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INSCRIBING CIRCLE CHAINS INSIDE AN ELLIPTICAL SEGMENT

GIOVANNI LUCCA

Abstract. In this paper we derive the conditions able to guarantee that a circle can be always inscribed, in any position, inside a generic elliptical segment. Moreover, the relevant geometrical construction is shown. Finally, an iterative procedure to build up a chain of mutually tangent circles inscribed in the elliptical segment is proposed.

1. Introduction

In a previous paper [1], we studied the problem of inscribing a chain of mutually tangent circles inside a circular segment; here we want to generalise the issue by considering an elliptical segment instead of a circular one. In particular, we pose the following question: given a generic elliptical segment is it always possible to inscribe, in a generic position, a circle inside it?
As far as we now, nothing about this issue is present in literature so, we started in studying it as a novel problem and we propose, in this paper, a solution.
It is convenient to deal with the problem in polar coordinates in function of the polar angle θ with $0 \leq \theta \leq 2\pi$; thus, the equation of the ellipse having major semi-axis a and minor semi-axis b is given by:

$$\varrho_e(\theta) = \frac{ab}{\sqrt{b^2 \cos^2\theta + a^2 \sin^2\theta}} \qquad a \geq b \tag{1}$$

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By remembering the expression of a generic straight line in polar coordinates [2], the one, intersecting the ellipse, and forming with it the elliptical segment is given by the following equation:

$$\varrho_r(\theta) = \frac{p}{\cos(\alpha - \theta)} \qquad p \neq 0 \tag{2}$$

with:

$$0 < p < \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} \tag{3}$$

In (2), α is the angle associated to the slope of the normal μ to the straigth line λ intersecting the ellipse in points A and B so forming the elliptical segment (see Fig.1) while p is the distance of λ from the origin O.
Condition (3), relevant to the parameter p, guarantees that the straight line λ intersects the ellipse in two points so forming an elliptical segment.
The polar angles θ_A and θ_B corresponding to the intersections of the straight line λ with the ellipse can be found by solving the equation obtained by equating (1) and (2) i.e.:

$$\frac{p}{\cos\alpha\cos\theta + \sin\alpha\sin\theta} = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \tag{4}$$

By solving equation (4) with respect to the the variable $\tan\theta$, one finally obtains the angles θ_A and θ_B corresponding to the extremes of the chord AB i.e.:

(5)

$$\theta_{B,A} = \arctan\left(\frac{\sin\alpha\cos\alpha \pm \sqrt{\sin^2\alpha\cos^2\alpha - \left(\frac{p^2}{b^2} - \sin^2\alpha\right)\left(\frac{p^2}{a^2} - \cos^2\alpha\right)}}{\frac{p^2}{b^2} - \sin^2\alpha}\right)$$

In relation to formula (5), it is useful to remark that θ_A and θ_B may vary inside the interval $[0, 2\pi]$ while the principal value of the *arctan* function is inside the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, attention must be paid to report the value calculated by means of the *arctan* function, to the interval $[0, 2\pi]$ that has been considered in this paper.

2. Circle tangent to the chord AB and to the ellipse

The starting point, in order to inscribe a generic circle inside the elliptical segment, is to determine centre coordinates and radius of such a circle.
To this aim, we have to remember that the circle centre must be equidistant from the chord and from the tangency point with the ellipse; this is equivalent to say that the centre must lie on the bisectrix of the angle formed by λ and by the tangent line to the ellipse in the tangency point between circle and ellipse.
Then, let us indicate by θ the angle corresponding to the generic tangency

