associated with a Reuleaux triangle and a pedal point M is the polar-dual of circle $\mathbf{c}$. Its type depends on the location of M on arc V_1V_2: it is an ellipse (resp. hyperbola) if it lies inside (resp. outside) circle $\mathbf{c}$. If it is on said circle, $\mathcal{C}^*$ is a parabola.*

Therefore one can (geometrically) construct any of its elements (vertices, other focus) as well as compute its axis and eccentricity.

Referring to Figure 7 and with the notation in Lemma 2.6.

Observation 2.1. *Let $\mathbf{C}'$ be reflection of $\mathbf{C}$ w.r. to line $V_1'V_2'$ and let D_1, D_2 be the two intersections between circles $\mathbf{c}$ and $\mathbf{C}'$. One can check that V_3' is the reflection of M w.r. to $V_1'V_2'$; therefore, M is located both on the reflection of the circumcircle of $\triangle V_1'V_2'V_3'$ w.r. to $V_1'V_2'$, and on arc V_1V_2 of the Reuleaux triangle. The location of M with respect to $\mathbf{C}'$ reveals the type of conic $\mathcal{C}^*$: it is an ellipses if M is on the arc D_1D_2 of circle $\mathbf{C}'$, a hyperbola, if $M \in V_2'D_2$ or $V_1'D_1$ and a parabola when M is either D_1 or D_2.*

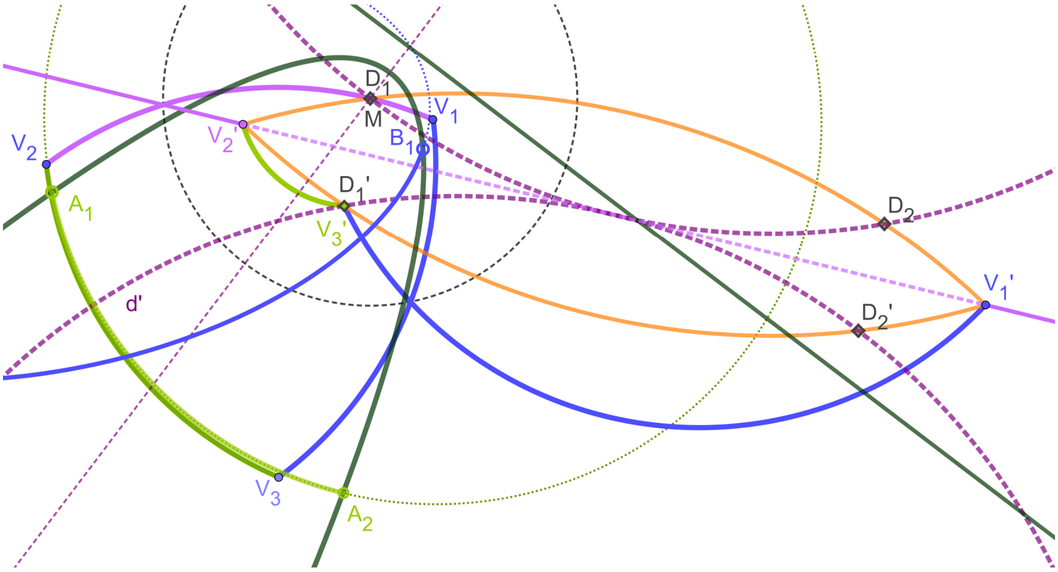

FIGURE 7. The endpoint conic $\mathcal{C}^*$ (dark green) is a parabola iff point M coincides with either D_1 or D_2; its directrix (dark green) is the polar of the circumcenter O of $\triangle V_1'V_2'V_3'$ (not shown).

