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Rational Triangles as a Bridge Between Geometry and Number Theory

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Abstract. The purpose of this article is twofold. Firstly, we show how the search for particular pairs of rational triangles which share two symmetric invariants leads to the problem of determining the set of rational points on some algebraic curve. Using deep theorems from the field of arithmetic geometry, we determine whether the latter set is finite or infinite. Moreover, in precisely two cases, we succeed in determining the set of rational points explicitly. Secondly, we introduce parameterizations for the side lengths of rational triangles which have an angle of 60° or an angle of 120°.

1. Introduction

The semiperimeter s, the inradius r and the circumradius R are the symmetric invariants of a triangle. For fixed $s, r, R \in \mathbb{R}^+$, satisfying the so-called Blundon's inequalities (see [1] for a geometric proof of these), there is a unique triangle having semiperimeter s, inradius r and circumradius R, up to congruence. Any symmetric polynomial expression in the elements of a triangle, such as side lengths or functions of the angles, can be expressed as a polynomial in s, r and R.

Given three distinct points I, H and O in the plane, Euler gave a solution to the so-called "triangle determination problem", i.e. the construction of a triangle which has these points as incenter, orthocenter and circumcenter, respectively (see for example [11]). Determining the invariants s, r and R is an important intermediary step in this construction.

From Blundon's inequalities, it is not hard to deduce that there are infinitely many non-similar triangles with rational side lengths having two of the corresponding symmetric invariants equal. However, the search for pairs of some particular triangles which have rational (or integral) side lengths and share some invariant(s) sparked the interests of number theorists, since these pairs are parameterized by rational points on algebraic curves. One such example can be found in previous work of the authors together with Siksek [2], where deep tools from arithmetic geometry have been used to build on previous work of Hirawaka and Matsumura [7]. The next theorem follows from the main results of [2, 7].

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Theorem 1.1 ([7],[2]). *Up to similarity, there is a unique pair consisting of a rational right triangle and a rational isosceles triangle which have two of the corresponding symmetric invariants* $\{s, r, R\}$ *of equal length. These have the same semiperimeter and the same inradius and their side lengths are (377, 135, 352) and (366, 366, 132), respectively.*

Ultimately, the proofs of the results mentioned above consist in the resolution of some Diophantine equations. Finding convenient parameterizations for the side lengths of triangles in those families plays a crucial role in the process of deriving the aforementioned Diophantine equations.

In this short article, we deduce parameterizations for three different families of rational triangles. These are the triangles which have an angle of 60°, angle of 120° or rational isosceles triangles having some particular symmetric invariants. Starting with these parameterizations, we emphasize connections with fascinating number theory problems and we present the new Diophantine questions they generate.

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2. Rational triangles with an angle of 60°

Theorem 2.1. *All nondegenerated rational triangles with an angle of* 60° *have side lengths of the form*

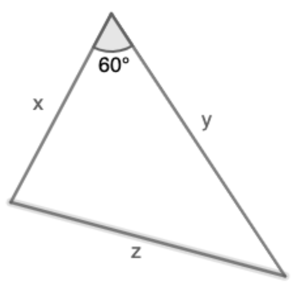
$$((1+2m-3m^2)k, 4mk, (1+3m^2)k) \tag{1}$$

where $k, m \in \mathbb{Q}^+$.

The area of such a triangle can be written as $K = m(1+2m-3m^2)k^2\sqrt{3}$, whereas its symmetric invariants are given by the formulas

$$\begin{aligned} s &= (1+3m)k, \\ r &= \tfrac{K}{s} = \tfrac{m(1+2m-3m^2)k\sqrt{3}}{1+3m}, \\ R &= \tfrac{xyz}{4K} = \tfrac{(3m^2+1)k}{\sqrt{3}}. \end{aligned} \tag{2}$$

Proof. Let $x, y, z \in \mathbb{Q}^+$ be the side lengths of the sought-after triangle, labeled such that the angle between x and y is 60°.



