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TILING A PLANE WITH TWO TYPES OF SEMI-REGULAR EQUILATERAL POLYGONS

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Abstract. In this paper, tiling a plane with equilateral semi-regular convex polygons is considered. Tiling a plane with this type of polygons can be observed from the aspect of a type of semi-regular equilateral polygons that join in one node. Therefore, we distinguish between the case when semi-regular equilateral polygons of one type meet in the node and the case when two or more types of semi-regular equilateral polygons meet in the node. As in the case of a semi-regular equilateral polygon, the interior angles are not equal, and tiling a plane with these polygons depends not only on the type of the polygon, but also on its interior angles that join in the node. In relation to the interior angles, the vertices of a semi-regular equilateral polygon with the same or different internal angles can be joined in the nodes. In this paper, the case of tiling a plane with equilateral semi-regular polygons when semi-regular polygons of different types join in the nodes is considered. The problem of determining the nodes in tiling a plane with two types of equilateral polygons is equivalent to determining the set of all solutions of the corresponding Diophantine equation in the form of $\alpha_1 \cdot t + \beta_1 \cdot s + \alpha_2 \cdot u + \beta_2 \cdot v = 2\pi$, where t, s, u, and v are the non-negative integers which are not simultaneously equal to zero, while $\alpha_1, \beta_1, \alpha_2$, and β_2 are the interior angles of the semi-regular equilateral polygons used for tiling a plane. Tiling a plane with semi-regular equilateral polygons with $2n$-sides is especially considered for the value of the characteristic angle δ that can be geometrically constructed. Also, an example of tiling a plane is analyzed when semi-regular quadrilaterals and hexagons meet in the nodes, for which a geometric interpretation is given.

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1. Introduction

The problem of tiling a plane is an old one, which even the mathematicians of ancient Egypt, Greece, Persia, China and other ancient civilizations were familiar with. Tiling comes down to dividing a plane into polygons that would completely cover it without having any overlaps or gaps, following certain regularity, depending on the type, shape and arrangement of the polygons. When tiling a plane, the goal is to divide the plane into polygons which have common sides and vertices. Then the polygons which have one side in common are called neighboring polygons, and the point in a plane in which the vertices of the neighboring polygons join are called the hub of plane partition. The hub is regular if angles of polygons that join in a hub are equal. We consider two hubs equal if number of angles that join in them are equal. More on the problem of tiling can be found in [4],[7] and a tiling catalog can be found in [1, 2, 3, 4].
Here, a special case of tiling a plane is considered – tiling with semi-regular polygons. In this case of tiling, semi-regular polygons of one, two or more types join in the hub. We observe the problem of tiling a plane with two types of semi-regular equilateral polygons. Equilateral semi-regular polygon with N sides of length a are designated as $\mathcal{P}_N^a$.

A simple polygon $\mathcal{P}_N \equiv A_1A_2...A_N$ that has equal all sides or equal all interior angles is called a semi-regular polygon [3],[9]. In terms of definition, we distinguish between two types of semi-regular polygons: equiangular (having equal interior angles and different sides) and equilateral (having equal sides, and different interior angles). We consider that vertex $A_i, i = 1, 2, ..., N, i \in \mathbb{N}$ of a polygon is in an even position, or odd position, if index i is an even, or an odd number, respectively. In this paper, we consider convex equilateral semi-regular polygons. The marking is as follows: -$N = n \cdot m$, is a number of sides in a semi-regular equilateral polygon, -n is a number of sides in a regular polygon $\mathcal{P}_n$, -a is a side in a semi-regular polygon $\mathcal{P}_N$, -$\mathcal{P}_k$ is a polygon with $m = k - 1$ sides constructed over each side A_jA_{j+1}, $j = 1, 2, \ldots, n$ of polygon $\mathcal{P}_n$ with which it has one side in common, and we call it the edge polygon of semi-regular polygon $\mathcal{P}_N$.
- $b_j, j \in \mathbb{N}$ is a side in a regular polygon $\mathcal{P}_n^i$ "inscribed" to a semi-regular polygon $\mathcal{P}_N \equiv A_1A_2...A_N$, constructed by joining its vertices in even (or odd) positions, -Interior angles of a semi-regular polygon at odd vertices are marked with α, and those at even vertices are marked with β (1). In addition to these interpreted marks, all other marks will be interpreted when mentioned in a given definition.
To a semi-regular equilateral polygon $\mathcal{P}_N \equiv A_1A_2...A_N$ with $N = 2 \cdot n, n \geq 2, n \in \mathbb{N}$ with equal sides there can be "inscribed" regular n-side polygons: by joining odd vertices, $\mathcal{P}_n^1 \equiv A_1A_3A_5...A_{2n-3}A_{2n-1}$, or even vertices $\mathcal{P}_n^2 \equiv A_2A_3A_4...A_{2n-2}A_{2n}$.
To analyze the metric properties of regular polygons [2, 4] it is sufficient for us to know one basic element, the length of a side, while for the semi-regular polygons this is not sufficient [9, 10, 12].

