



MODEST TRIANGLES AND PERIAMBIC POINTS

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Abstract. The major periambic points P and Q are radical centers associated with the six circles having a triangle's vertices as centers and its sides as radii. We offer an alternate construction of P and Q which is valid for “modest” triangles, in which the ratio of two side lengths lies in the closed interval $[-1 + \sqrt{2}, 1 + \sqrt{2}]$.

1. INTRODUCTION

The radical center of three circles is the point of concurrency of the three radical axes determined by pairs of the circles. The major periambic points P and Q are radical centers associated with a (counterclockwise-labeled) triangle ABC : P is the radical center of the periambic circles P_A , P_B , and P_C , having centers A , B , and C and radii $AB = c$, $BC = a$, and $CA = b$, respectively. P lies on each of the radical axes p_a , p_b , and p_c (called p -lines) determined by circle pairs $\{P_B, P_C\}$, $\{P_C, P_A\}$, and $\{P_A, P_B\}$, respectively. Q is the radical center of periambic circles Q_A , Q_B , and Q_C , with centers A , B , and C , and radii b , c , and a , respectively. Q lies on each of the radical axes q_a , q_b , and q_c (called q -lines) determined by circle pairs $\{Q_C, Q_B\}$, $\{Q_A, Q_C\}$, and $\{Q_B, Q_A\}$, respectively. These definitions are given in [2], where it is shown, for instance, that P , Q , the orthocenter, and the circumcenter of triangle ABC form a parallelogram.

In [2], “perpendicularity” is a dominant theme; each of the 12 real radical axes defined by pairs of periambic circles is at right angles to a side of triangle ABC . The configuration shown in Figure 1, and summarized in the following conjecture due to Marty Getz [1], offers a step toward defining the points P and Q by means of non-perpendicular elements. We bear in mind that the cyclic relabeling $(a, A) \rightarrow (b, B) \rightarrow (c, C) \rightarrow (a, A)$ of sides and vertices yield in total three versions of each result below; for brevity and clarity, only one version will be presented.

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Conjecture. Given triangle ABC , let p_a intersect P_A at J_A (on the same side of BC as A) and at K_A . Let CJ_A and CK_A intersect P_B at L_B and M_B , respectively. Then L_B, P , and M_B are collinear.

This conjecture could be a theorem, except for one problem: its first sentence. As Figure 2 shows, the radical axis p_a may not intersect the circle P_A .

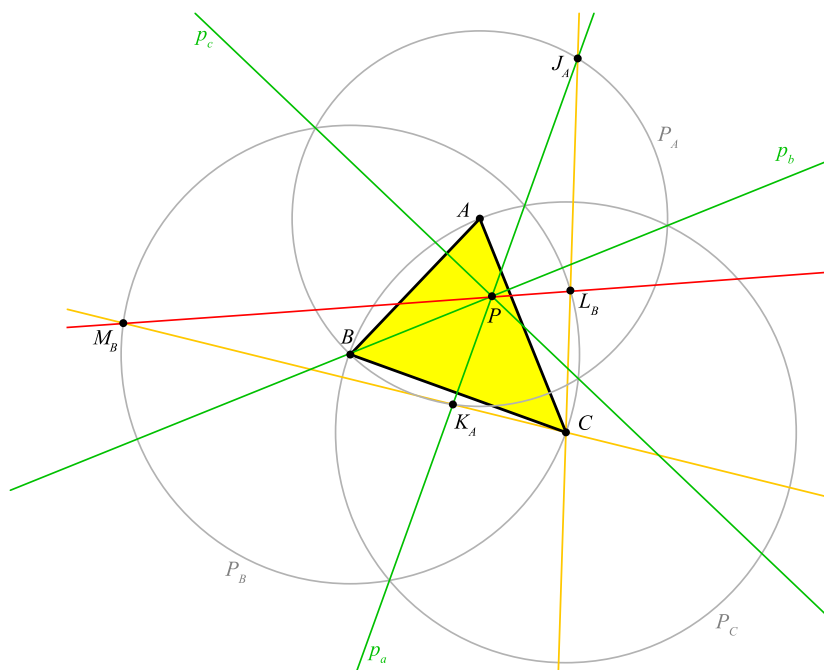


FIGURE 1. Radical axis p_a intersects periambic circle P_A at J_A and K_A . CJ_A and CK_A intersect P_B at L_B and M_B , respectively. The conjecture claims that L_B, P , and M_B are collinear.