3. Some Elementary Properties

3.1. Collinearity and Tangencies.

Referring to Figure 8:

Proposition 3.1. *The negative pedal curve $\mathcal{N}$ of the Reuleaux triangle consists of two elliptic arcs $\mathcal{E}_A$ and $\mathcal{E}_B$ and a point P_0, the antipode of M w.r. to the center of the circle where M is located. The two ellipses are centered on the vertices of the Reuleaux triangle, V_1 and V_2, have one common focus at M, and their semi-axes are of length r.*

Proof. By hypothesis, M belongs to arc V_1V_2 of the circle centered in V_3 that passes through V_2 and V_3. Hence, if P is any point on this arc and we draw the perpendicular p through P on PM, all these lines will pass through a fixed point P_0, which is he antipode of M w.r. to center V_3.

The second part derives directly from the general construction of the negative pedal curve of a circle. See Proposition A.4 in the Appendix.

Proposition 3.2. *The minor axes of $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ pass through P_0.*

Proof. By the definition of the negative pedal curve, if we regard V_1 as a point on arc V_1V_2 of the circle centered on V_3 on which the pedal point M lies, then P_0V_1 will be perpendicular to MV_1. Since V_1 is the center of $\mathcal{E_A}$ and since line MV_1 is its major axis, its minor axis will be along P_0V_1. Similarly, the minor axis of $\mathcal{E_B}$ is P_0V_2.

Proposition 3.3. *Points A_2, B_2 and V_3 are collinear and line A_2B_2 is a common tangent to $\mathcal{E}_A$ and $\mathcal{E}_B$.*

Proof. By construction, the negative pedal curve of arc V_2V_3 is the elliptic arc $\mathcal{E_A}$, delimited by A_1 and A_2. This implies that MV_3 and A_2V_3 are perpendicular, as well as MV_3 and B_2V_3. Thus points A_2, V_3 and B_2 are collinear. Also by construction, the perpendicular to MV_3 at V_3 is tangent to $\mathcal{N}$ at A_2 (resp. B_2) when V_3 is regarded as a point in the V_2V_3 (resp. V_1V_3) arc. Hence the points A_2, V_3 and B_2 are collinear ($\angle A_2V_3B_2 = 180°$) and A_2B_2 is the common tangent to $\mathcal{E_A}$ and $\mathcal{E_B}$, at A_2 and B_2, respectively.

Proposition 3.4. *Point A_1 is on P_0V_2 and B_1 is on P_0V_1.*

Proof. If we regard V_1 as a point on arc V_1V_2 of the circle centered on V_3 whose negative pedal curve is P_0, then, necessarily, $V_1P_0 \perp MV_1$. Similarly, if we regard V_1 as a point on arc V_1V_3 of the circle centered on V_2 whose negative pedal is $\mathcal{E_B}$, then by $\mathcal{N}$'s construction $B_1V_1 \perp MV_1$. Since this perpendicular must be unique, P_0, B_1, and V_1 are collinear as will be P_0, A_1, and V_2.

Proposition 3.5. *The line joining the intersection points of $\mathcal{E}_A$ and $\mathcal{E}_B$ is the perpendicular bisector of segment $[f_Af_B]$ and also passes through P_0.*

Proof. Let U_1, U_2 denote the points where $\mathcal{E_A}$ and $\mathcal{E_B}$ intersect. In order to prove that P_0, U_1, and U_2 are collinear, we show each one lies on the perpendicular bisector of $[f_Af_B]$. Since U_1 (resp. U_2) is on $\mathcal{E_A}$ (resp. $\mathcal{E_B}$), whose foci are M and f_A (resp. M and f_B), with major axis of length $2r$, then

$$U_1f_A + U_1M = 2r; \qquad U_2f_B + U_1M = 2r.$$

This implies that $U_1f_A = U_1f_B$ and $U_2f_A = U_2f_B$, hence both U_1 and U_2 belong to the perpendicular bisector of $[f_Af_B]$. Since we've already shown that $P_0V_1 \perp MV_1$, and since V_1 is the midpoint of Mf_A, this means that P_0V_1 is the perpendicular bisector of $[Mf_A]$ and this implies that $P_0f_A = P_0M$. Similarly, $P_0f_B = P_0M$, and hence $P_0f_A = P_0f_B$. Therefore P_0 is also on the perpendicular bisector of $[f_Af_B]$, ending the proof.