Therefore, in addition to side a of a semi-regular polygon, for the analysis of the metric properties we will use another element of it, and that is the

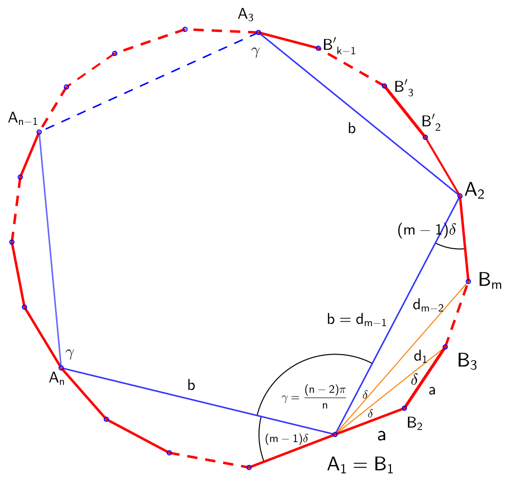

FIGURE 1. Basic elements of equilateral semi-regular polygon $\mathcal{P}_{2n}^{a,\delta}$ of side a and angle δ.

angle between side a of the semi-regular polygon and side b of its "inscribed" regular polygon, which we mark with δ, i.e. $\delta = \angle(a, b)$ (Figure1) [3]. To show that a semi-regular equilateral $2n-$ side polygon is given by side a and angle δ we write: $\mathcal{P}_{2n}^{a,\delta}$. If $\gamma = \frac{(n-2)\pi}{n}, n \geq 2$ is the interior angle of the "inscribed" regular polygon $\mathcal{P}_n^1$, then $\alpha = \gamma + 2\delta = \frac{(n-2)\pi}{n} + 2\delta$ gives interior angles by odd vertices, and $\beta = \pi - 2\delta$ gives the ones at even vertices of the semi-regular polygon $\mathcal{P}_{2n}^a$ of side a, where $\delta = \angle(b, a)$ marks the angle between the sides of polygons $\mathcal{P}_n^1$ and $\mathcal{P}_{2n}$ (Figure1). Here, we consider that a regular polygon with $n = 2$ sides (segment) is "inscribed" to a semi-regular equilateral quadrilateral (rhombus).
Next, we consider those values of angle δ for which $\mathcal{P}_N$ is a convex semi-regular equilateral polygon. We find the values of angle δ for which a semi-regular equilateral polygon $\mathcal{P}_{2n}, n \geq 2, n \in \mathbb{N}$ is convex, from the inequality which connects the definition of convexity with the values of interior angles of the semi-regular polygon [1].
That is, from inequality $\alpha < \pi$ and $\beta < \pi$ and the values of angles α, β we find that for all $\delta \in \langle 0, \frac{\pi}{n} \rangle$ a semi-regular equilateral polygon $\mathcal{P}_{2n}$ is convex, while for $\delta = \frac{2\pi}{n}$ it is convex and regular. The following is true:
An equilateral semi-regular $2n$-side polygon of side a and angle δ is:

(1) convex, if $\delta \in \langle 0, \frac{\pi}{n} \rangle$. For $\delta = \frac{\pi}{2n}$ it is regular and convex.
(2) non-convex, if $\delta \in \langle \frac{\pi}{n}, \frac{\pi}{2} \rangle$.
(3) if $\delta > \frac{\pi}{2}$, then semi-regular polygon $\mathcal{P}_{2n}$ is not defined [14].

Let us mark semi-regular polygon $\mathcal{P}_N$ with $N = n \cdot m$ sides as determined with the characteristic elements n, m, δ and interior angles α, β, with $\mathcal{P}_N \vdots \left\{ {n,m \atop \delta} \right\}_\alpha^\beta$ [9, 11, 13, 14].
First let us specify the lemma which is valid for the interior angels of semi-regular equilateral polygons $\mathcal{P}_N$ and the theorem related to their convexity.

More on proof of the lemma and theorem can be found in [9, 10, 12, 15]. The lemma which is valid for the interior angles reads:

Lemma 1.1. *Semi-regular equilateral convex polygon $\mathcal{P}_N$ with $N = n \cdot m$ sides and angle δ, has n interior angles, at vertices of the inscribed regular polygon $\mathcal{P}_n$, equal to angle α, and the following is valid:*

$$\alpha = \frac{(n-2)\pi}{n} + 2(m-1)\delta \tag{1}$$

and $(m-1) \cdot n$ interior angles, at vertices of "edge" isosceles polygons $\mathcal{P}_k$, equal to angle β, and the following applies:

$$\beta = \pi - 2\delta, \delta > 0, \quad m, n, k \in \mathbb{N}, m, n \geq 2, m = k-1 \tag{2}$$

[12].

Theorem 1.1. *Equilateral semi-regular polygon $\mathcal{P}_N$ with $N = n \cdot m$ sides is convex if*

$$\delta \in \left\langle 0; \frac{\pi}{N-n} \right\rangle; \delta \neq \frac{\pi}{N}; n, m \geq 2 \in \mathbb{N} \tag{3}$$

Note that for $\delta = \frac{\pi}{N}$ the convex semi-regular polygon is regular, so that this value of the angle is excluded from further consideration [9].

2. MAIN RESULT

2.1. **Tiling a Plane with Two Types of Semi-Regular Equilateral Polygons.** Let it be that there are given two semi-regular equilateral polygons $\mathcal{P}_{N_1}$ and $\mathcal{P}_{N_2}$ with their characteristic elements

$$\mathcal{P}_{N_1}\colon\left\{\begin{matrix} n_1, m_1 \\ \delta_1 \end{matrix}\right\}_{\alpha_1}^{\beta_1} \quad \text{i} \quad \mathcal{P}_{N_2}\colon\left\{\begin{matrix} n_2, m_2 \\ \delta_2 \end{matrix}\right\}_{\alpha_2}^{\beta_2},$$

where $N_i = n_i \cdot m_i; i = 1, 2$ are the number of sides of semi-regular polygon, and $n_i, m_i \geq 2$ are the natural numbers. Labels α_i, β_i designate the corresponding interior angles of the semi-regular polygon, determined by relations $\alpha_i = \frac{(n_i-2)\pi}{n_i} + 2(m_i-1)\delta_i$ and $\beta_i = \pi - 2\delta_i, \quad i = 1, 2$