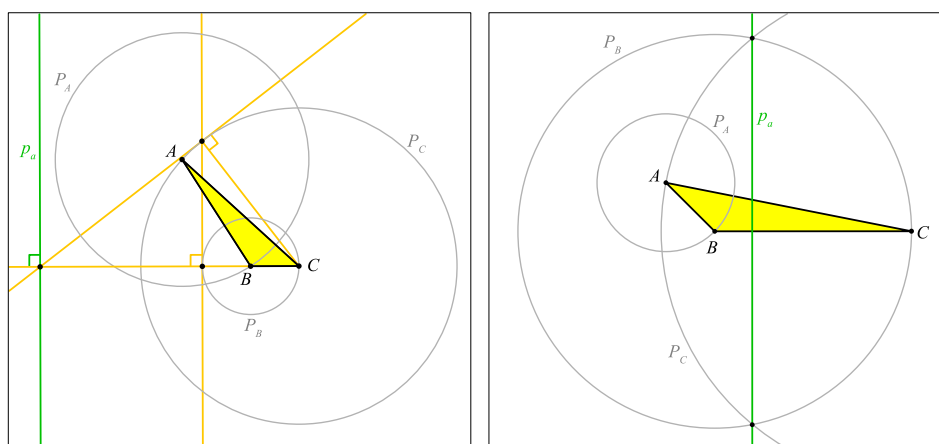


FIGURE 2. Two cases in which p_a and P_A fail to intersect. Gold lines on the left are for locating p_a .

2. MODEST TRIANGLES

To work the conjecture into something provable, we pause to investigate the conditions under which the points J_A and K_A are well-defined.

Proposition 2.1. *In triangle ABC , the radical axis p_a intersects the periambic circle P_A if, and only if,*

$$(1) \quad -1 + \sqrt{2} \leq \frac{c}{a} \leq 1 + \sqrt{2}$$

or, equivalently,

$$(2) \quad -1 + \sqrt{2} \leq \frac{a}{c} \leq 1 + \sqrt{2} .$$

We call a triangle *ca*-modest if it satisfies inequality (1). By condition (2), a *ca*-modest triangle is also *ac*-modest. Apropos of our comment on cyclic relabeling of vertices and sides, we note that, in an *ab*-modest triangle, p_b and P_B intersect; in a *bc*-modest triangle, p_c and P_C intersect. A triangle in which all three *p*-lines intersect their corresponding periambic circles, as in Figure 1, will simply be called modest.

Proof. Conjugate multiplication in the reciprocal of (1) yields (2); we prove the latter. Without loss of generality, in the cartesian plane let A, B , and C have coordinates $(x_1, x_2), (0, 0)$, and $(a, 0)$, respectively, with $x_2 > 0$ and $a > 0$. Let p_a intersect the x -axis at $T : (t, 0)$. Since T has equal powers with respect to P_B and P_C , we have

$$(3) \quad \begin{aligned} BT^2 - a^2 &= CT^2 - b^2 \\ t^2 - a^2 &= (a - t)^2 - b^2 \\ t &= \frac{2a^2 - b^2}{2a} . \end{aligned}$$

The coordinates of A , considered as a point on P_C , satisfy $(a - x_1)^2 + x_2^2 = b^2$. Substituting for b^2 in (3) and rearranging, we obtain

$$(4) \quad \begin{aligned} t &= \frac{2a^2 - ((a - x_1)^2 + x_2^2)}{2a} \\ &= \frac{a^2 + 2ax_1 - (x_1^2 + x_2^2)}{2a} \\ &= \frac{a^2 + 2ax_1 - c^2}{2a} . \end{aligned}$$

Let $U : (u, x_2)$ and $V : (v, x_2)$ be the left and right endpoints, respectively, of the diameter of P_A which is parallel to the x -axis. Clearly

$$u = x_1 - c \quad \text{and} \quad v = x_1 + c .$$

For p_a to intersect P_A , we require

$$(5) \quad u = x_1 - c \leq t \leq x_1 + c = v .$$

The left-hand inequality becomes

$$(6) \quad \begin{aligned} x_1 - c &\leq \frac{a^2 + 2ax_1 - c^2}{2a} \\ 0 &\leq a^2 + 2ac - c^2 , \end{aligned}$$

which, for nonzero c , yields

$$(7) \quad a \geq c(-1 + \sqrt{2}) .$$

Similarly, the right-hand inequality in (5) yields

$$(8) \quad \frac{a^2 + 2ax_1 - c^2}{2a} \leq x_1 + c$$

$$0 \leq -a^2 + 2ac + c^2 ,$$

having positive solution

$$(9) \quad a \leq c(1 + \sqrt{2}) .$$

Results (7) and (9) combine to establish inequality (2). Conversely, assuming (1), we have

$$a^2(3 - 2\sqrt{2}) \leq c^2 \text{ and}$$

$$2a^2(-1 + \sqrt{2}) \leq 2ac ,$$

from which we recover (8); likewise, assuming (2), we can construct (6). It then follows that (5) is satisfied.