3.2. **Triangles and Homotheties.** Referring to Figure 8:

Proposition 3.6. *The two sides of triangle $\triangle f_AP_0f_B$, incident on P_0, contain points A_2 and B_2. The other side contains points A_1 and B_1.*

Proof. The construction of the negative pedal curve of arc V_2V_3 implies $A_1V_2 \perp MV_2$. Since V_2 is the center of the $\mathcal{E}_A$, A_1V_2 is the perpendicular bisector of $[Mf_B]$ hence $A_1f_B = A_1M$. Since A_1 lies on $\mathcal{E}_A$, $MA_1 + f_AA_1 = 2r$, hence $f_BA_1 + f_AA_1 = f_Af_B$. Therefore, triangle inequality implies f_B, A_1, and f_A must be collinear. A similar proof applies to B_1. In order to prove

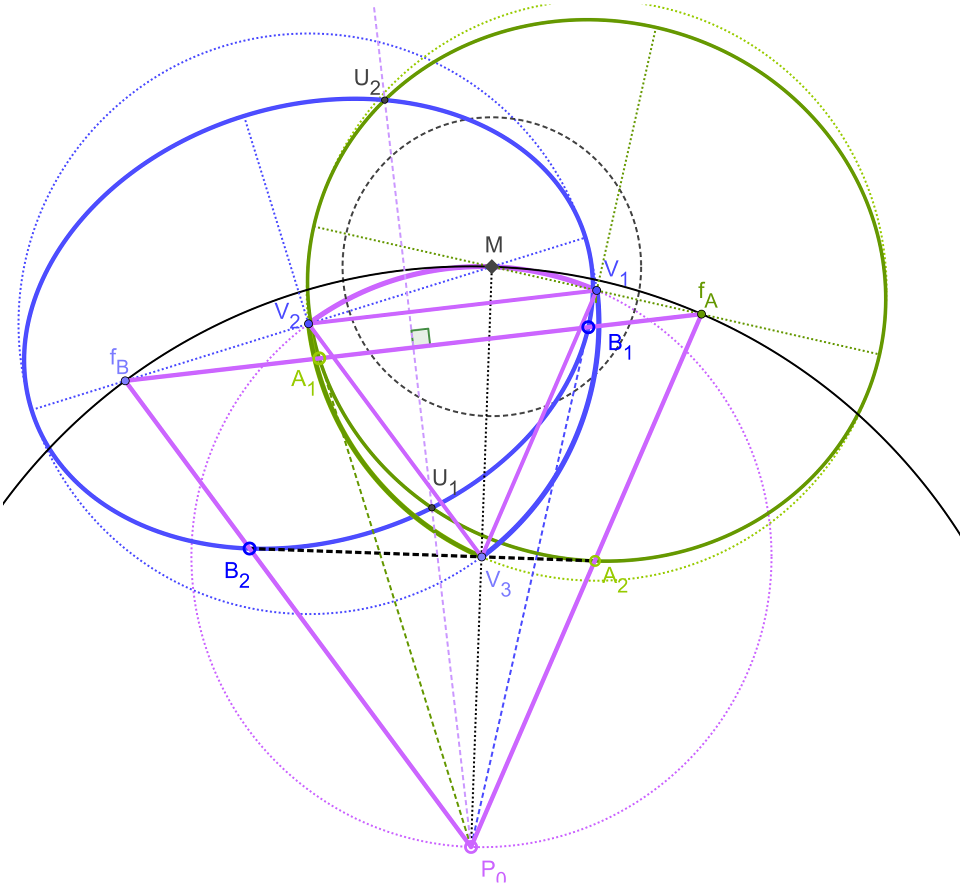

Figure 8. The negative pedal curve $\mathcal{N}$ w.r. to pedal point M consists of two arcs of ellipses $\mathcal{E}_A$ and $\mathcal{E}_B$ (green and blue), centered on Reuleaux vertices V_1, V_2, respectively. They have a common focus at M, and the other foci are f_A, f_B. Their major axes have length of $2r$, equal to the diameters of the three Reuleaux circles (dashed). Points P_0, A_1, V_2 P_0, B_1, V_1 are collinear and along their minor axis. The lines P_0A_1 and P_0B_1 are tangent to $\mathcal{E}_A$ and $\mathcal{E}_B$, respectively. A_2B_2 is tangent to both ellipses and $A_2, B_2, V3$ are collinear. The circle (black) passing through M, f_A and f_B $\mathcal{E}_A, \mathcal{E}_B$ (green and blue) is centered on P_0 (antipodal of M w.r. to V_3). The distance between the foci f_A and f_B is constant. Triangle $\mathcal{T} = \triangle f_A f_B P_0$ is equilateral and its sides pass through (i) A_2, (ii) B_2, (iii) A_1, B_1, respectively. Both intersections U_1, U_2 of $\mathcal{E}_A$ with $\mathcal{E}_B$ lie on the perpendicular bisector of $f_A f_B$, hence are collinear with P_0. $\mathcal{T}$ and $\triangle V_1V_2V_3$ are homothetic (homothety center M and homothety ratio 2)).