Using the cosine theorem, we find

$$z^2 = x^2 - xy + y^2. \tag{3}$$

Dividing by z^2, the latter is equivalent to

$$a^2 - ab + b^2 = 1, \tag{4}$$

which in turn can be written as

$$\left(\frac{2a-b}{2}\right)^2 + 3\left(\frac{b}{2}\right)^2 = 1, \tag{5}$$

which under the obvious substitution becomes an equation of the form

$$u^2 + 3v^2 = 1. \tag{6}$$

The equation (6) represents an ellipse in the coordinate system uOv and our initial problem has been translated into finding all rational points situated on the graph of this ellipse. It is known that ellipses are curves of genus 1, therefore if they contain one rational point, they contain infinitely many such. Moreover, in that case, the rational points can be parameterized. Let us explain how one obtains this parameterization.

Is is easy to see that the point $(u_0, v_0) = (-1, 0)$ lies on the ellipse (4). Notice that all rational points (u, v) on the ellipse must lie on lines that pass through the fixed point $(-1, 0)$ and have rational slopes. The bundle of such lies is described by all the points (u, v) satisfying the equation $v = m(u+1)$ for some $m \in \mathbb{Q}$. A point on the ellipse (6) which additionally lies on such a line must satisfy the equation $u^2 + 3r^2(u+1)^2 = 1$ and hence

$$(1+3m^2)u^2 + 6m^2u + 3m^2 - 1 = 0, \tag{7}$$

for some $m \in \mathbb{Q}$. If we regard (7) as an second degree equation in u, its roots are -1 and $\frac{1-3m^2}{1+3m^2}$. We just showed that all of the rational points different from $(u, v) \neq (-1, 0)$ on the ellipse (6) can be written as $(u, v) = \left(\frac{1-3m^2}{1+3m^2}, \frac{2m}{1+3m^2}\right)$, for some $m \in \mathbb{Q}$.

Replacing in the initial substitution

$$\frac{2a-b}{2} = \frac{1-3m^2}{1+3m^2} \text{ and } \frac{b}{2} = \frac{2m}{1+3m^2},$$

we get $a = \frac{1+2m-3m^2}{1+3m^2}$ and $b = \frac{4m}{1+3m^2}$, for some $m \in \mathbb{Q}$, hence the desired parameterization.

3. Rational triangles with an angle of 120°

Theorem 3.1. *All nondegenerated rational triangles with an angle of* 120° *have side lengths of the form* $((1-2m-3m^2)k, 4mk, (1+3m^2)k)$ *where* $k, m \in \mathbb{Q}^+$.

In terms of these parameters, the area of such a triangle can be written as $K = m(1-2m-3m^2)k^2\sqrt{3}$. The symmetric invariants are given by the formulas

$$s = (1+m)k,$$

$$r = \tfrac{K}{s} = m(1-3m)k\sqrt{3}, \tag{8}$$

$$R = \tfrac{xyz}{4K} = \tfrac{k(3m^2+1)}{\sqrt{3}}.$$

Proof. Let $x, y, z \in \mathbb{Q}^+$ be the side lengths of such a triangle, labeled such that the angle between the sides of length x and y is $120°$. Using the cosine theorem, we find that

$$z^2 = x^2 + xy + y^2. \tag{9}$$

Dividing by z^2, the latter is equivalent to

$$a^2 + ab + b^2 = 1, \tag{10}$$

for some $a, b \in \mathbb{Q}^+$, which is equivalent to

$$\left(\frac{2a+b}{2}\right)^2 + 3\left(\frac{b}{2}\right)^2 = 1. \tag{11}$$

Under the obvious substitution this gives rise again to the equation (6), namely

$$u^2 + 3v^2 = 1.$$

As in the proof of Theorem 2.1 we show that

$$\frac{2a+b}{2} = \frac{1-3m^2}{1+3m^2} \text{ and } \frac{b}{2} = \frac{2m}{1+3m^2},,$$

which yields that $a = \frac{1-3m^2-2m}{1+3m^2}$ and $b = \frac{4m}{1+3m^2}$, for some rational number m. The conclusion follows easily.