For δ angle, let us choose the values which can be geometrically constructed as follows:

$$\delta_i = \delta(n_i, m_i) = \frac{\pi}{2^{m_i} \cdot n_i(m_i - 1)}, \quad \text{and} \quad \delta_i \in (0, \frac{\pi}{N_i - n_i}), \delta_i \neq \frac{\pi}{N_i}.$$

Let $\mathcal{P}_{N_1}$ i $\mathcal{P}_{N_2}$ be the convex equilateral semi-regular polygons. If it is possible to tile a plane with those polygons,then there must exist some non-negative integres numbers t, s, u, v, which are not all simultaneously equal to zero, so that for each hub the following relation is applicable:

$$\alpha_1 \cdot t + \beta_1 \cdot s + \alpha_2 \cdot u + \beta_2 \cdot v = 2\pi. \tag{4}$$

Relation (4) presents the meeting point in the hub of the following:

1. t- interior angles equal to angle α_1,
2. s- interior angles equal to angle β_1 of semi-regular polygon $\mathcal{P}_{N_1}$,

(3) u- interior angles equal to angle α_2, and
(4) v- interior angles equal to angle β_2.

Each solution (t, s, u, v) of that equation corresponds to one hub, and the set of all solutions of the equation represents all different hubs which occur by tiling a plane with those semi-regular polygons. If in relation (4) we change values of interior angles, with the mentioned labels, we get the following series of relations

$$\begin{aligned} \alpha_1 \cdot t + \beta_1 \cdot s + \alpha_2 \cdot u + \beta_2 \cdot v &= 2\pi \quad \Leftrightarrow \\ (\frac{(n_1-2)\pi}{n_1} + 2(m_1-1)\delta_1)t &+ \\ (\pi - 2\delta_1)s + (\frac{(n_2-2)\pi}{n_2} + 2(m_2-1)\delta_2)u + (\pi - 2\delta_i)v &= 2\pi \quad \Leftrightarrow \\ (\frac{(n_1-2)\pi}{n_1} + \frac{2\pi}{2^{m_1}\cdot n_1})t + (\pi - \frac{2\pi}{2^{m_1}\cdot n_1(m_1-1)})s &+ \\ (\frac{(n_2-2)\pi}{n_2} + \frac{2\pi}{2^{m_2}\cdot n_2})u + (\pi - \frac{2\pi}{2^{m_2}\cdot n_2(m_2-1)})v &= 2\pi \quad \Leftrightarrow \\ (\frac{n_1-2}{n_1} + \frac{2}{2^{m_1}\cdot n_1})t + (1 - \frac{2}{2^{m_1}\cdot n_1(m_1-1)})s &+ \\ (\frac{n_2-2}{n_2} + \frac{2}{2^{m_2}\cdot n_2})u + (1 - \frac{2}{2^{m_2}\cdot n_2(m_2-1)})v &= 2. \end{aligned}$$

wherefrom we get Diophantine equations

$$At + Bs + Cu + Dv = 2 \tag{5}$$

where

$$\begin{aligned} A &= \frac{2^{m_1}\cdot(n_1-2)+2}{2^{m_1}\cdot n_1} \\ B &= \frac{2^{m_1}n_1\cdot(m_1-1)-2}{2^{m_1}\cdot n_1(m_1-1)} \\ C &= \frac{2^{m_2}\cdot(n_2-2)+2}{2^{m_2}\cdot n_2} \\ D &= \frac{2^{m_2}n_2\cdot(m_2-1)-2}{2^{m_2}\cdot n_2(m_2-1)}. \end{aligned}$$

For different values of co-efficients A, B, C, D there are different Diophantine equations. For equation alone (5) the solution is [3], if $d = NZD(A, B, C, D) = 1$, if A, B, C, D are relatively simple.

2.2. **Tiling with Semi-Regular Equilateral Polygons $\mathcal{P}_{2n}$.** Let as analyze tiling a plane with two types of semi-regular polygons $\mathcal{P}_{2n}, m = 2, n \geq 2 \in \mathbb{N}$ which can be geometrically constructed [9]. Let angle $\delta = \frac{\pi}{2^m\cdot n}$. Let $\mathcal{P}_{2n_1}, n_1 \geq 2 \in \mathbb{N}$,and $\mathcal{P}_{2n_2}, n_2 \geq 2 \in \mathbb{N}$ be the given semi-regular polygons with its characteristic elements of internal angles being: $\alpha_1, \beta_1, \alpha_2, \beta_2$,

$$\mathcal{P}_{2n_1}\vdots\begin{Bmatrix} n_1, 2 \\ \delta_1 \end{Bmatrix}_{\alpha_1}^{\beta_1} \quad \text{i} \quad \mathcal{P}_{2n_2}\vdots\begin{Bmatrix} n_2, 2 \\ \delta_2 \end{Bmatrix}_{\alpha_2}^{\beta_2}.$$