that P_0, B_2, and f_A are collinear, we simply show that $P_0 f_A = P_0A_2 + A_2 f_A$. As noted above, A_2V_3 is perpendicular to P_0M and V_3 is its midpoint. Hence A_2V_3 is the perpendicular bisector of $[P_0M]$; so $P_0A_2 = MA_2$. Since A_2 lies on $\mathcal{E}_A$, we have:

$$P_0A_2 + A_2 f'_A = MA_2 + A_2 f'_A = 2r$$

The proof for B_2 is similar.

Proposition 3.7. *Triangles $\triangle f_A f_B P_0$ and $\triangle V_1V_2V_3$ are homothetic at ratio 2, and with M the homothety center. Hence, $\triangle f_A f_B P_0$ is equilateral and the distance between f_A and f_B is $2r$. Furthermore, their barycenters X_2 and X'_2 are collinear with M.*

Proof. Points V_1, V_2, V_3 are the midpoints of Mf_A, Mf_B, and P_0M, respectively. Thus, V_1V_2 is a mid-base of $\triangle f_AMf_B$, V_2V_3 is a mid-base of $\triangle f_BP_0f_A$ and V_3V_1 is a mid-base of $\triangle P_0Mf_A$. Hence $\triangle f_Af_BP_0$ and $\triangle V_1V_2V_3$ are homothetic with ratio 2, and homothety center M. Therefore $\triangle f_Af_BP_0$ is equilateral and the distance between f_A and f_B is the same as the diameter $2r$ of the circles that form the Reuleaux triangle.

Thus, $\triangle f_Af_BP_0$ is equilateral with sides twice that of the original triangle: $f_Af_B = 2V_1V_2$. This shows that the distance between the pair of foci of $\mathcal{E}_\mathcal{A}$ and $\mathcal{E}_\mathcal{B}$ is constant and equal to the length of their major axes. Note that lines V_1f_A, V_2f_B, P_0V_3 intersect at M, hence the two triangles are perspective at M. Due to the parallelism of their sides, their medians will be respectively parallel; let X_2 and X_2' denote the barycenters of triangles $\triangle V_1V_2V_3$ and $\triangle f_Af_BP_0$, respectively. The barycenter divides the medians in equal proportions, which guarantees $\triangle MX_2'V_2 \sim \triangle MX_2f_B$. Since M, V_2, f_B are collinear, so are M, X_2', X_2.