4. Rational points on curves of higher genus

For any bivariate polynomial $f \in \mathbb{Q}[X,Y]$, let $C_f := \{(x,y) \in \overline{\mathbb{Q}}^2 : f(x,y) = 0\}$ be an affine algebraic curve. The points of C_f with coordinates in $\mathbb{Q}$ are called rational and, in general, for any $S \subseteq \overline{\mathbb{Q}}$, we denote by $C_f(S) = C_f \cap S^2$. Curves can be classified by their genus, a non-negative integer associated to their projectivization. The genus is a geometric invariant. A very deep result in number theory is the following theorem.

Theorem 4.1 (Faltings, 1983)**.** *If $f \in \mathbb{Q}[X,Y]$ defines an irreducible curve C_f of genus $g(C_f) \geq 2$, then $C_f(\mathbb{Q})$ is finite.*

Although Faltings' theorem represents a milestone in number theory, the result is "ineffective", which means that the proof of the theorem does not even allow one to control the size of the set $C_f(\mathbb{Q})$ known to be finite. Therefore it cannot be used to explicitly determine $C_f(\mathbb{Q})$. Effectively finding rational points on curves is an incredible difficult task and a very active topic of research.

For every smooth, projective and absolutely irreducible curve C of genus g defined over $\mathbb{Q}$, the Jacobian J_C is a g-dimensional abelian variety, *functorially* associated to C. Fixing a point $P_0 \in C(\mathbb{Q})$, the curve C can be identified as a subvariety of J_C via the Abel-Jacobi map $\iota : C \hookrightarrow J_C$ with base point P_0. The famous Mordell-Weil theorem gives that, as is the case for elliptic curves, the set of $\mathbb{Q}$ rational points of J_C has the structure of a finitely generated abelian group, i.e.

$$J_C(\mathbb{Q}) \equiv T \oplus \mathbb{Z}^r,$$

where T is a finite abelian group and r is a positive integer, called the *rank*.

A famous theorem due to Chabauty and Coleman [5] is the following.

Theorem 4.2. *Let C be a smooth, projective and absolutely irreducible curve of genus g over $\mathbb{Q}$, with Jacobian J. Assume that the rank r of the Mordell-Weil group $J_C(\mathbb{Q})$ is strictly less than g. Then, there is an algorithm for determining the set of rational points $C(\mathbb{Q})$. Moreover, if p is a prime of good reduction for C such that $p > 2g$, then*

$$\#C(\mathbb{Q}) \leq \overline{C}(\mathbb{F}_p) + 2g - 2.$$

Here we denoted by $\overline{C}$ the curve obtained by reducing modulo p the coefficients of the equation defining C.

We will apply the theorem above to show that there is no pair of rational triangles consisting of one triangle with angle of 120 and an isosceles triangle which have the same perimeter and the same area (equivalently, the same semiperimeter and the same inradius). We show in Theorem 6.2 that these are parameterized by rational points on a hyperelliptic curve of genus 2, whose Jacobian variety has rank 1. The algorithm mentioned in Theorem 4.2 will be applied to find these rational points.

A beautiful presentation the method of Chabauty and Coleman works can be found in the expository article of McCallum and Poonen [8] and an algorithm suitable to the set-up that we encounter in the proof of Theorem 6.2 is implemented in `Magma`.

A very brief summary of the ideas that make the method work is as follows. For any prime p of good reduction for C^{proj}, the closure of $J_C(\mathbb{Q})$ in $J_C(\mathbb{Q}_p)$ (under the p-adic topology) can be described as the locus where certain power series vanish. It turns out that, under a natural embedding, the image of C^{proj} in J_C meets this closure in a finite set, a set that must contain $C^{\mathrm{proj}}(\mathbb{Q})$.

