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The converse of the Newton-Gauss theorem

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Abstract. In this note we give a proof for a natural converse of the Newton-Gauss theorem. To be precise, we show that the only triple (k_1, k_2, k_3) of real numbers, different from 0 and 1, such that the points dividing the diagonals AC, BD and EF of every complete quadrilateral $ABCD : EF$ in the signed ratios k_1, k_2 and k_3, are collinear is the triple $(-1, -1, -1)$. The latter corresponds to the midpoints of the diagonals in the complete quadrilateral.

1. Introduction

Consider a quadrilateral $ABCD$ such that no pair of its sides are parallel. The opposite sides AB and CD meet in E and AD and BC meet in F. The configuration $ABCD : EF$ is called *the complete quadrilateral* obtained from $ABCD$ with the diagonals $[AC]$, $[BD]$ and $[EF]$. The famous Newton-Gauss theorem asserts that the midpoints of the diagonals of any complete quadrilateral are collinear. The line on which these midpoints sit is called the *Newton-Gauss line* of $ABCD$.

Concerning proofs of the aforementioned result, two of them are based on elementary methods of Euclidean geometry. The first such makes use of a parallelogram property, whereas the second is an application of Menelaos' theorem [4]*Theorem 3.6.1. Other proofs include one based on projective geometry, one using the vector method [4]*Theorem 1.3.14, and one which involves complex coordinates (see [1] and [4]).

There are many properties of complete quadrilaterals which involve the Newton-Gauss line. For the interested reader, we recommend [7] and [13]*pp. 365–376 from the numerous references available.

As it is noticed in [3], the property asserted by the Newton-Gauss theorem does not depend on the configuration, i.e. it is not specific to the geometry of the quadrilateral. The assertion is in fact a property of the set $\{A, B, C, D\}$ consisting of four distinct points in the plane such that no three of them are collinear. Every such quadrangle has three Newton-Gauss lines corresponding to the three possible choices one can make when selecting the vertices defining the "diagonals". Of course, in case some of the lines are

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parallel then we think about their intersection projectively, i.e. they will meet in a point on the “line at infinity".

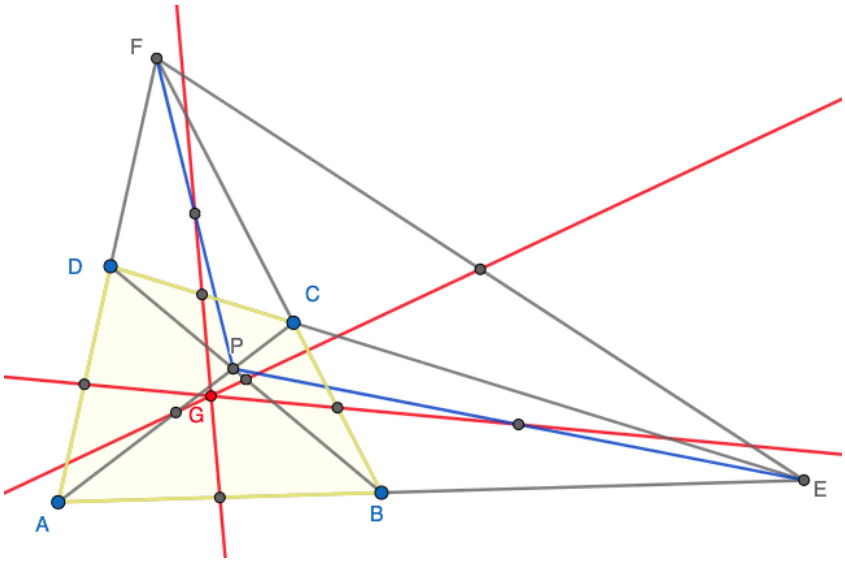

FIGURE 1. The Newton-Gauss lines of $\{A, B, C, D\}$

The Newton-Gauss lines of the quadrangle $\{A, B, C, D\}$ are illustrated in Figure 1 and these correspond to the complete quadrilaterals $ABCD : EF$, $ACDB : PE$ and $ACBD : PF$ which can be obtained from the four points $ABCD$.

A short proof of the collinearity results implying the existence of these Newton-Gauss lines is based on the following idea. After applying an affine transformation, one can assume without losing generality that the complex affixes of the four points are $A(0), B(1), C(z)$ and $D(i)$, where $z \in \mathbb{C}^*$ and $z \neq 1, i$. For a justification of the existence of such an affine transform, the curious reader could consult [11], where everything is explained in the language of Cartesian coordinates.