Let us suppose that $n_2 \geq n_1$. For each hub in tiling a planewith semi-regular polygons $\mathcal{P}_{2n_1}$ i $\mathcal{P}_{2n_2}$ the following relation applies,

$$\alpha_1 \cdot t + \beta_1 \cdot s + \alpha_2 \cdot u + \beta_2 \cdot v = 2\pi,$$

and integers $t, s, u, v \geq 0$ are not all simultaneously equal to zero. If we change values of internal angels $\alpha_1, \beta_1, \alpha_2, \beta_2$ and $m = 2$, we then have that

$$\begin{aligned}
(6) \qquad \alpha_1 \cdot t &+ \beta_1 \cdot s + \alpha_2 \cdot u + \beta_2 \cdot v = 2\pi \\
(\frac{(n_1-2)\pi}{n_1} + 2\delta_1)t &+ (\pi - 2\delta_1)s \\
+(\frac{(n_2-2)\pi}{n_2} + 2\delta_2)u &+ (\pi - 2\delta_i)v = 2\pi \\
(\frac{(n_1-2)\pi}{n_1} + \frac{2\pi}{2^m \cdot n_1})t &+ (\pi - \frac{2\pi}{2^m \cdot n_1})s \\
&+ (\frac{(n_2-2)\pi}{n_2} + \frac{2\pi}{2^m \cdot n_2})u + (\pi - \frac{2\pi}{2^m \cdot n_2})v = 2\pi \\
(\frac{n_1-2}{n_1} + \frac{1}{2 \cdot n_1})t &+ (1 - \frac{1}{2 \cdot n_1})s + (\frac{n_2-2}{n_2} \\
&+ \frac{1}{2 \cdot n_2})u + (1 - \frac{1}{2 \cdot n_2})v = 2 \\
\frac{2(n_1-2)+1}{2n_1}t &+ \frac{(2n_1-1)}{2n_1}s + \frac{2(n_2-2)+1}{2n_2}u + \frac{2n_2-1}{2n_2}v = 2.
\end{aligned}$$

By multiplying these relations by $4n_1n_2$, the calculation gives the following equation

$$(7) \quad (2n_1n_2 - 3n_2)t + (2n_1n_2 - n_2)s + (2n_1n_2 - 3n_1)u + (2n_1n_2 - n_1)v = 4n_1n_2.$$

Equation (7) represents Diophantine equation in which unknowns are t, s, u, v natural numbers $n_1, n_2 \geq 2$. By not losing anything at the generality of it, let us assume that $n_2 > n_1$. For the sake of keeping the solution of the equation simple, let us put that

$$A = 2n_1n_2 - 3n_2, B = 2n_1n_2 - n_2, C = 2n_1n_2 - 3n_1, D = 2n_1n_2 - n_1.$$

Equation (7) transforms into:

$$(8) \qquad At + Bs + Cu + Dv = 4n_1n_2.$$

In order to solve equation (8), observe that $C - A = 3(n_2 - n_1)$,$C = A + 3(n_2 - n_1)$. Similarly we can see that $D = B + (n_2 - n_1)$. If this is changeed to (8), the folowing equation is obtained:

$$(9) \qquad At + Bs + (A + 3(n_2 - n_1))u + (B + (n_2 - n_1))v = 4n_2n_1.$$

If we introduce transformations

$$(10) \qquad t + u = t', s + v = s'$$

equation (9) then transforms into:

$$At' + Bs' + 3(n_2 - n_1)u + (n_2 - n_1)v = 4n_1n_2$$
$$At' + Bs' + (n_2 - n_1)(u + v) + 2(n_2 - n_1)u = 4n_1n_2$$
$$\text{stavimo li da je} \quad u + v = u' \text{ tada je}$$
$$At' + Bs' + (n_2 - n_1)u' + 2(n_2 - n_1)u = 4n_1n_2$$

and from this relation we get the unknown:

$$u = \frac{2n_1n_2}{n_2 - n_1} - \frac{A}{2(n_2 - n_1)}t' - \frac{B}{2(n_2 - n_1)}s' - \frac{u'}{2} \tag{11}$$

From transformation $u + v{=}u'$ we get variable v:

$$v = \frac{A}{2(n_2 - n_1)}t' + \frac{B}{2(n_2 - n_1)}s' + 3\frac{u'}{2} - \frac{2n_1n_2}{n_2 - n_1} \tag{12}$$

If we first insert the obtained value for u into transformation $t + u{=}t'$ we get variable t :

$$t = (1 + \frac{A}{2(n_2 - n_1)})t' + \frac{B}{2(n_2 - n_1)}s' + \frac{u'}{2} - \frac{2n_1n_2}{n_2 - n_1}, \tag{13}$$

and then form transformation $s + v{=}s'$ we get variable s :

$$s = -\frac{A}{2(n_2 - n_1)}t' + (1 - \frac{B}{2(n_2 - n_1)})s' - 3\frac{u'}{2} + \frac{2n_1n_2}{n_2 - n_1} \tag{14}$$

We determine non-negative integers $t', s'u'$ from condition $t, s, u, v \geq 0$, based on what we form the following system of equations:

$$\begin{aligned}
(1 + \frac{A}{2(n_2 - n_1)})t' + \frac{B}{2(n_2 - n_1)}s' + \frac{u'}{2} - \frac{2n_1n_2}{n_2 - n_1} &\geq 0 \\
-\frac{A}{2(n_2 - n_1)}t' + (1 - \frac{B}{2(n_2 - n_1)})s' - 3\frac{u'}{2} + \frac{2n_1n_2}{n_2 - n_1} &\geq 0 \\
-\frac{A}{2(n_2 - n_1)}t' - \frac{B}{2(n_2 - n_1)}s' - \frac{u'}{2} + \frac{2n_1n_2}{n_2 - n_1} &\geq 0 \\
\frac{A}{2(n_2 - n_1)}t' + \frac{B}{2(n_2 - n_1)}s' + 3\frac{u'}{2} - \frac{2n_1n_2}{n_2 - n_1} &\geq 0.
\end{aligned} \tag{15}$$