A description of the `Magma` implementation of the aforementioned algorithm can be found at

`https://magma.maths.usyd.edu.au/magma/handbook/text/1507`.

5. Pairs of triangles sharing two symmetric invariants

In this section we derive three Diophantine equations whose rational solutions are closely connected with pairs consisting of rational triangles, one with an angle of 60° and one with an angle of 120° which have two symmetric invariants of the same length. These pairs are considered modulo the equivalence relation of similarity (scaling). In Theorems 5.1-3 we prove whether, up to similarity, there are finitely or infinitely many such pairs.

Theorem 5.1. *Up to scaling, the pairs consisting of a rational triangle with an angle of 60° and a rational triangle with an angle of 120° which have the same semiperimeter and the same inradius are parameterized by a subset of the infinitely many rational points on an elliptic curve.*

Remark 5.1. *We point out that given a pair of triangles, having the same semiperimeter and the same inradius is equivalent to having the same perimeter and the same area.*

Proof. Suppose we have a pair of such triangles. From the formulas presented in (2) and (8) it follows that their parameterizations are given by positive rational numbers k, m, k', m' such that the following equations hold:

$$\begin{cases} (1+3m)k = (1+m')k' \text{ and} \\ \frac{m(1+2m-3m^2)k}{1+3m} = m'(1-3m')k' \end{cases}.$$

Scaling if necessary, we can assume that $k' = 1$. Substituting k from the first equation into the second, we obtain the equation

$$\frac{m(1+2m-3m^2)}{(1+3m)^2} + m'\left(\frac{m(1+2m-3m^2)}{(1+3m)^2} - 1\right) + 3m'^2 = 0.$$

Regarding the last equation as a quadratic in m', this has rational solutions if and only if its discriminant

$$\frac{m^4 + 40m^3 - 18m^2 - 8m + 1}{(3m+1)^2}$$

is a rational square. That happens if and only if the numerator is a rational square, i.e. if there exists $y \in \mathbb{Q}$ such that

$$y^2 = m^4 + 40m^3 - 18m^2 - 8m + 1.$$

The projective closure of the affine curve defined by the latter equation is an elliptic curve. Using the computer algebra package `Magma`, we show that this elliptic curve is isomorphic to the one given by the equation

$$E : y^2 + xy + y = x^3 - x^2 - 2x.$$

It is recorded in [9] (label 99.a2) that the Mordell-Weil group $E(\mathbb{Q})$ is isomorphic to

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.$$

Every rational point on E pulls back to a rational point on (m, y) on our curve.

□

Theorem 5.2. *There are no pairs consisting of a rational triangle with an angle of* $60°$ *and a rational triangle with an angle of* $120°$ *which have the same inradius and the same circumradius.*

Proof. Let k, m, k', m' be positive rationals parameterizing a pair of such triangles. Using the formulas (2) and (8), it follows that these parameters must satisfy the system of equations

$$\begin{cases} \frac{m(1+2m-3m^2)k}{1+3m} = m'(1-3m')k', \\ \\ (3m^2+1)k = k'(3m'^2+1). \end{cases}$$

Scaling, we can assume that $k' = 1$. Via the reduction method, we find that m, m' must satisfy the equation

$$\frac{3(2m^2+m+1)}{m(m-1)} \cdot m'^2 - \frac{3m^2+1}{m(m-1)} m' - 1 = 0.$$

This quadratic in m' has rational solutions if and only if its discriminant

$$\frac{33m^4 - 12m^3 + 6m^2 - 12m + 1}{(m-1)^2 m^2}$$

is a rational square. It follows that there exists $y \in \mathbb{Q}$ such that

$$y^2 = 33m^4 - 12m^3 + 6m^2 - 12m + 1.$$

Let C be the affine curve defined by the equation above. As in the proof of the previous theorem, the projective closure of C defines a hyperelliptic curve of genus 1. Using the computer algebra package `Magma`, we show that the latter is birationally equivalent to the elliptic curve given by the equation