4. Conclusion

We studied some proprieties of the negative pedal curve to an object with a remarkable symmetry: the equilateral Reuleaux triangle. The five endpoints of this curve determine a conic with one focus on pedal point M, for any choice of M on the third arc of the Reuleaux.

To prove that we adopted an inversive approach, based on the fact that the reciprocal of a conic w.r. to a circle is a circle, iff the center of inversion is the focus of said conic.

Since points on the original curve convert to lines tangent to the reciprocal curve, it sufficed to show that the five lines tangent to the inverted sides of the Reuleaux, at their endpoints, are tangent to some circle. Our proof relies on Poncelet's porism.

One may also consider an asymmetric Reuleaux triangle delimited by three circles whose radii r_1, r_2, r_3 are distinct, and whose centers O_i are not necessarily vertices of an equilateral triangle. Preliminary experiments show that some properties of the equilateral Reuleaux still in the asymmetric case. Using the notation in Figure 8, one observes that for any choice of $O_1, O_2, O_3, r_1, r_2, r_3$:

(1) the distance between foci $|f_Af_B|$ does not depend on the location of M.
(2) V_2, A_1, P_0 as well as V_1, B_1, P_0 are collinear.
(3) the line through the two intersections U_1, U_2 of $\mathcal{E}_A$ with $\mathcal{E}_B$ is perpendicular to f_Af_B.
(4) line A_2B_2 is tangent to both $\mathcal{E}_A$ and $\mathcal{E}_B$.

Nevertheless, in this general setting, the focus of the endpoint conic no longer coincides with the pedal point M. Below, some open questions we could not yet answer synthetically:

- What is the location of the focus of the endpoint conic if the Reuleaux triangle is asymmetric? Is it still geometrically meaningful?
- When does the focus of the endpoint conic of an arbitrary Reuleaux triangle coincides with the pedal point?