This system of linear inequations is solved by Fourier-Matzkin method [5, 6, 8]. Let us first transform the system with elementary methods into the following form:

$$\begin{aligned}
u' &\geq \frac{4n_1n_2}{n_2 - n_1} - \frac{B}{2(n_2 - n_1)}s' - 2(1 + \frac{A}{2(n_2 - n_1)})t' \\
u' &\geq \frac{4n_1n_2}{3(n_2 - n_1)} - \frac{B}{3(n_2 - n_1)}s' + \frac{A}{3(n_2 - n_1)}t' \\
u' &\leq \frac{4n_1n_2}{3(n_2 - n_1)} + \frac{2}{3}(1 - \frac{B}{2(n_2 - n_1)})s' - \frac{A}{3(n_2 - n_1)}t' \\
u' &\leq \frac{4n_1n_2}{n_2 - n_1} - \frac{B}{n_2 - n_1}s' - \frac{A}{n_2 - n_1}
\end{aligned} \tag{16}$$

This system of inequations is equivalent to this system:

$$\begin{aligned}
&\max\{\frac{4n_1n_2}{n_2-n_1} - \frac{B}{2(n_2-n_1)}s' - 2(1+\frac{A}{2(n_2-n_1)})t';\\
&\qquad \frac{4n_1n_2}{3(n_2-n_1)} - \frac{B}{3(n_2-n_1)}s' + \frac{A}{3(n_2-n_1)}t'\}\\
&\le u' \le \min\{\frac{4n_1n_2}{3(n_2-n_1)} + \frac{2}{3}(1-\frac{B}{2(n_2-n_1)})s' - \frac{A}{3(n_2-n_1)}t';\\
&\qquad \frac{4n_1n_2}{n_2-n_1} - \frac{B}{n_2-n_1}s' - \frac{A}{n_2-n_1}t'\}\\
&\max\{\frac{4n_1n_2}{n_2-n_1} - \frac{B}{2(n_2-n_1)}s' - 2(1+\frac{A}{2(n_2-n_1)})t';\\
&\qquad \frac{4n_1n_2}{3(n_2-n_1)} - \frac{B}{3(n_2-n_1)}s' + \frac{A}{3(n_2-n_1)}t'\} \le\\
&\min\{\frac{4n_1n_2}{3(n_2-n_1)} + \frac{2}{3}(1-\frac{B}{2(n_2-n_1)})s' - \frac{A}{3(n_2-n_1)}t';\\
&\qquad \frac{4n_1n_2}{n_2-n_1} - \frac{B}{n_2-n_1}s' - \frac{A}{n_2-n_1}t'\}.
\end{aligned}$$

The last inequation from the given system is equivalent to the following system of inequations:

$$\begin{aligned}
&\frac{4n_1n_2}{n_2-n_1} - \frac{B}{2(n_2-n_1)}s' - 2(1+\frac{A}{2(n_2-n_1)})t'\\
&\le \frac{4n_1n_2}{3(n_2-n_1)} + \frac{2}{3}(1-\frac{B}{2(n_2-n_1)})s' - \frac{A}{3(n_2-n_1)}t'\\
&\frac{4n_1n_2}{n_2-n_1} - \frac{B}{2(n_2-n_1)}s' - 2(1+\frac{A}{2(n_2-n_1)})t'\\
&\le \frac{4n_1n_2}{n_2-n_1} - \frac{B}{n_2-n_1}s' - \frac{A}{n_2-n_1}t'\\
&\frac{4n_1n_2}{3(n_2-n_1)} - \frac{B}{3(n_2-n_1)}s' + \frac{A}{3(n_2-n_1)}t'\\
&\le \frac{4n_1n_2}{3(n_2-n_1)} + \frac{2}{3}(1-\frac{B}{2(n_2-n_1)})s' - \frac{A}{3(n_2-n_1)}t'\\
&\frac{4n_1n_2}{3(n_2-n_1)} - \frac{B}{3(n_2-n_1)}s' + \frac{A}{3(n_2-n_1)}t'\\
&\le \frac{4n_1n_2}{n_2-n_1} - \frac{B}{n_2-n_1}s' - \frac{A}{n_2-n_1}t'.
\end{aligned}$$

After simplifying all of those inequations, we get the simplified form of the latter system, as follows:

$$\begin{aligned}
(A+3(n_2-n_1))t' - (n_2-n_1-B)s' - 4n_1n_2 &\ge 0\\
t' &\ge 0\\
At' - (n_2-n_1)s' &\le 0\\
2At' + Bs' - 4n_1n_2 &\le 0
\end{aligned}$$

Since $A = 2n_1n_2 - 3n_2$ i $B = 2n_1n_2 - n_2$, we get that the latter system is equivalent to the following system of inequations:

$$\begin{aligned} n_1(2n_2-3)t' - (2n_2(1-n_1)-n_1)s' - 4n_1n_2 &\geq 0 \\ t' &\geq 0 \\ (2n_1n_2-3n_2)t' - (n_2-n_1)s' &\leq 0 \\ 2(2n_1-3)t' + (2n_1-1)s' - 4n_2 &\leq 0 \end{aligned}$$

Based on that, the system of inequations in (16) is equivalent to the following system:

$$\begin{aligned} (17) \qquad & \max\{\frac{4n_1n_2}{n_2-n_1} - \frac{B}{2(n_2-n_1)}s' - 2(1+\frac{A}{2(n_2-n_1)})t'; \\ & \frac{4n_1n_2}{3(n_2-n_1)} - \frac{B}{3(n_2-n_1)}s' + \frac{A}{3(n_2-n_1)}t'\} \\ \leq u' \leq & \min\{\frac{4n_1n_2}{3(n_2-n_1)} + \frac{2}{3}(1-\frac{B}{2(n_2-n_1)})s' - \frac{A}{3(n_2-n_1)}t'; \\ & \frac{4n_1n_2}{n_2-n_1} - \frac{B}{n_2-n_1}s' - \frac{A}{n_2-n_1}t'\} \\ n_1(2n_2-3)t' - (2n_2(1-n_1)-n_1)s' - 4n_1n_2 &\geq 0 \\ t' &\geq 0 \\ (2n_1n_2-3n_2)t' - (n_2-n_1)s' &\leq 0 \\ 2(2n_1-3)t' + (2n_1-1)s' - 4n_2 &\leq 0 \end{aligned}$$

From the last three inequations we determine the domain of values for t', s', and from the first one, the domain of values for u'. The domain of values for variables t' and s' is determined from the last three inequations by using a graphical method. After determining the values of non-negative integers t', s' from the first inequation, let us now determine the value of non-negative integer u'.Based on the calculated values from syste$t + u{=}t', s + v{=}s', u + v{=}u'$ we determine the values of non-negative integerst, s, u, v which meet the initial Diophantine equation. The set of all solutions of that equation represent the set of all different hubs which appear when tiling a plane with semi-regular polygons.