$$E : y^2 - 12xy - 384y = x^3 + 60x^2 + 768x.$$

The same computer algebra package can be used to show that $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$ and generated by $P = (0,0)$. The points $\{P, 2\cdot P, \ldots, 5\cdot P, \infty\}$ on E pull back to

$$(m, y) \in \{(-1,8), (1,4), (1,-4), (-1,-8), (0,-1), (0,1)\}$$

on the curve C. As m must be a positive rational, we find that $m = 1$. This value of m does not correspond to a non-degenerate rational triangle with an angle of $60°$, since from (2) we see that the inradius of such a triangle would be equal to 0. The conclusion follows. □

A slightly different phenomena occurs when we restrict our attention to pairs of such triangles which have the same perimeter and the same circumradius. The next theorem shows that these pairs are parameterized by infinitely many points on an elliptic curve.

Theorem 5.3. *Up to similarity, the pairs consisting of a rational triangle with an angle of* $60°$ *and a rational triangle with an angle of* $120°$ *which have the same semiperimeter and the same circumradius are parameterized by a subset of the infinitely many rational points on an elliptic curve.*

Proof. Let k, m, k', m' be positive rationals parameterizing a pair of such triangles. Using again the equations (2) and (8), we get that these parameters must satisfy simultaneously the following equations:

$$\begin{cases} (1+3m)k = (1+m')k' \text{ and} \\ (3m^2+1)k = (3m'^2+1)k' \end{cases}.$$

Since we work under the equivalence relation of similarity, we can scale everything to ensure that $k' = 1$. Replacing k from the first equation into the second and using the fact that $1 + 3m$ is not zero for non-degenerate triangles, we get

$$3m'^2 - \frac{3m^2+1}{3m+1}\cdot m' - \frac{3(m-1)m}{3m+1} = 0.$$

As before, this quadratic in m' has rational solutions if and only if its discriminant

$$\frac{9m^4 + 108m^3 - 66m^2 - 36m + 1}{(3m+1)^2}$$

is a rational square. The latter holds if and only if the numerator is a rational square, i.e. there exists $y \in \mathbb{Q}$ such that

$$y^2 = 9m^4 + 108m^3 - 66m^2 - 36m + 1.$$

The last equation defines an affine curve C which has rational points, for example $(m, y) = (0, 1)$. Sending one of these points to the point at infinity, one can find a birational map between the hyperelliptic curve above and an elliptic curve. Using the computer algebra package `Magma`, we show that this elliptic curve is isomorphic to the one given by the elliptic curve

$$E : y^2 - 36xy - 13824y = x^3 + 780x^2 + 152064x$$

It can be proved that recorded that the Mordell-Weil group

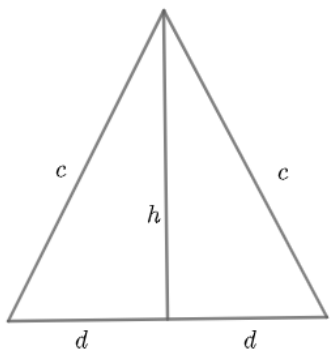
$$E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$$

and that the free part is generated by $P(-396, -432) \in E(\mathbb{Q})$. The infinitely many rational points on $E(\mathbb{Q})$ pull back to infinitely many points on C. Fixing a positive coordinate corresponding to m, we get two possible rational values for m'. Each one of these determines a value for k, hence we find such a pair of triangles. Of course, some of the obtained values for m, k, m' might give degenerate triangles.

□

6. Rational isosceles triangles

Let us write $(c, c, 2d)$, with $c, d \in \mathbb{Q}^+$ with $c > d$, for the triple of side lengths of a rational isosceles triangle.



In this section we will derive new Diophantine equations arising from the following geometric question.

Question. Up to similitude, how many pairs consisting of a rational isosceles triangle and a rational triangle with an angle of 60° having two corresponding symmetric invariants of equal length?

