INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 10 (2021), No. 1, 74–78

A GENERALIZATION OF FEUERBACH'S THEOREM

TRAN QUANG HUNG AND NGUYEN THI THUY DUONG

Abstract. Given a triangle with the nine-point circle Ω, we construct a radical circle from the pedal triangle of an arbitrary point which is tangent to Ω. Feuerbach's theorem is a particular case where the pedal triangle is an intouch triangle.

1. Introduction

In 1822, the 22 year-old German mathematician Feuerbach published a theorem on the nine-point circle which can be stated as follows

Theorem 1 (Feuerbach, 1822)**.** *The nine-point circle of a nonequilateral triangle is tangent to both its incircle and its three excircles.*

The theorem appeared in a small book Feuerbach wrote after receiving a doctorate, and immediately became the highlight of this book. It is also one of the most significant legacies of the short-lived geometer before he passed at age 34.

The fisrt author of this note has given an extension of Feuerbach's Theorem on the right triangles; see [7].

In this paper the authors shall develop a direct generalisation of Theorem 1 based on the following theorem regarding Thebault's orthogonal circles in [2].

Theorem 2 (Thébault's general theorem [4])**.** *If the distances O_2O_3, O_3O_1, O_1O_2 between the centers of three circles (O_1), (O_2), (O_3), having radii R_1, R_2, R_3, satisfy a relation of the form*

$$|O_2O_3| \cdot R_1 \pm |O_3O_1| \cdot R_2 \pm |O_1O_2| \cdot R_3 = 0, \tag{1}$$

then the circumcircle (O) of triangle $O_1O_2O_3$ is tangent to the radical circle (O') of the circles (O_1), (O_2), (O_3), and the radical center O' is at the center of one of the tritangent circles of the triangle determined by the radical axes of circle (O) in association with each of the circles (O_1), (O_2), (O_3).

This theorem was also proven in details in [4]. We now proceed to generalize Theorem 1.

2010 *Mathematics Subject Classification.* 51M04, 51-03.

Key words and phrases. Feuerbach's theorem, general theorem, pedal triangle, tangent circles

Received: 29.06.2020. In revised form: 20.12.2020. Accepted: 10.10.2020.

Theorem 3 (Generalization of Feuerbach's theorem)**.** *Let ABC be a triangle with A_1, B_1, and C_1 are midpoints of the sides BC, CA, and AB, respectively. Let $A_0B_0C_0$ be the pedal triangle of an arbitrary point P. Then the radical circle ω of the circles (A_1, A_1A_0), (B_1, B_1B_0), and (C_1, C_1C_0) is tangent to the nine-point circle Ω of triangle ABC.*

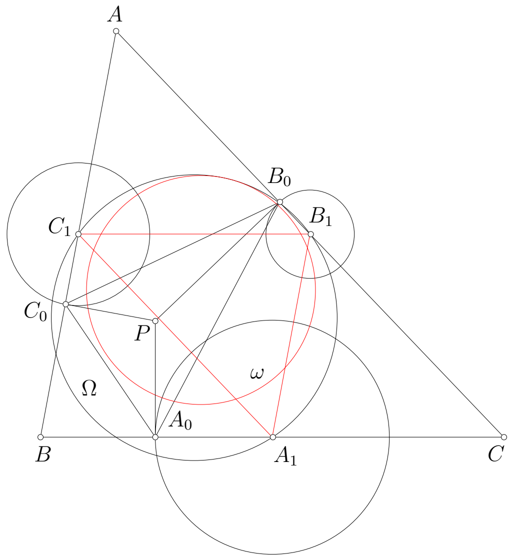

FIGURE 1. Generalization of Feuerbach's theorem

After proving this generalized theorem, we will apply it to a particular case in order to construct new tangent circles in ABC. We will also propose two important theorems that describe more clearly the radical center mentioned in Theorem 3. With this specified description we will then obtain an explicit construction for the tangent circle to the nine-point circle in Theorem 3.