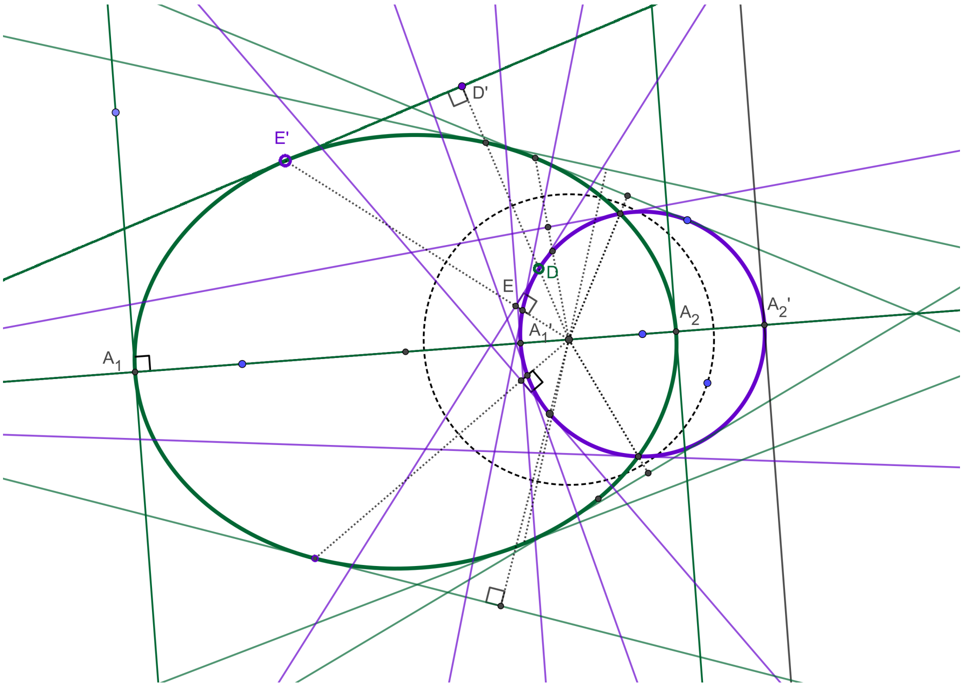

Figure 9. The dual of Γ (violet) w.r. to its inversion circle (dashed black) is the curve Γ^* (dark green), the envelope of polars $E'D'$ of points D on Γ, as well as loci of E', the poles of the tangents ED to Γ, as D sweeps Γ. The dual of a circle is a conic. Γ^* is an ellipse iff M is inside Γ, a hyperbola iff M is outside Γ, and a parabola, iff M is on Γ. Its focus is M, the inversion center (not illustrated), and the directrix is the polar of its center. Its vertices are the inverses of A_1' and A_2', the diameter of Γ through M.

- Are there any special locations of M on arc V_1V_2 for which it still the focus of the endpoint conic?
- Is there a poristic family of Reuleaux triangles whose endpoint conic has a focus on M (i.e. for any choice of the pedal point on the third arc of the Reuleaux)?
- What are the bounds on the eccentricity of the endpoint conic associated with some Reuleaux triangle?

Appendix A. Duality and the Negative Pedal Curve

Here we review concepts and results on polar duals; see [2].

Let a circle $\mathcal{I}$ be called the inversion circle and its center M the inversion point. Assume all inversions, the poles and polars below are performed w.r. to $\mathcal{I}$.

The following result provides two equivalent definitions for the dual curve:

Theorem A.1. *Let Γ be a regular curve, Γ_1^* the locus of the poles of its tangents, Γ_2^* the envelope of the polars of its points. Then $\Gamma_1^* = \Gamma_2^*$ and simply denote it Γ^*.*

Γ^* is a regular curve and the polars of its points are the tangents to Γ, while the poles of the tangents of Γ^* are points of Γ.

Further more $[\Gamma^*]^* = \Gamma$.