2.3. **Tiling a plane with semi-regular quadrilaterals and hexagons.** Let us consider tiling a plane with semi-regular equilateral quadrilaterals $\mathcal{P}_4$, (when $n_1 = 2, m = 2$) and semi-regular hexagons $\mathcal{P}_6$, (when $n_2 = 3, m = 2$ and angle δ). In this case appropriate Diophantine equation is

$$(18) \qquad 3t + 9s + 6u + 10v = 24$$

Let us determine all non-negative integer solutions of that equation. Since $3, 9, 6, 10$ are relatively simple, therefore $NZD(3, 9, 6, 10)$ =d= 1 and $d|24$ equation has a solution. We can notice that the given equation can be written in the form of:

$$\begin{aligned} 3t + (3\cdot 3+1)v + 6u + (1\cdot 6+3)s &= 24 \\ 3(t+3v) + (v+s) + 6(u+s) + 2s &= 24. \end{aligned}$$

Let us put that

$$t+3v=t' \quad \text{i} \quad u+s=u' \quad \text{i} \quad v+s=s' \tag{19}$$

The last equation transforms into

$$3t'+s'+6u'+2s=24. \tag{20}$$

Form that equation we can get that

$$s=-\frac{3}{2}t'-\frac{1}{2}s'-3u'+12 \tag{21}$$

If we put the obtained values for s first into this transformation: $u+s=s'$, and then into this transformation: $v+s=s'$ form (19) we get that:

$$u=\frac{3}{2}t'+\frac{1}{2}s'+4u'-12, \tag{22}$$

and then

$$v=\frac{3}{2}t'+\frac{3}{2}s'+3u'-12 \tag{23}$$

and, lastly, from relation $t+3v=t'$ we get that:

$$t=-t'-\frac{9}{2}s'-9u'+36 \tag{24}$$

Equations (21-24) provide solutions for Diophantine equation (18), depending on variables $t', s;, u'$. In order to determine the values of non-negative integers (t,s,u,v), let us first determine the domain of definition for non-negative variables $t', s'.u'$ from the condition $t,s,u,v\geq 0$. Based on this requiremen and equations (21-24), we get the following system of inequations:

$$\begin{aligned}
-t'-\frac{9}{2}s'-9u'+36 &\geq 0\\
-\frac{3}{2}t'-\frac{1}{2}s'-3u'+12 &\geq 0\\
\frac{3}{2}t'+\frac{1}{2}s'+4u'-12 &\geq 0\\
\frac{3}{2}t'+\frac{3}{2}s'+3u'-12 &\geq 0.
\end{aligned}$$

Let us write this system of inequations in the form of:

$$\begin{aligned}
t' &\leq -\frac{9}{2}s'-9u'+36\\
t' &\leq -\frac{1}{3}s'-2u'+8\\
t' &\geq -\frac{1}{3}s'-\frac{8}{3}u'+8\\
t' &\geq -s'-2u'+8.
\end{aligned}$$

This system of inequations is equivalent to this system:

$$max\{-\frac{1}{3}s' - \frac{8}{3}u' + 8, -s' - 2u' + 8\} \leq$$
$$t' \leq min\{-\frac{9}{2}s' - 9u' + 36, -\frac{1}{3}s' - 2u' + 8\}$$
$$max\{-\frac{1}{3}s' - \frac{8}{3}u' + 8, -s' - 2u' + 8\} \leq$$
$$min\{-\frac{9}{2}s' - 9u' + 36, -\frac{1}{3}s' - 2u' + 8\}.$$

Inequation:

$$max\{-\frac{1}{3}s' - \frac{8}{3}u' + 8, -s' - 2u' + 8\} \leq min\{-\frac{9}{2}s' - 9u' + 36, -\frac{1}{3}s' - 2u' + 8\}$$

is equivalent to this system of inequations:

$$\begin{aligned} -\frac{1}{3}s' - \frac{8}{3}u' + 8 &\leq -\frac{9}{2}s' - 9u' + 36 \\ -\frac{1}{3}s' - \frac{8}{3}u' + 8 &\leq -\frac{1}{3}s' - 2u' + 8 \\ -s' - 2u' + 8 &\leq -\frac{9}{2}s' - 9u' + 36 \\ -s' - 2u' + 8 &\leq -\frac{1}{3}s' - 2u' + 8. \end{aligned}$$

From this, after calculation, we get the folloing system of linear inequations:

$$\begin{aligned} 25s' + 38u' - 168 &\leq 0 \\ s' + 2u &\leq 0 \\ s' &\geq 0 \\ u' &\geq 0. \end{aligned}$$

Based on this system of inequations, from which we got the values for t', s', u', the following applies:

$$max\{-\frac{1}{3}s' - \frac{8}{3}u' + 8, -s' - 2u' + 8\} \leq t'$$
$$\leq min\{-\frac{9}{2}s' - 9u' + 36, -\frac{1}{3}s' - 2u' + 8\}$$
$$\begin{aligned} 25s' + 38u' - 168 &\leq 0 \\ s' + 2u &\leq 0 \\ s' &\geq 0 \\ u' &\geq 0. \end{aligned}$$

The last four inequations are solved graphically, and the values of integers for variables s', u' are determined.
The cut off domain is given in (Figure 2) and it represents a domain in which the values of non-negative integers of variables u' and s' are found. Figure 2 shows that $u' \in \{0, 1, 2, 3, 4\}$ and $s' \in \{0, 1, 2, 3, 4, 5, 6\}$ are the pairs of non-negative integers (u', s'), which meet the last four inequations of the system, and which are shown as points in the innner part of the polygon.