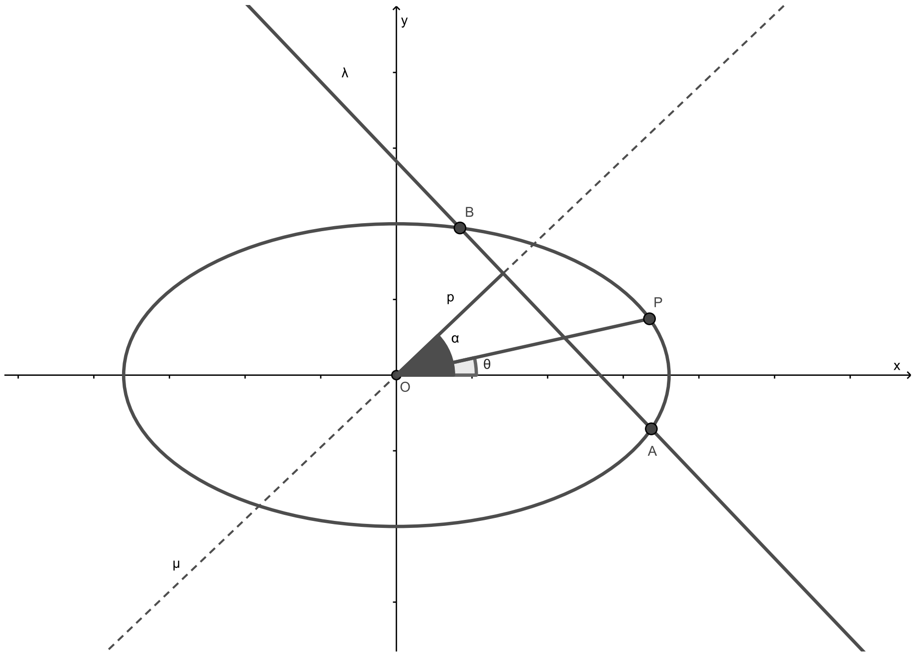

FIGURE 1. Example of elliptical segment formed by the chord AB and the arc of ellipse $\widehat{AB}$.

point P of the circle with the ellipse. Its coordinates are:

$$\begin{cases} x_e(\theta) = \dfrac{ab\cos\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \\ y_e(\theta) = \dfrac{ab\sin\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \end{cases} \tag{6}$$

Hence, the straight line tangent to the ellipse in P is [3]:

$$\frac{x x_e(\theta)}{a^2} + \frac{y y_e(\theta)}{b^2} = 1 \tag{7}$$

The equation of the straight line λ, by taking into account of (2) and that $\varrho_r(\theta)\cos\theta = x$ and $\varrho_r(\theta)\sin\theta = y$, is:

$$x\cos\alpha + y\sin\alpha - p = 0 \tag{8}$$

Therefore, by using equations (7) and (8), it is possible to calculate the bisectrix; its equation is:

$$x\cos\alpha + y\sin\alpha - p - \frac{-b^2x_e(\theta) - a^2y_e(\theta) + a^2b^2}{\sqrt{b^4x_e^2(\theta) + a^4y_e^2(\theta)}} = 0 \tag{9}$$

The centre of the circle tangent to both the chord AB and to the the ellipse in P can be found by considering the intersection between the bisectrix given by (9) and by the normal to the ellipse in P given by equation:

$$a^2 y_e(\theta)\, x - b^2 x_e(\theta)\, y + \left(a^2 - b^2\right) x_e(\theta)\, y_e(\theta) = 0 \tag{10}$$

Finally, the centre of the circle tangent to both the chord and the ellipse, is solution of the following system:

(11)

$$\begin{cases} \dfrac{\left[\cos\alpha\sqrt{b^4 x_e^2(\theta) + a^4 y_e^2} + b^2 x_e\right] x + \left[\sin\alpha\sqrt{b^4 x_e^2(\theta) + a^4 y_e^2} + a^2 y_e\right] y}{p\sqrt{b^4 x_e^2(\theta) + a^4 y_e^2} + a^2 b^2} = 1 \\ \\ a^2 y_e(\theta)\, x - b^2 x_e(\theta)\, y = \left(a^2 - b^2\right) x_e(\theta)\, y_e(\theta) \end{cases}$$

By solving system (11), and by remebering the relationships (6), we obtain that the centre coordinates $(X_c(\theta), Y_c(\theta))$ are:

(12)

$$\begin{cases} X_c(\theta) = \dfrac{p\, b^2\sqrt{b^2\cos^2\theta + a^2\sin^2\theta} + \sin\alpha\, ab\left(a^2 - b^2\right)\sin\theta + ab\sqrt{b^4\cos^2\theta + a^4\sin^2\theta}}{\left[b^2\cos\theta\cos\alpha + a^2\sin\theta\sin\alpha + \sqrt{b^4\cos^2\theta + a^4\sin^2\theta}\right]\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\cos\theta \\ \\ Y_c(\theta) = \dfrac{p\, a^2\sqrt{b^2\cos^2\theta + a^2\sin^2\theta} - \cos\alpha\, ab\left(a^2 - b^2\right)\cos\theta + ab\sqrt{b^4\cos^2\theta + a^4\sin^2\theta}}{\left[b^2\cos\theta\cos\alpha + a^2\sin\theta\sin\alpha + \sqrt{b^4\cos^2\theta + a^4\sin^2\theta}\right]\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\sin\theta \end{cases}$$

Equations (12) are the parametric representation, in polar form, of the locus of the centres of the circles that are tangent to both the chord AB and the ellipse; we name such a curve by Γ. The radius $R(\theta)$ of the circles can be calculated by considering the distance between the straight line λ, expressed by (8) and the centre $(X_c(\theta), Y_c(\theta))$. After some calculations, one obtains:

(13)

$$R(\theta) = \left|\frac{2ab\cos(\alpha - \theta) - \sqrt{2}p\sqrt{a^2 + b^2 - (a^2 - b^2)\cos 2\theta}}{\sqrt{2}\sqrt{a^4 + b^4 - (a^4 - b^4)\cos 2\theta} + (a^2 + b^2)\cos(\alpha - \theta) - (a^2 - b^2)\cos(\alpha + \theta)}\right| \cdot \sqrt{\frac{a^4 + b^4 - \left(a^4 - b^4\right)\cos 2\theta}{a^2 + b^2 - (a^2 - b^2)\cos 2\theta}}$$

Nevertheless, this circle, even if tangent to both the chord and the ellipse is not necessarly also inscribed inside the elliptical segment, see for example Fig.2. Thus, a further constraint is needed.

3. Inscribing a circle inside the elliptical segment

In this paragraph, we derive the inscribability conditions of a circle inside an elliptical segment, but, firstly, it necessary to derive some useful formulas by studying the possible intersections between the ellipse and the circle having centre and radius given by (12) and (13) respectively.

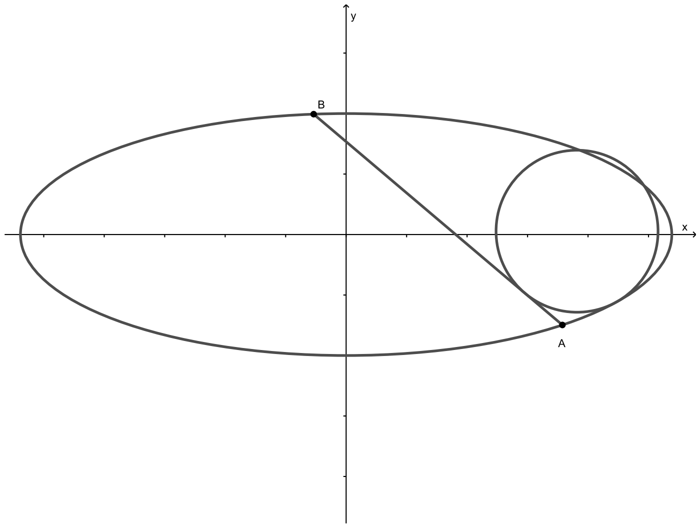

FIGURE 2. Example of circle tangent to both chord and ellipse but not inscribed in the elliptical segment.