What can we say about the same question, if 60° is replaced by 120°?

From the formulas (2) and (8) it follows easily that if an isosceles triangle with rational side lengths shares two symmetric invariants with a rational triangle which has an angle of 60° or 120°, then the altitude h of the isosceles triangle must be of the form $l\sqrt{3}$, where $l \in \mathbb{Q}^+$.

If this is the case, we have that $d^2 + 3l^2 = c^2$. We encountered this ellipse in the proof of Theorem 2.1. Following the discussion in the aforementioned proof, it is easy to deduce that there are $\alpha, \beta \in \mathbb{Q}^+$ with $\alpha^2 < 1/\sqrt{3}$ such that

$$(c, c, 2d) = ((1 + 3\alpha^2)\beta, (1 + 3\alpha^2)\beta, 2(1 - 3\alpha^2)\beta). \tag{12}$$

The length of the altitude is $h = l\sqrt{3} = 2\alpha\beta\sqrt{3}$ and the symmetric invariants of such a triangle are given by the formulas

$$\begin{aligned} s &= 2\beta \\ r &= \tfrac{K}{s} = \tfrac{2\alpha(1-3\alpha^2)\beta^2\sqrt{3}}{2\beta} = \alpha(1 - 3\alpha^2)\beta\sqrt{3} \\ R &= \tfrac{xyz}{4K} = \tfrac{\beta^3(1+3\alpha^2)^2 2(1-3\alpha^2)}{8\alpha(1-3\alpha^2)\beta^2\sqrt{3}} = \tfrac{\beta(1+3\alpha^2)^2}{4\alpha\sqrt{3}} \end{aligned} \tag{13}$$

Theorem 6.1. *Up to scaling, there are finitely many pairs consisting of a rational triangle with an angle of* 60° *and a rational isosceles triangle, having the same semiperimeter and the same inradius.*

Proof. Let k, m, α, β be positive rationals parameterizing a pair of such triangles. Using the equations (2) and (13), we get that these parameters must satisfy simultaneously the following system of equations

$$\begin{cases} (1 + 3m)k = 2\beta, \\ \frac{m(1+2m-3m^2)k}{1+3m} = \alpha(1 - 3\alpha^2)\beta. \end{cases}$$

Scaling, we can assume that $\beta = 1$ so $k = \frac{2}{1+3m}$ from the first equation. Substituting this into the second equation, we get

$$-2(m - 1)m + (3m + 1)(3\alpha^3 - \alpha) = 0$$

We implicitly assumed that $1 + 3m \neq 0$, as the triangles we are looking for are non-degenerate.

Regarding the last equation as a quadratic in m, we deduce that it has rational solutions if and only if its discriminant

$$81\alpha^6 - 54\alpha^4 + 60\alpha^3 + 9\alpha^2 - 20\alpha + 4$$

is a rational square. Therefore, the values for α must give rational points on the affine curve

$$C : y^2 = 81\alpha^6 - 54\alpha^4 + 60\alpha^3 + 9\alpha^2 - 20\alpha + 4.$$

Using `Magma`, we computed that the genus of the projective closure of this curve is 2, therefore by Theorem 4.1 it has finitely many rational points. The conclusion follows.

□

In the next theorem we are able to go beyond Falting's theorem. We will use the full power of Theorem 4.2 to explicitly determine the set of rational points on a hyperelliptic curve of genus 2. Looking at the parameterizations, we will deduce that these points do not correspond to pairs of triangles.