For the given values of u' and s' from inequation $max\{-\frac{1}{3}s' - \frac{8}{3}u' + 8, -s' - 2u' + 8\} \leq t' \leq min\{-\frac{9}{2}s' - 9u' + 36, -\frac{1}{3}s' - 2u' + 8\}$ let us determine the values of non-negative integers for variable t'. The obtained non-negative values of variables t', s', u' are show in the Table below, (Table 1).

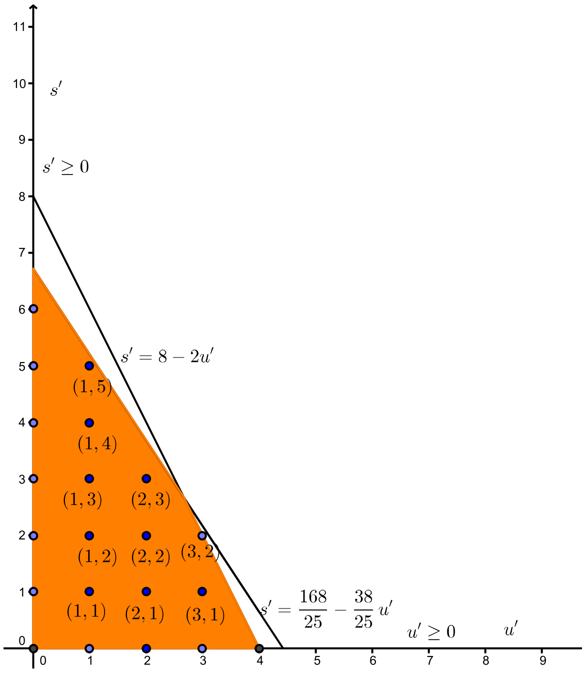

FIGURE 2. Number of non-negative values of variables u' and s'

Table.1. Non-negative values of variables t', s', u'.

t'	8	7	6	6	4	2	0	0
s'	0	3	6	0	0	0	0	2
u'	0	0	0	1	2	3	0	3

If we put the given values from Table 1 into equation (19), respectively, then, by solving the system of equations:

$$\begin{aligned} t + 3v &= t' \\ v + s &= s' \\ u + s &= u' \end{aligned}$$

for each choice of corresponding variables t', s', u', we get the non-negative integer values of constants t, s, u, v which meet Diophantine equation $3t + 9s + 6u + 10v = 24$. For example, for $t' = 8, s' = 0, u' = 0$, the system of equation reads:

$$\begin{aligned} t + 3v &= 8 \\ v + s &= 0 \\ u + s &= 0. \end{aligned}$$

From this system, we get that $t = 8 - 3u, s = -u, v = u$ i.e. the four items that satisfy the system are $(t, s, u, v) = (8 - 3u, -u, u, u) = (8, 0, 0, 0) +$

$u(-3,-1,1,1), u \in \mathbb{Z}$. If we put those values into equation $3t+9s+6u+10v = 24$ we get that $u = 0$. Therefore, the non-negative solution of Diophantine equation is $(t,s,u,v) = (8,0,0,0)$. Similarly, for $t' = 6, s' = 6, u' = 0$ the solution of the system is:

$$\begin{aligned} t + 3v &= 6 \\ v + s &= 6 \\ u + s &= 0. \end{aligned}$$

is $s = -u, v = u + 6, t = -12 - 3u, \quad u \in \mathbb{Z}^+$. When we put those values into Diophantine equation, we get that $u = 0$, but the obtained value $t = -12$ does not meet the non-negative requirement, therefore, that solution is rejected.
Similarly, let us check the remaining values of variables t', s', u'. After checking, we find that the set of the non-negative solutions of Diophantine equation, $3t + 9s + 6u + 10v = 24$ for the given values of angle δ :

$$\mathcal{R} = \{(8,0,0,0),(6,0,1,0),(4,0,2,0),(2,0,3,0),(0,0,4,0),(0,2,1,0)\}.$$

Based on that, we can conclude that six different nodes appear when tiling a plane with semi-regular quadrilaterals $\mathcal{P}_4$ and hexagons$\mathcal{P}_6$, with given angle δ.

Example 2.1. *As for the second example of tiling a plane with semi-regular equilateral quadrilaterals and hexagons, let us use the polygons with the following characteristic elements:*

$$\mathcal{P}_4 \vdots \left\{ \begin{matrix} 2,2 \\ \frac{\pi}{6} \end{matrix} \right\}_{60}^{120}; \mathcal{P}_6 \vdots \left\{ \begin{matrix} 3,2 \\ \frac{\pi}{4} \end{matrix} \right\}_{150}^{90}.$$

If we calculate coefficients A, B, C, D for the given elements, we get that $A = \frac{1}{3}, B = \frac{2}{3}, C = \frac{5}{6}, D = \frac{1}{2}$, and we get Diophantine equation

$$2x + 4y + 5z + 3t = 12. \tag{25}$$

Let us determine all solutions for this equation.