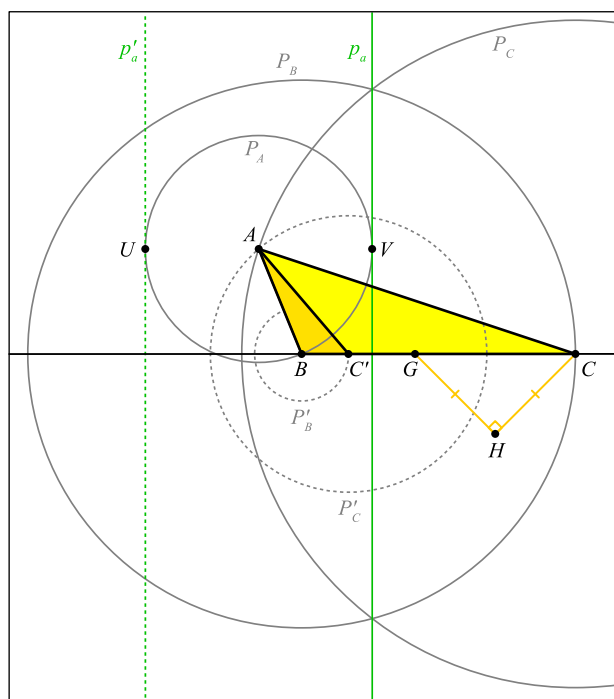


FIGURE 3. Example showing ca - and $c'a'$ -modest triangles ABC and ABC' , respectively, for which the extrema of the ratios of Proposition 2.1 are achieved when A , B , and $\angle CBA = \angle C'BA$ are fixed. Solid green line p_a is the radical axis of solid gray circles P_B and P_C ; dotted green line p'_a is the radical axis of dotted gray circles P'_B and P'_C .

Figure 3 shows the configurations resulting when c/a attains the values $-1 + \sqrt{2}$ and $1 + \sqrt{2}$. In this construction,

$$c = AB = BG = GH = HC ,$$

and $\angle GHC$ is a right angle, so $a = BC = c + c\sqrt{2}$; the radical axis p_a (solid green line) is tangent to P_A on the circle's right. On the other hand, in triangle ABC' where a', b', c' are the sides opposite A, B, C' , respectively, C' is constructed so that $a' = BC' = -c + c\sqrt{2}$, and the resulting radical axis p'_a (dotted green line) is tangent to P_A on its left. From the ratios c/a and c'/a' and the shared angle CBA , it is clear that triangles ABC and ABC' are inversely similar.

One can construct an isosceles triangle in which two of the three p -lines are tangent to their corresponding periambic circles. In Figure 4,

$$b = c = a(1 + \sqrt{2}) ,$$

and Proposition 2.1 is satisfied by a/b , b/a , c/a , and a/c . If B were nearer C than is shown in Figure 4, p_a would not intersect P_A , nor would p_b intersect P_B .

3. A THEOREM FROM THE LIMITED CONJECTURE

Restricted to the relevant type of modest triangle, the conjecture which began this note becomes a theorem.