One can construct three circles having the diagonals of a complete quadrilateral as diameters. A stronger result asserts that these three circles are coaxial, meaning that all three distinct pairs that can be formed with two of these circles have the same radical axis. We recall that the radical axis of two circles is orthogonal on the line determined by their centers. In this view, the aforementioned result known as the *Gauss-Bodenmiller* theorem (see [12]*page 175) implies that collinearity of the three centers.

An equivalent form of this theorem is the following: The orthocenters of the four triangles constituted by the sides of a complete quadrilateral (i.e. triangles ADE, ABF, BCE and CDF in Figure 1) lie on a line perpendicular to the Newton-Gauss line of the quadrilateral.

A short history of this result is presented in the paper [8] and an analytic proof together with various consequences is given in [9]. The extension to the n-dimensional Euclidean space is treated in [10] and some non-Euclidean versions are discussed in [14].

Both of these results are very useful tools in solving geometry olympiad problems, as the reader could convince himself by consulting [2] and [6].

The purpose of this note is to give a solution to the following natural problem, which clearly represents a converse of the Newton-Gauss theorem.

Firstly, recall that the point $M \in AB$, $M \neq A, B$, divides the segment $[AB]$ in the ratio $k \neq 0, 1$ if $\overline{MA} = k\overline{MB}$. In this case, the position vector of M can be expressed as

$$\overline{r}_M = \frac{\overline{r}_A - k\overline{r}_B}{1-k}.$$

In fact, we will use this formula in terms of complex coordinates (i.e. affixes) of the points. More precisely, if z_A, z_B, z_M are the complex coordinates of A, B and M, as above, then we have

$$z_M = \frac{z_A - kz_B}{1-k}.$$

Problem. Determine all triples (k_1, k_2, k_3) of real numbers, different from 0 and 1, such that the points dividing the diagonals AC, BD and EF of every complete quadrilateral $ABCD : EF$ in the signed ratios k_1, k_2 and k_3, respectively are collinear.

We remark that from the Newton-Gauss theorem it follows that the triple $(-1, -1, -1)$ has this property. The main theorem proved in this note asserts that the $(-1, -1, -1)$ is the only such triple.

2. Main theorem

In this section we proceed to proving our main result.

Theorem 2.1. *If the triple (k_1, k_2, k_3) satisfies the property described in the above problem, then $(k_1, k_2, k_3) = (-1, -1, -1)$.*

Proof. Consider the complete quadrilateral $ABCD : EF$. As it is explained in the expository paper [11], applying an affine transformation, we can assume without losing generality that the complex affixes of the vertices of $ABCD$ are $A(0), B(1), C(z)$ and $D(i)$, where $z = x + iy$, $x, y \in \mathbb{R}$. The generality is not lost due to the fact that affine transformations preserve ratios of "internal lengths" and concurrence of lines (see [11]*Table 1).

Let M, N and P be the points dividing the diagonals $[AC]$, $[BD]$ and $[EF]$ in the ratios k_1, k_2 and k_3, respectively.

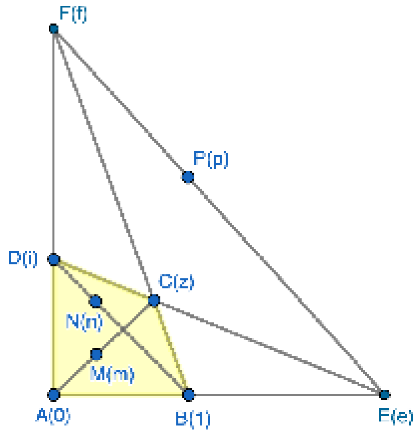

FIGURE 2. The complete quadrilateral after affine transformation

To find the affix e of E, we determine the intersection of the lines AB and CD. This affix must be real, since it lies on the line AB. Since it belongs to CD, we find that it must satisfy the equation $e = \alpha z + (1-\alpha)i$ for some $\alpha \in \mathbb{R}$. On one hand, taking real parts we see that $e = \alpha x$. By considering the imaginary parts, we must have that $\alpha y + 1 - \alpha = 0$ and hence $\alpha = \frac{1}{1-y}$. We thus found that $e = \frac{x}{1-y} \in \mathbb{R}$.