2. Proof of generalized theorem

Proof. Using signed length of segment, we see that $\overline{CB} = \overline{B_1C_1}$. Also applying the Pythagorean theorem, we have

$$\begin{aligned}|PB|^2 - |PC|^2 &= |A_0B|^2 - |A_0C|^2\\ &= \left(\overline{A_0B} + \overline{A_0C}\right)\left(\overline{A_0B} - \overline{A_0C}\right)\\ &= 2 \cdot \overline{A_0A_1} \cdot \overline{CB}\\ &= 4 \cdot \overline{A_0A_1} \cdot \overline{B_1C_1}. \qquad (2)\end{aligned}$$

By analogy, we get

$$|PC|^2 - |PA|^2 = 4 \cdot \overline{B_0B_1} \cdot \overline{C_1A_1}. \qquad (3)$$

and

$$|PA|^2 - |PB|^2 = 4 \cdot \overline{C_0C_1} \cdot \overline{A_1B_1}. \qquad (4)$$

From (2), (3), and (4), we obtain

$$4 \cdot \overline{A_0A_1} \cdot \overline{B_1C_1} + 4 \cdot \overline{B_0B_1} \cdot \overline{C_1A_1} + 4 \cdot \overline{C_0C_1} \cdot \overline{A_1B_1} \tag{5}$$

$$= \left(|PB|^2 - |PC|^2\right) + \left(|PC|^2 - |PA|^2\right) + \left(|PA|^2 - |PB|^2\right) = 0 \tag{6}$$

or

$$A_1A_0 \cdot B_1C_1 \pm B_1B_0 \cdot C_1A_1 \pm C_1C_0 \cdot A_1B_1 = 0. \tag{7}$$

Note that A_1A_0, B_1B_0, and C_1C_0 are the radii of circles (A_1, A_1A_0), (B_1, B_1B_0), and (C_1, C_1C_0), respectively. Then from (7) and Theorem 2, radical circle ω of the circles (A_1, A_1A_0), (B_1, B_1B_0), and (C_1, C_1C_0) is tangent to the circumcircle of triangle $A_1B_1C_1$ which is the nine-point circle Ω of ABC. □

3. More detailed description for the radical center

In order to construct the radical circle in the Theorem 3, we need a way to construct the radical center of the circles (A_1, A_1A_0), (B_1, B_1B_0), and (C_1, C_1C_0). The following two theorems from [6] help us describe in more details the radical center of these circles.

Theorem 4 (Tran Quang Hung, post #5 in [6]). *Let ABC be a triangle with A_1, B_1, and C_1 as midpoints of the sides BC, CA, and AB, respectively. Let $A_0B_0C_0$ be a pedal triangle of any point P. Let Q be the radical center of the circles (A_1, A_1A_0), (B_1, B_1B_0), and (C_1, C_1C_0). Let R be the isogonal conjugate of Q with respect to triangle $A_0B_0C_0$. Then the line PR passes through circumcenter O of triangle ABC.*

Theorem 5 (Alexander Skutin, post #8 in [6]). *Let ABC be a triangle with A_1, B_1, and C_1 as midpoints of the sides BC, CA, and AB, respectively. Let $A_0B_0C_0$ be the pedal triangle of any point P. Let Q be the radical center of the circles (A_1, A_1A_0), (B_1, B_1B_0), and (C_1, C_1C_0). Let I be the circumcenter of the pedal triangle of P with respect to triangle $A_0B_0C_0$, and G be the centroid of $A_0B_0C_0$. Then three points Q, G, and I are collinear and $|QG| = 2 \cdot |GI|$.*

4. Some applications

In [4], Victor Thébault showed that if $P = I$ where I is the incenter of triangle ABC, then incircle (I) clearly coincides with the radical circle of (A_1, A_1A_0), (B_1, B_1B_0), and (C_1, C_1C_0). Thus, (I) is tangent to nine-point circle of ABC.

If $P = I_a$ where I_a is the A-excenter of triangle ABC, then excircle (I_a) clearly coincides with the radical circle of (A_1, A_1A_0), (B_1, B_1B_0), and (C_1, C_1C_0). Thus (I_a) is tangent to nine-point circle of ABC.

If $P = H$ where H is the orthocenter of triangle ABC, we obtain the application (B), in [4].

We give another application of Theorem 3.