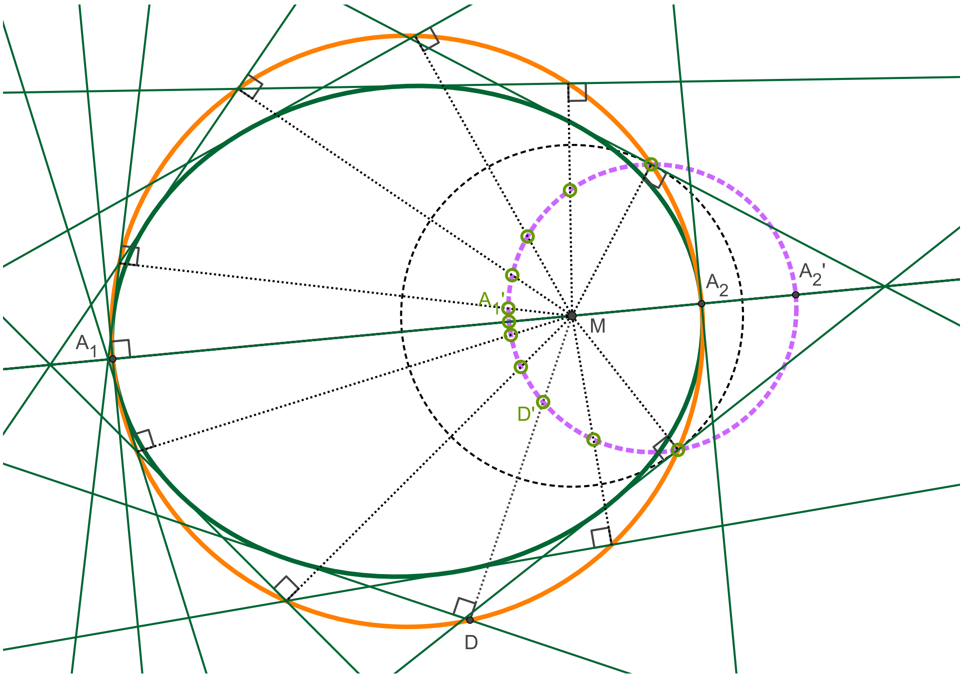

FIGURE 10. The negative pedal curve $\mathcal{N}(\Gamma)$ (green) of a circle Γ (orange) w.r. to $\mathcal{I}$ (dashed black) is the envelope of lines DE' (green) where D is a generic point on Γ and $DE' \perp MD$. Since DE' is (also) the polar of D', the inverse of D, then $\mathcal{N}(\Gamma)$ is the envelope of the polars of points of its inverse circle Γ'. Therefore, $\mathcal{N}(\Gamma)$ is the dual of Γ', hence is a conic with a focus at M. Its vertices are A_1, A_2, the diameter of Γ through M. Here, $\mathcal{N}(\Gamma)$ is an ellipse since M is inside Γ.

This theorem, whose proof is based on the fundamental pole-polar theorem justifies the dual definition of the curve Γ^* either as a locus of points or as an envelope of lines, and specifies who the points and the tangents at a dual curve are.

For more details on poles, polars and polar reciprocity, see e.g. [2].
Referring to Figure 9:

Proposition A.1. *The dual (or the polar dual, or the reciprocal) of a circle Γ w.r. to an inversion circle centered at M is a conic Γ^* whose:*

- *focus coincides with the inversion center;*
- *vertices are the inverses of the endpoints of the diameter of Γ passing through the inversion center;*
- *directrix is the polar of the center of Γ.*

The dual conic Γ^* is and ellipse (resp. hyperbola) if M is inside (resp. outside) Γ and a parabola if M is on said curve.

Proposition A.2. *The dual of a conic Γ is a circle iff the inversion center is a focus of Γ.*

If this is the case, (i) the inverses of the vertices of Γ are a pair of antipodal points on the dual circle Γ^*; (ii) the center of Γ^* is the pole of the directrix of Γ.

These remarkable results are classic; see [1] or [2, Art.309], for a proof and more details.