Let us start by considering the following system:

$$\begin{cases} (x - X_c(\theta))^2 + (y - Y_c(\theta))^2 = R^2(\theta) \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \end{cases} \tag{14}$$

After some algebraical steps, one gets the following fourth degree equation with respect to the variable x:

(15)

$$\left(1 - \frac{b^2}{a^2}\right)^2 x^4 - 4X_c(\theta)\left(1 - \frac{b^2}{a^2}\right)x^3 +$$
$$+ \left\{4X_c^2(\theta) - 2\left(1 - \frac{b^2}{a^2}\right)\left[b^2 + X_c^2(\theta) + Y_c^2(\theta) - R^2(\theta)\right] + 4Y_c^2(\theta)\frac{b^2}{a^2}\right\}x^2 +$$
$$- 4X_c(\theta)\left[b^2 + X_c^2(\theta) + Y_c^2(\theta) - R^2(\theta)\right]x + \left[b^2 + X_c^2(\theta) + Y_c^2(\theta) - R^2(\theta)\right]^2 +$$
$$- 4Y_c^2(\theta)\, b^2 = 0$$

By taking into account that the circle is, by construction, tangent to the ellipse in $(x_e(\theta), y_e(\theta))$, equation (15) can factorised as follows:

$$(x - x_e(\theta))^2 \cdot \left[\begin{array}{c} \left(1 - \dfrac{b^2}{a^2}\right)^2 x^2 + 2\left(1 - \dfrac{b^2}{a^2}\right)\left[x_e(\theta)\left(1 - \dfrac{b^2}{a^2}\right) - 2X_c(\theta)\right]x + \\ + \dfrac{\left[b^2 + X_c^2(\theta) + Y_c^2(\theta) - R^2(\theta)\right]^2 - 4Y_c^2(\theta)\, b^2}{x_e^2(\theta)} \end{array}\right] = 0 \tag{16}$$

In order that further and distinct intersection points exist between the circle and the ellipse, in addition to the tangency point, the discriminant $\Delta(\theta)$ of the second degree expression with respect to x, inside square parentheses in formula (16), must be positive i.e.:

(17)

$$\frac{\Delta(\theta)}{4\left(1-\frac{b^2}{a^2}\right)^2} =$$

$$\left[x_e(\theta)\left(1-\frac{b^2}{a^2}\right)-2X_c(\theta)\right]^2 - \frac{\left[b^2+X_c^2(\theta)+Y_c^2(\theta)-R^2(\theta)\right]^2-4b^2Y_c^2(\theta)}{x_e^2(\theta)} > 0$$

The expression (17), after a certain number of steps, can be transformed into the following more convenient form:

(18)

$$\frac{\Delta(\theta)}{4\left(1-\frac{b^2}{a^2}\right)^2} = -Y_c(\theta)\cdot$$

$$\cdot\left[\left(\frac{4y_e^2(\theta)-4b^2}{x_e^2(\theta)}\right)Y_c(\theta)+\frac{4y_e(\theta)\left\{b^2-x_e^2(\theta)+2X_c(\theta)\,x_e(\theta)-y_e^2(\theta)\right\}}{x_e^2(\theta)}\right]$$

By substituting in (18) the relevant expressions for $x_e(\theta)$, $y_e(\theta)$, $X_c(\theta)$, $Y_c(\theta)$, given respectively by (6) and (12), one obtains a very long and complicated formula that, by the aid of a computer algebra system, can be written in the following simpler form:

(19)

$$\frac{\Delta(\theta)}{4} = \frac{4\sin^2\theta\left(1-\frac{b^2}{a^2}\right)^2}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)\left(\sqrt{b^4\cos^2\theta+a^4\sin^2\theta}+a^2\sin\alpha\sin\theta+b^2\cos\alpha\cos\theta\right)^2}\cdot$$

$$\cdot\, b\left[a\sqrt{b^4\cos^2\theta+a^4\sin^2\theta}+a\sin\alpha\sin\theta\left(a^2-b^2\right)+bp\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}\right]\cdot$$

$$\cdot\left[-pa^2\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}+\cos\alpha\cos\theta\,ab\left(a^2-b^2\right)-ab\sqrt{b^4\cos^2\theta+a^4\sin^2\theta}\right]$$

We have to notice that the first factor in the first line of (19) is always positive; also the second factor in the second line is always positive (see Appendix for the demonstration). Hence, the sign of (19) is determined by the sign of the third factor appearing in the third line.
We can now state the condition of inscribability that holds two alternative cases:

Theorem 3.1. *Sufficient condition in order to inscribe, in any position, a circle inside an elliptical segment, is that one either the other of the following inequalities holds:*

(20) $$\frac{p}{|\cos\alpha|} > a \quad CASE\ I$$

$$(21)\qquad \begin{aligned} & 2b^2p^2\left(a^2+b^2\right) < \left(p^2+b^2\right)\left[b^4+a^2p^2- \mid \cos\alpha \mid^2 \left(a^2-b^2\right)^2\right] + \\ & + 2pb\left(a^2-b^2\right) \mid \cos\alpha \mid \sqrt{\left(a^2-b^2\right)\left[\left(a^2-b^2\right) \mid \cos\alpha \mid^2 + b^2 - p^2\right]} \\ & when \quad \frac{p}{\mid \cos\alpha \mid} < a \qquad\qquad CASE\ II \end{aligned}$$

Proof.
In CASE I, equation (20) simply means that the chord AB intersects the cartesian axis that contains the major axis of the ellipse, outside the ellipse itself. We have to show that the discriminant $\Delta(\theta)$ is always negative. By considering the third factor in (19), we have that the following inequality holds:

(22)

$$- pa^2\sqrt{b^2\cos^2\theta + a^2\sin^2\theta} + \cos\alpha\cos\theta\, ab\left(a^2-b^2\right) - ab\sqrt{b^4\cos^2\theta + a^4\sin^2\theta} <$$
$$< -pa^2b \mid \cos\theta \mid + ab\left(a^2-b^2\right)\cos\alpha\cos\theta - ab^3 \mid \cos\theta \mid$$

In the second member of the inequality (22), the second addendum can be written as follows:

(23)

$$\begin{cases} \cos\alpha\cos\theta = - \mid \cos\alpha \mid\mid \cos\theta \mid & \text{if the cosines have opposite sign} \\ \cos\alpha\cos\theta = \mid \cos\alpha \mid\mid \cos\theta \mid & \text{if the cosines have same sign} \end{cases}$$

therefore, in the first case, substitution of the first equation of (23) inside the second member of (22) holds:

$$- \left[pa^2b + ab\left(a^2-b^2\right) \mid \cos\alpha \mid + ab^3\right] \mid \cos\theta \mid < 0$$

while, in the second case, substitution of the second equation of (23), inside the second member of (22) and, by exploiting (20), holds:

$$- \left[pb^3 + ab^3\right] \mid \cos\theta \mid < 0$$

So, one can conclude that in CASE I the discriminant $\Delta(\theta)$ is always negative.
In CASE II, let us start again from the third factor in (19) that can be written as follows:

(24)

$$- \mid \cos\theta \mid \left[pa^2\sqrt{b^2 + a^2\tan^2\theta} \mp \mid \cos\alpha \mid ab\left(a^2-b^2\right) + ab\sqrt{b^4 + a^4\tan^2\theta}\right]$$

By looking at (24), when one has the sign + before the second addendum inside the expression between square parentheses, it clear that $\Delta(\theta) < 0$. Therefore, let us consider the case when one has the sign $-$ before the second addendum inside the expression between square parentheses in (24). The study of the positive sign of (24) and then of the positive sign of $\Delta(\theta)$, is then reduced to the study of the following inequality:

$$pa^2\sqrt{b^2+a^2\tan^2\theta} - \mid \cos\alpha \mid ab\left(a^2-b^2\right) + ab\sqrt{b^4+a^4\tan^2\theta} < 0 \tag{25}$$

It is convenient to make the following variable change:

$$w = \tan^2\theta \qquad w > 0 \tag{26}$$

So, we have to solve the following irrational equation:

$$pa\sqrt{b^2+a^2w} - b\left(a^2-b^2\right) \mid \cos\alpha \mid + b\sqrt{b^4+a^4w} = 0 \tag{27}$$

The root $\bar{w}$ of (27) is:

(28)

$$\bar{w} = \frac{b^2}{a^4\left(b^2-p^2\right)^2} \cdot$$
$$\cdot \begin{bmatrix} -\left(p^2+b^2\right)\left\{b^4+a^2p^2 - \mid \cos\alpha \mid^2 \left(a^2-b^2\right)^2\right\} + 2p^2b^2\left(a^2+b^2\right) + \\ -2pb\left(a^2-b^2\right) \mid \cos\alpha \mid \sqrt{\left(a^2-b^2\right)\left[\left(a^2-b^2\right) \mid \cos\alpha \mid^2 + b^2 - p^2\right]} \end{bmatrix}$$

For brevity reasons, we have omitted the derivation of (28).
Thus, being the expression in (27) a monotonically increasing function with respect to the variable w, with the only root given by (28), inequality (25) is verified when $w < \bar{w}$ that is:

$$\tan^2\theta - \bar{w} < 0 \tag{29}$$

Concluding, expression (24) is positive and so does $\dfrac{\Delta(\theta)}{4}$ only if the following conditions holds :

$$\arctan\left(-\sqrt{\bar{w}}\right) < \theta < \left(\arctan\sqrt{\bar{w}}\right) \quad \text{and} \quad \bar{w} > 0 \tag{30}$$

Thus, only if both the conditions in(30) are satisfied, we have two further intersections points between circle and ellipse.
In particular, from the second condition in (30), the expression in (28) between square parentheses must be positive so that we obtain:

(31)

$$2b^2p^2\left(a^2+b^2\right) > \left(p^2+b^2\right)\left[b^4+a^2p^2 - \mid \cos\alpha \mid^2 \left(a^2-b^2\right)^2\right] +$$
$$+ 2pb\left(a^2-b^2\right) \mid \cos\alpha \mid \sqrt{\left(a^2-b^2\right)\left[\left(a^2-b^2\right) \mid \cos\alpha \mid^2 + b^2 - p^2\right]}$$

Conversely, if (31) is not satisfied i.e.:

(32)
$$2b^2p^2\left(a^2+b^2\right) < \left(p^2+b^2\right)\left[b^4+a^2p^2- \mid \cos\alpha \mid^2 \left(a^2-b^2\right)^2\right]+ \\ +2pb\left(a^2-b^2\right) \mid \cos\alpha \mid \sqrt{\left(a^2-b^2\right)\left[\left(a^2-b^2\right) \mid \cos\alpha \mid^2 +b^2-p^2\right]}$$

the circle has no further intersection points with the ellipse (so that it is actually inscribed in the elliptical segment) and inequality (21) represents the condition of inscribability in CASE II.

□

It is useful to remark that if $a=b$ (that is the ellipse degenerates into a circle) inequality (32) becomes:

$$\left(a^2-p^2\right)^2>0 \qquad (33)$$

which is always verified; so, it is always possible to inscribe a circle, in any position, inside a circular segment.

4. Some remarks about the locus Γ

In this paragraph we add two interesting remarks about the locus Γ defined in par.2.
In the first one we show how the shape of the curve Γ (defined in par.2) is related to the condition of inscribability of a circle inside the elliptic segment. First of all, we have to notice that $Yc\,(\theta)$ and the discriminant $\Delta\,(\theta)$ have in common the points where they vanishes (provided they exist) inside the interval $[\theta_A,\theta_B]$. This property can be deduced by looking at (18), (19) and at the numerator of the second expression in (12).
One can also notice that, if the chord AB intersects the x-axis, then $Yc\,(\theta)$ vanishes for $\theta=0$ or $\theta=\pi$. Moreover, if inequality (31) is satisfied we have two further zeros of $Yc\,(\theta)$ that is: $\theta_1=\sqrt{\bar{w}}$ and $\theta_2=-\sqrt{\bar{w}}$.
One can also verify that $X_c\,(\theta_1)=X_c\,(\theta_2)$.
Consequently, being: $\left(X_c\,(\theta_1)\,,Y_c\,(\theta_1)\right)=\left(X_c\,(\theta_2)\,,Y_c\,(\theta_2)\right)$, we have that this point, located at the intersection between the curve Γ and the x-axis is a double point or node. The existence of this point is strictly related to satisfaction of inequality (31) expressing the condition of non-inscribability; thus we can say that if the curve Γ has a double point, then it is not possible to inscribe a circle, in any position, inside an elliptical segment.
In Fig.3 it is shown an example.