Theorem 6. *Let ABC be a triangle inscribed in a circle ω. Na is Nagel point ([5]) of triangle ABC. X is the midpoint of arc BC containing A of ω. ω_a is the circle center X and radius XA. From Na draw tangent lines NaA_1, NaA_2 to circle ω_a with A_1 and A_2 on ω_a. Define similarly the points*

B_1, B_2, C_1, and C_2. Then six points A_1, A_2, B_1, B_2, C_1, and C_2 lie on a circle Ω, and Ω is tangent to ω.

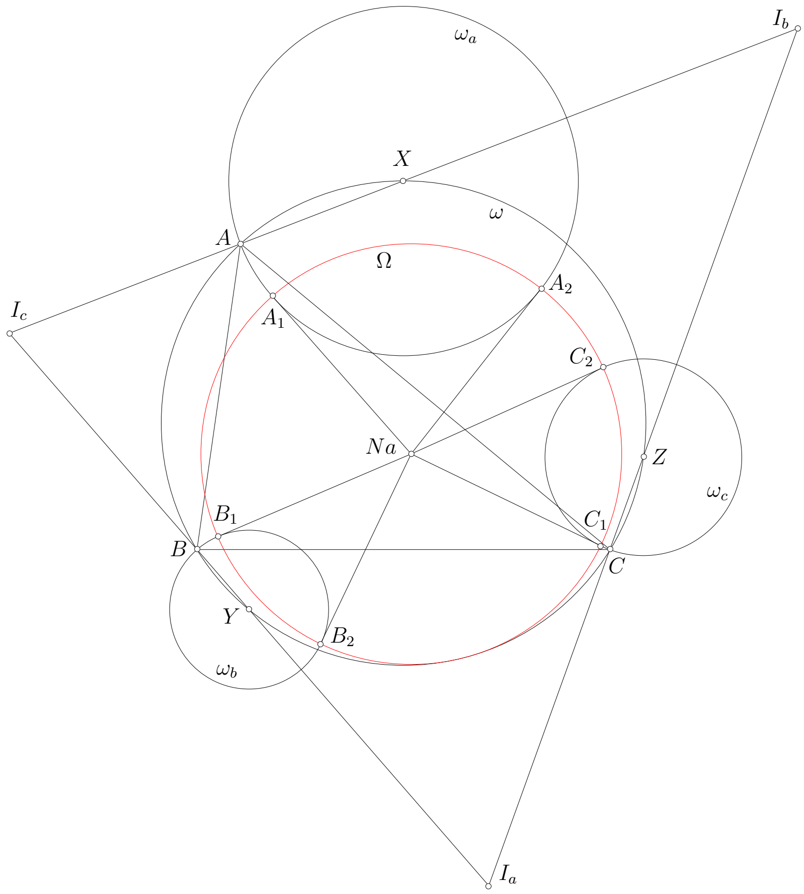

FIGURE 2. An application with Nagel point

Proof. Let I_a, I_b, and I_c be the excenters at the vertices A, B, and C, respectively. It is not hard to see that I is the orthocenter of triangle $I_aI_bI_c$, and X, Y, Z are midpoints of sides I_bI_c, I_cI_a, I_aI_c, respectively. Thus ω is the nine-point circle of triangle $I_aI_bI_c$. Following [6] (with detailed proof of Luis González in post #2), Na is the radical center of ω_a, ω_b, and ω_c. Therefore,

$$NaA_1 = NaA_2 = NaB_1 = NaB_2 = NaC_1 = NaC_2. \tag{8}$$

In other words, six points A_1, A_2, B_1, B_2, C_1, and C_2 lie on circle Ω which is the radical circle of ω_a, ω_b, and ω_c. Therefore, X, Y, Z are midpoints of sides I_bI_c, I_cI_a, I_aI_c, respectively. By applying Theorem 3, we get Ω is tangent to ω. This completes the proof. □

5. Conclusion

We have extended Feuerbach's theorem by using the concept of radical circles. Simple as this extension may be, it is by no means trivial. Note that

Victor Thébault's general theorem plays a crucial role in this whole proof, its application to the pedal triangle in this generalization of Feuerbach's theorem is pivotal.

Acknowledgements. The authors would like to thank Luis González and Alexander Skutin for their assistance with this article.

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High School for Gifted Students, Vietnam National University, Hanoi, Vietnam
Email address: `tranquanghung@hus.edu.vn`

Concord College, Shropshire, United Kingdom
Email address: `nt.thuyduong3402@gmail.com`