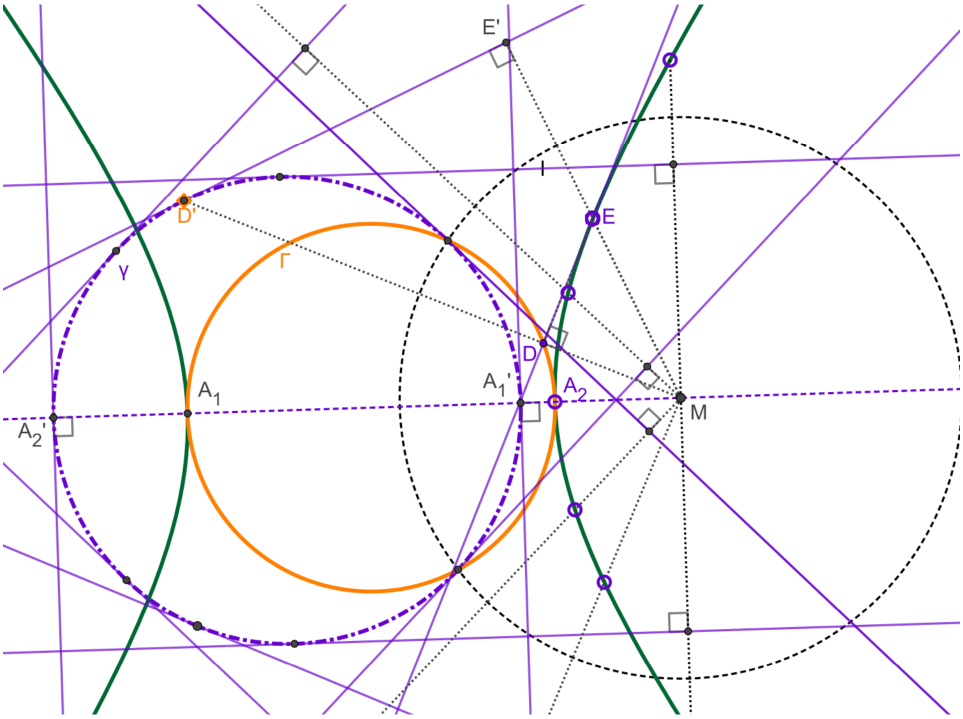

FIGURE 11. $\mathcal{N}(\Gamma)$ (green hyperbola), the negative pedal curve of a circle Γ (orange), being the dual of circle Γ' (dashed violet), the inverse of Γ, is also the locus of the poles E of tangents in D' to Γ' as D sweeps Γ. The line DE is the tangent to $\mathcal{N}(\Gamma)$ passing through D. $\mathcal{N}(\Gamma)$ is an ellipse (see Fig 10) iff $M \in [A_1A_2]$; $\mathcal{N}(\Gamma)$ is a hyperbola iff M is not on segment $[A_1A_2]$. The negative pedal curve of a circle is never a parabola.

There is a natural intertwining between the negative pedal curve, inversion, and polar reciprocity.

Proposition A.3. *The negative pedal curve of* Γ *w.r. to a pedal point is the reciprocal of its inverse,* Γ' *w.r. to a circle centered on that pedal point:* $\mathcal{N}(\Gamma) = \left[\Gamma'\right]^*$. *Therefore (i)* $\mathcal{N}(\Gamma)$ *is the locus of the poles of the tangents to its inverted curve,* Γ'*; (ii) the polars of the points of a negative pedal curve* $\mathcal{N}(\Gamma)$ *are the tangents to its inverted curve* Γ'.

Proof. First we prove that the dual of Γ' is contained in the negative pedal of Γ. Let S be a point in Γ'; then S is an inverse of some point L in Γ; $S = L'$; hence, the polar of S is the perpendicular in point S' to the line joining M and S'; since $S' = (L')' = L$, the polar of $S = L'$ is the perpendicular in L to line ML. Since inversion is bijective (in fact, it is an involution), if S sweeps Γ', L sweeps Γ, and lines ML are the set of all tangent lines to the negative pedal curve of Γ.

The reverse inclusion is similar, if we refer to negative pedal curves as an envelope of lines.

Thus, the negative pedal curve, initially defined as an envelope of lines, can also be constructed as a “point curve”, i.e. as the locus of the poles of the tangents to its inverse Γ'.

Below we describe the negative pedal curve of a circle. Refer to figure 11.

Proposition A.4. *The negative pedal $\mathcal{N}(\Gamma)$ of a circle Γ, w.r. to a pedal point M not located on Γ, is a conic, whose (i) focus is M;*
center is the center of circle Γ; (ii) vertices are the intersection points of the line that joins the pedal point M and the center of Γ, with the circle Γ; (iii) focal axis is the diameter of Γ.

$\mathcal{N}(\Gamma)$ will be an ellipse (resp. a hyperbola), if the pedal point is interior (resp. exterior) to Γ.

The negative pedal curve of a circle, w.r. to a point on the circumference, reduces to a point, the antipode of M.

Proof. First assume that M is not on the circle. Draw the negative pedal curve of the circle as follows:

(1) let Γ', be the inverse of Γ; then Γ' is a circle whose diameter is $[A_1'A_2']$ where A_1' and A_2' are the inverses of vertices A_1 and A_2 of Γ.

(2) perform the dual of Γ' to obtain a conic whose focus is M (the inversion center) and whose vertices are the inverses of A_1' and A_2', respectively, i.e., A_1 and A_2.

Then $\mathcal{N}(\Gamma) = \left[\Gamma'\right]^*$. By the above, the conic will be an ellipse (resp. hyperbola), if M is inside (resp. outside) Γ.

If the pedal point M is on the circle, then the inverse is a line, whose reciprocal is a point, its pole.

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