Similarly, to find the affix f of F we determine the intersection of AD and BC. One on hand f must be equal to $\alpha z + (1-\alpha)$ for some $\alpha \in \mathbb{R}$, since F lies on BC. From the equation of the line AD we also find that $f = \beta i$ for some $\beta \in \mathbb{R}$.

Considering real and imaginary parts of the equality $\alpha z + (1-\alpha) = i\beta$, we find that $\alpha = \frac{1}{1-x}$ and hence $\beta = y\alpha = \frac{y}{1-x}$. We just showed that $f = \frac{iy}{1-x}$.

From the hypothesis it follows that the affixes of M, N and P are

$$m = \frac{-k_1 z}{1-k_1},\ n = \frac{1-k_2 i}{1-k_2} \text{ and } p = \frac{\frac{x}{1-y} - k_3 \frac{iy}{1-x}}{1-k_3}$$

respectively. These three points are collinear if and only if

$$\begin{vmatrix} m & \overline{m} & 1 \\ n & \overline{n} & 1 \\ p & \overline{p} & 1 \end{vmatrix} = 0. \tag{1}$$

Multiplying the lines in the above determinant with the non-zero scalars $1-k_1$, $1-k_2$ and $1-k_3$ respectively, we obtain that (1) holds if and only if

$$\begin{vmatrix} -k_1 z & -k_1 \overline{z} & 1-k_1 \\ 1-k_2 i & 1+k_2 i & 1-k_2 \\ \frac{x}{1-y} - k_3 \frac{iy}{1-x} & \frac{x}{1-y} + k_3 \frac{iy}{1-x} & 1-k_3 \end{vmatrix} = 0. \tag{2}$$

Adding the first column to the second one and simplifying the latter by 2, we get

$$
\left| \begin{array}{ccc} -k_1 z & -k_1 x & 1-k_1 \\ 1-k_2 i & 1 & 1-k_2 \\ \frac{x}{1-y} - k_3 \frac{iy}{1-x} & \frac{x}{1-y} & 1-k_3 \end{array} \right| = 0. \tag{3}
$$

Separating the real and imaginary parts in the first column of the determinant in (3), one can decompose it as a sum of two determinants. It can be easily deduced that

$$
\left| \begin{array}{ccc} -k_1 y & -k_1 x & 1-k_1 \\ -k_2 & 1 & 1-k_2 \\ -k_3 \frac{y}{1-x} & \frac{x}{1-y} & 1-k_3 \end{array} \right| = 0. \tag{4}
$$

Any quadrilateral with non-parallel opposite sides can be transformed mapped, using affine transformations, in a quadrilateral $ABCD$ with affixes $0, 1, z, i$, where $z \neq 1+i$. As the ratios k_1, k_2 and k_3 are preserved under affine transformations, the relation obtained in (4) must hold for any $(x, y) \in \mathbb{R}^2 \setminus \{(1,1)\}$.

As the determinant in (4) is a polynomial in $\mathbb{R}[x, y]$ which vanishes at infinitely many values x, y, we deduce that it is in fact the zero polynomial. That is, after expanding the determinant we get

$$
-k_1(1-k_3)y(1-x)(1-y) - k_2(1-k_1)x(1-x) + k_1k_3(1-k_2)xy(1-y)+
$$

$$
+k_3(1-k_1)y(1-y) + k_1(1-k_2)xy(1-x) - k_1k_2(1-k_3)x(1-x)(1-y)
$$

must be the zero polynomial in $\mathbb{R}[x, y]$.

The coefficient of x in the above expression is

$$
-k_2(1-k_1) - k_1k_2(1-k_3).
$$

Equating this expression to zero we get that $k_1k_3 = 1$. The coefficient of y is equal to

$$
-k_1(1-k_3) + k_3(1-k_1),
$$

from which we deduce that $k_1 = k_3$.

Combining the above relations with the hypothesis that $k_1 \neq 1$ and $k_2 \neq 1$, we get that $k_1 = k_3 = -1$. Now, the coefficient of xy is equal to

$$
-2 + 1 - k_2 - (1-k_2) - 2k_2.
$$

The latter is zero if and only if $k_2 = -1$, and the desired conclusion follows.

□

The following result is an immediate consequence of our theorem.

Corrolary 2.1. *For every triple* $(k_1, k_2, k_3) \neq (-1, -1, -1)$ *consisting of real numbers different from* 0 *and* 1, *there exists a complete quadrilateral such that the points dividing the diagonals in ratios* k_1, k_2, k_3 *are not collinear.*

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