Coming to the second remark, let us consider the particular case when $a=b$ so that the ellipse becomes a circle. Under this hypothesis, the parametric equations of the locus Γ expressed by (12) become:

$$\begin{cases} X_c\,(\theta)=\dfrac{p+a}{1+\cos\left(\alpha-\theta\right)}\cos\theta \\ Y_c\,(\theta)=\dfrac{p+a}{1+\cos\left(\alpha-\theta\right)}\sin\theta \end{cases} \qquad (34)$$

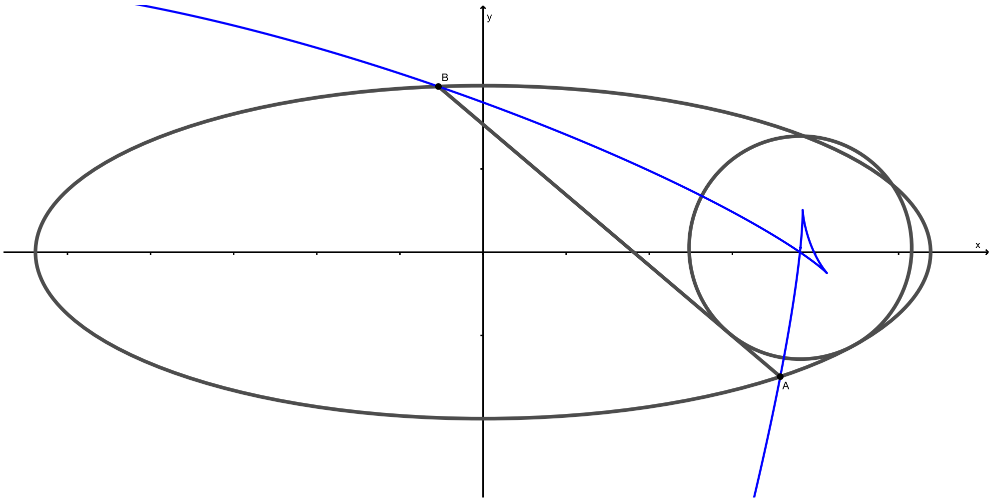

Figure 3. Example of curve Γ, with a double point and circle not inscribed in the elliptical segment.

Equation (34) is the parametric representation, in polar coordinates, of a parabola with focus in the origin, axis coincident with the straight line μ (previously defined and shown in Fig.1), directrix placed at a distance $p+a$ from the focus and parallel to the chord AB. We notice that this result is consistent to the one shown in [1].

5. Constructing circle chains

If the parameters a, b, p and α characterising the ellipse and the chord AB satisfy the inequalities shown in Theorem 3.1, it is then possible to inscribe, inside the elliptical segment, an infinite chain of mutually tangent circles where each circle is tangent to the preceeding and successive one.
The starting point, is to arbitrarly choose, an angle $\theta^0 \in (\theta_A, \theta_B)$; to this angle, it correspond an inscribed circle having centre coordinates given by $\left(X_c^0, Y_c^0\right)$ and radius R^0 according to (12) and (13) respectively. It is convenient to name this arbitrary starting circle, characterised by index 0, *origin circle*; starting from the origin circle, it is possible to construct two chains: one directed toward point B and the other toward A.
Thus, we have to determine two sequences composed by those specific values of the angle θ, i.e. $\left\{\theta_B^i\right\}$ and $\left\{\theta_A^i\right\}$, with $i \in \mathbb{N}$, that generate the values of the centre coordinates and radius relevant to the circles forming the two chains.
The equation enabling to calculate the value of θ^{i+1} in function of θ^i is the Pythagorean Theorem that connects centre coordinates and radius of two consecutive circles that is:

$$\left[X_c\left(\theta^{i+1}\right) - X_c\left(\theta^i\right)\right]^2 + \left[Y_c\left(\theta^{i+1}\right) - Y_c\left(\theta^i\right)\right]^2 = \left[R\left(\theta^{i+1}\right) + R\left(\theta^i\right)\right]^2 \tag{35}$$