Theorem 6.2. *There is no pair consisting of a rational triangle with an angle of* 120° *and a rational isosceles triangle, having the same semiperimeter and the same inradius.*

Proof. Let k, m, α, β be positive rationals parameterizing a pair of such triangles. Using the equations (8) and (13), we get that these parameters must satisfy simultaneously the following system of equations

$$\begin{cases} (1+m)k = 2\beta, \\ \\ m(1-3m)k = \alpha(1-3\alpha^2)\beta. \end{cases}$$

Scaling, we can assume that $k = 1$, hence $\beta = \frac{1+m}{2}$. Substituting this into the second equation, we get that

$$2m(1-3m) = \alpha(1-3\alpha^2)(1+m).$$

Looking at this equation as a quadratic in m, it follows that this has rational solutions if and only if its discriminant $9\alpha^6 - 6\alpha^4 + 84\alpha^3 + \alpha^2 - 28\alpha + 4$ is a rational square. That is, there exists $y \in \mathbb{Q}$ such that

$$y^2 = 9\alpha^6 - 6\alpha^4 + 84\alpha^3 + \alpha^2 - 28\alpha + 4. \tag{14}$$

The last equation defines the affine part of a projective hyperelliptic curve C of genus 2. Using `Magma`, we show that the rank of the Jacobian of C is equal to 1. Now, we will apply the algorithm intrinsic in Theorem 4.2 for determining the set $C(\mathbb{Q})$. This is implemented as the function `Chabauty()` in the same computer algebra package `Magma`. Calling this function, we compute that the rational points in $C(\mathbb{Q})$, represented in the weighted projective plane $\mathbb{P}^2_{(1,3,1)}$, are

$(1 : -3 : 0), (1 : 3 : 0), (0 : -2 : 1), (0 : 2 : 1), (1 : -8 : 1), (1 : 8 : 1), (7 : -2343 : 6)$, and $(7 : 2343 : 6)$.

The first two points on the list above (the points at infinity) are not visible on the affine model given by the equation (14). However, the other points correspond to pairs (α, y) given by $(0, -2)$, $(0, 2)$, $(1, -8)$, $(1, 8)$, $(7/6, -781/72)$ and $(7/6, 781/72)$ respectively. We see that none of the positive values for α, namely 1 and $7/6$ are less than or equal to $1/\sqrt{3}$ as required. In fact, if one plugs either one of these two values in the formulas (12) then one will obtain a side of negative length. The conclusion follows. □

Theorem 6.3. *Up to scaling, there are finitely many pairs consisting of a rational triangle with an angle of* 60° *and a rational isosceles triangle, having the same semiperimeter and the same circumradius.*

Proof. Let k, m, α, β be positive rationals parameterizing a pair of such triangles. Using the equations (2) and (13), we get that these parameters must satisfy simultaneously the following system of equations

$$\begin{cases} (1+3m)k = 2\beta, \\ \\ (3m^2+1)k = \frac{\beta(1+3\alpha^2)^2}{4\alpha}. \end{cases}$$

Scaling, we can assume that $k = 1$. It follows that $\beta = \frac{1+3m}{2}$ and substituting this into the second equation we get

$$3m^2 + 1 = \frac{(1+3m)(1+3\alpha^2)^2}{8\alpha}.$$

The above quadratic in m has rational solutions if and only if its discriminant

$$\frac{3(243\alpha^8 + 324\alpha^6 + 288\alpha^5 + 162\alpha^4 + 192\alpha^3 - 220\alpha^2 + 32\alpha + 3}{64\alpha^2}$$

is a rational square. Therefore, there must be $y \in \mathbb{Q}$ such that

$$y^2 = 3(243\alpha^8 + 324\alpha^6 + 288\alpha^5 + 162\alpha^4 + 192\alpha^3 - 220\alpha^2 + 32\alpha + 3.$$

The previous equation defines an affine curve whose projectivization has genus 3, hence the conclusion follows by Faltings' theorem.