$$\begin{aligned} 2x + 4y + 5z + 3t &= 12 \\ 2x + 5z + 4y + 3t &= 12 \\ 2(x + 2z) + 3(y + t) + y + z &= 12. \end{aligned}$$

If we put that $x + 2z = x'$ and $y + t = y'$ the last equation can be written in the form of

$$\begin{aligned} 2x' + 3y' + y + z &= 12 \\ y + z &= 12 - 2x' - 3y' \quad \text{stavimo li da je z=z' nalazimo da je} \\ y &= 12 - 2x' - 3y' - z' \end{aligned}$$

Given the introduced shifts, we find that the solution of the equation is given by the following:

$$\begin{aligned} y &= 12 - 2x' - 3y' - z' \\ t &= 2x' + 4y' + z' - 12 \\ x &= x' - 2z' \\ z &= z'. \end{aligned} \tag{26}$$

The value of non-negative variables x', y', z' is determined from requirement $x \geq 0, y \geq 0, z \geq 0, t \geq 0$, or

$$(27)\qquad \begin{aligned} 12 - 2x' - 3y' - z' &\geq 0 \\ 2x' + 4y' + z' - 12 &\geq 0 \\ x' - 2z' &\geq 0 \\ z' &\geq 0. \end{aligned}$$

This system of inequations is equivalent to this system:

$$(28)\qquad \begin{aligned} max\{0, 12 - 2x' - 4y'\} \leq z' &\leq min\{\frac{x'}{2}, 12 - 2x' - 3y'\} \\ max\{0, 12 - 2x' - 4y'\} &\leq min\{\frac{x'}{2}, 12 - 2x' - 3y'\}. \end{aligned}$$

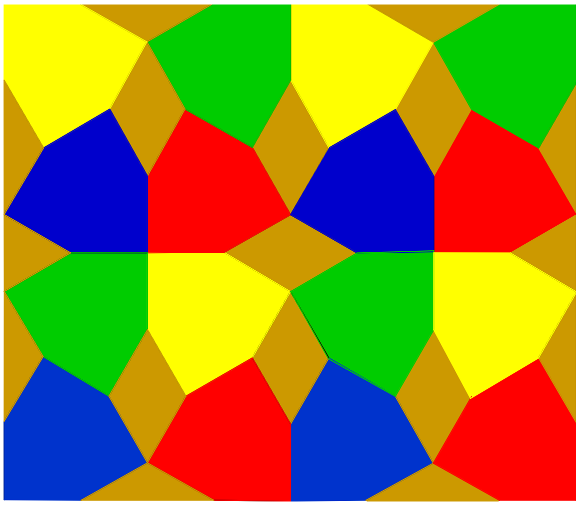

Figure 3. Tiling a plane with convex semi-regular polygons $\mathcal{P}_4, \mathcal{P}_6$.

That system of inequations is equivalent to this system:

$$(29)\qquad \begin{aligned} max\{0, 12 - 2x' - 4y'\} \leq z' &\leq min\{\frac{x'}{2}, 12 - 2x' - 3y'\} \\ x' &\geq 0 \\ y' &\geq 0 \\ 2x' + 3y' &\geq 0 \\ 5x' + 8y' &\geq 0. \end{aligned}$$

From the last four inequations, we determine the integer values of variables x', y', and from the first inequation, we determine the integer value of variable z'. We get the following set of non-negative variables

$$\begin{aligned} \mathcal{R}_1 &= \{(x', y', z')|(2,2,1),(2,2,0),(3,2,0), \\ &, \quad (4,1,0),(4,1,1),(5,0,2),(6,0,0),(0,3,0),(0,4,0)\}. \end{aligned}$$

If we put those variables into 5.9, after calculation, we get all non-negative integer solutions of Diophantine equation $2x+4y+5z+3t = 12$, which are the following: $\mathcal{R}_u = \{(x, y, z, t)|(0,1,1,1),(2,2,0,0),(3,0,0,2),(4,1,0,0),(2,0,1,1),$ $(1,0,2,0),(6,0,0,0),(0,3,0,0),(0,0,0,4)\}$.

This example of tiling a plane is given in Figure3.
An example of tiling a plane with semi-regular convex and non-convex equilateral polygons is given in Figure 4.

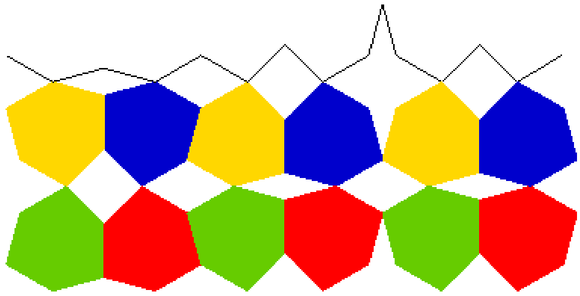

FIGURE 4. Tiling a plane with $\mathcal{P}_4, \mathcal{P}_6$ and the non-convex semi-regular polygons

3. CONCLUSION

Tiling a plane with semi-regular equilateral polygons when two or more semi-regular polygons meet at vertices is a problem that can be observed in other cases as well as in the here presented cases. These types of tiling a plane are special cases and can be observed in the case when semi-regular equilateral polygons of different types and different sides meet in one node, which has not been considered in this paper, as well as in cases when more than two semi-regular polygons of equal sides meet in the nodes. The considered case of tiling a plane with two types of semi-regular equilateral polygons with 2n-sides showed that a plane can be tiled with two types of semi-regular polygons with 2n-sides only in case of tiling a plane with semi-regular quadrilaterals and semi-regular hexagons. Tiling a plane with two types of semi-regular polygons can also be considered, among which are the convex and non-convex semi-regular polygons. An example of such tiling a plane is shown in Figure 4.

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