Hence, if one knows the value of θ^i, by solving equation (35), it is possible

to calculate the value of θ^{i+1}; therefore, by means of two different iterative procedures, the sequences $\{\theta_B^i\}$ and $\{\theta_A^i\}$ can be determined. For both the sequences the common starting value is θ^0.
We have to remark that, having in mind equations (12) and (13), equation (35) assumes a complicated expression; hence, due to its complexity, it has to be solved numerically. Moreover, when calculating θ_B^{i+1} in function of the preceeding value θ_B^i, we have to bear in mind that $\theta_B^{i+1} \in (\theta_B^i, \theta_B)$; similarly, when calculating θ_A^{i+1}, in function of the preceeding value θ_A^i, it must be $\theta_A^{i+1} \in (\theta_A, \theta_A^i)$.
In Fig.4 an example of circle chain inscribed inside an elliptical segment is shown.
The dashed line is the locus Γ while the black circle is the *origin circle*.

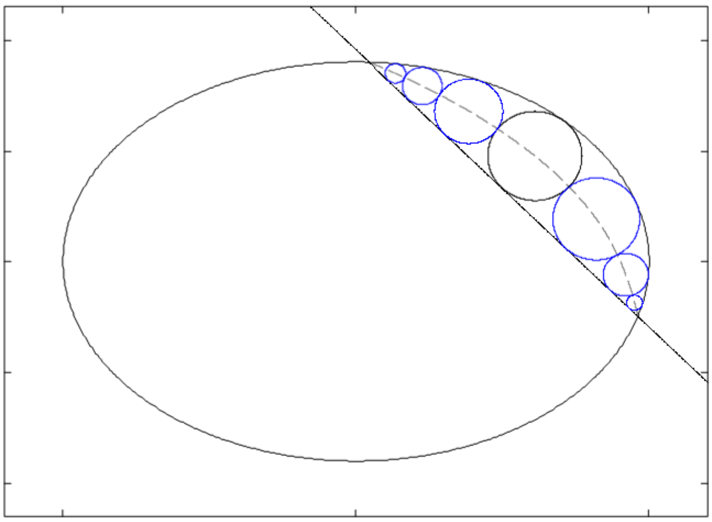

FIGURE 4. Example of circle chain inscribed in the elliptical segment.

APPENDIX

In this Appendix, we show that the second factor in formula (19) i.e.:

$$b\left[a\sqrt{b^4\cos^2\theta + a^4\sin^2\theta} + a\sin\alpha\sin\theta\left(a^2-b^2\right) + bp\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}\right] \tag{36}$$

is always positive.
The expression between square parentheses in (36) is surely greater than:

$$b\left[a^3 \mid \sin\theta \mid \pm a \mid \sin\alpha \mid\mid \sin\theta \mid \left(a^2-b^2\right) + bpa \mid \sin\theta \mid\right] \tag{37}$$

In expression (37) the $+$ holds if $\sin\alpha$ and $\sin\theta$ have the the same sign, while $-$ holds if they have opposite sign.

Hence in the first case, expression (37) becomes:

$$b\left[a^3 + a \mid \sin\alpha \mid \left(a^2 - b^2\right) + bpa\right] \mid \sin\theta \mid \tag{38}$$

which is positive because $a > b$.
In the secon case, expression (37) becomes:

$$b\left[a^3 \left(1 - \mid \sin\alpha \mid\right) + ab^2 + bpa\right] \mid \sin\theta \mid \tag{39}$$

which is positive because $1- \mid \sin\alpha \mid > 0$.
So, we can conclude that expression (36) is always positive.

References

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Piacenza, ITALY
E-mail address: vanni_lucca@inwind.it