□

Theorem 6.4. *Up to scaling, there are finitely many pairs consisting of a rational triangle with an angle of* 120° *and a rational isosceles triangle, having the same semiperimeter and the same circumradius.*

Proof. Let k, m, α, β be positive rationals parameterizing a pair of such triangles. Using the equations (8) and (13), we get that these parameters must satisfy simultaneously the following system of equations

$$\begin{cases} (1+m)k = 2\beta, \\ k(3m^2+1) = \frac{\beta(1+3\alpha^2)^2}{4\alpha} \end{cases}$$

As before, by scaling we can assume that $k = 1$. This gives that $\beta = \frac{1+m}{2}$, therefore replacing it into the second equation we have

$$3m^2 + 1 = \frac{(1+m)(1+3\alpha^2)^2}{8\alpha}.$$

This has rational solutions if and only if its discriminant

$$\frac{1 + 96\alpha - 756\alpha^2 + 576\alpha^3 + 54\alpha^4 + 864\alpha^5 + 108\alpha^6 + 81\alpha^8}{64\alpha^2}$$

is a rational square, i.e. there exists $y \in \mathbb{Q}^2$ such that

$$y^2 = 1 + 96\alpha - 756\alpha^2 + 576\alpha^3 + 54\alpha^4 + 864\alpha^5 + 108\alpha^6 + 81\alpha^8.$$

The conclusion follows from Falting's theorem, as before. This is because the projective closure of the affine curve given by the last equation defines a hyperelliptic curve of genus 3.

□

The last two theorems are similar to the ones above, but they are about pairs of triangles having the same inradius and the same circumradius.

Theorem 6.5. *Up to scaling, there are finitely many pairs consisting of a rational triangle with an angle of* 60° *and a rational isosceles triangle, having the same inradius and the same circumradius.*

Proof. Let k, m, α, β be positive rationals parameterizing a pair of such triangles. Using the equations (2) and (13), we get that these parameters must satisfy simultaneously the following system of equations

$$\begin{cases} m(1-m)k = \alpha(1-3\alpha^2)\beta, \\ \\ (3m^2+1)k = \frac{\beta(1+3\alpha^2)^2}{4\alpha}. \end{cases}$$

Scaling, we can assume that $\beta = 1$. Replacing k from the first equation into the second, we obtain

$$(3m^2+1)\cdot \alpha(1-3\alpha^2) - m(1-m)\cdot \frac{(1+3\alpha^2)^2}{4\alpha} = 0.$$

Regarding this as a a quadratic equation in m, we find, as before that its discriminant must be a rational square. This implies that α gives rise to a rational point on the affine curve

$$C : y^2 = -1215\alpha^8 + 1404\alpha^6 - 186\alpha^4 - 4\alpha^2 + 1.$$

The projective closure of C has genus 3. By Faltings' theorem, the latter contains only finitely many rational points.

□

Theorem 6.6. *Up to scaling, there are finitely many pairs consisting of a rational triangle with an angle of* $120°$ *and a rational isosceles triangle, having the same inradius and the same circumradius.*

Proof. Finally, let k, m, α, β be positive rationals parameterizing a pair of such triangles. Using the equations (8) and (13), we get that these parameters must satisfy simultaneously the following system of equations

$$\begin{cases} m(1-3m)k = \alpha(1-3\alpha^2)\beta, \\ k(3m^2+1) = \frac{\beta(1+3\alpha^2)^2}{4\alpha} \end{cases}$$

Scaling we can assume that $k = 1$, hence $\beta = \frac{m(1-3m)}{\alpha(1-3\alpha^2)}$. Substituting this into the second equation, we get that

$$3m^2 + 1 = \frac{m(1-3m)(1+3\alpha^2)^2}{4\alpha^2(1-3\alpha^2)}.$$

Regarding this as a quadratic in m, it has rational solutions if and only if its discriminant

$$\frac{(-351\alpha^8 + 1692\alpha^6 - 282\alpha^4 - 36\alpha^2 + 1)}{16\alpha^4(3\alpha^2-1)^2}$$

is a rational square. In particular, its numerator must be a rational square, i.e. there exists $y \in \mathbb{Q}$ such that

$$y^2 = -351\alpha^8 + 1692\alpha^6 - 282\alpha^4 - 36\alpha^2 + 1.$$

The equation above defines the affine part of a projective hyperelliptic curve C of genus 3, which by Faltings' theorem has only finitely many rational points.

□

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