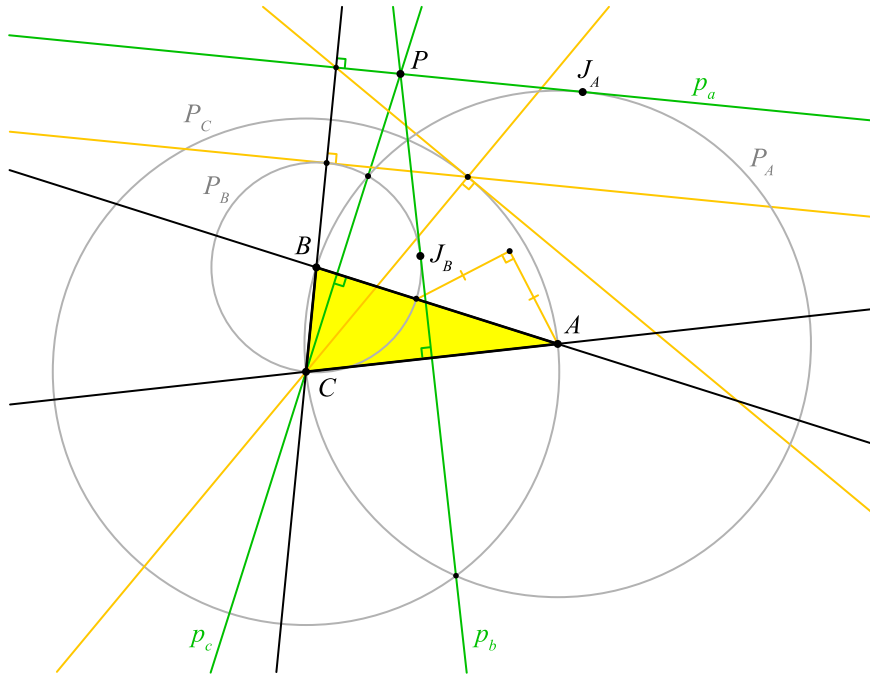


FIGURE 4. Isosceles triangle ABC in which $b = c = a(1 + \sqrt{2})$. In this case, p_a and p_b are tangent to P_A and P_B , respectively. For any shorter side a , p_a and p_b do not intersect their corresponding periambic circles.

Theorem 3.1. *Let ABC be a ca -modest triangle. Let p_a intersect P_A at J_A (on the same side of BC as A) and at K_A . Let CJ_A and CK_A intersect P_B at L_B and M_B , respectively. Then $L_B, P, \text{ and } M_B$ are collinear.*

Proof. By requiring triangle ABC to be ca -modest, we guarantee the existence of the points J_A and K_A , per Proposition 2.1. Figures 5 and 6 illustrate the following construction for the cases in which P_B and P_C do and do not intersect, respectively. Let $L_B M_B$ intersect radical axis p_a at Z . By definition, Z has equal powers with respect to circles P_B and P_C .

Let S , different from C , be the intersection point of BC with P_B , and

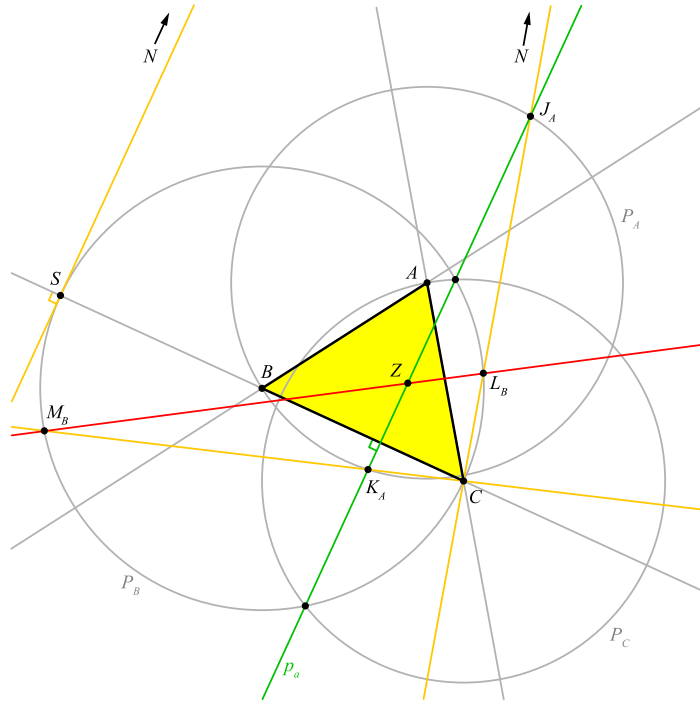


FIGURE 5. Construction for the proof of Theorem 3.1, in which P_B and P_C intersect.

let the tangent to P_B at S intersect CJ_A at N . Arcs and angles measured counterclockwise are positive; the notation arc XY represents the positive arc length from X to Y on a circle, with $0 < \text{arc } XY < 2\pi$ for $X \neq Y$. We have

$$\begin{aligned} \angle ZJ_A L_B &= \angle SNC = \frac{1}{2} (\text{arc } SC - \text{arc } L_B S) \\ &= \frac{1}{2} \text{arc } CL_B = \angle CM_B L_B \\ &= \angle K_A M_B Z . \end{aligned}$$

From this we see that $\triangle ZM_B K_A$ and $\triangle ZJ_A L_B$ are inversely similar. Therefore,

$$\frac{ZK_A}{ZM_B} = \frac{ZL_B}{ZJ_A}$$

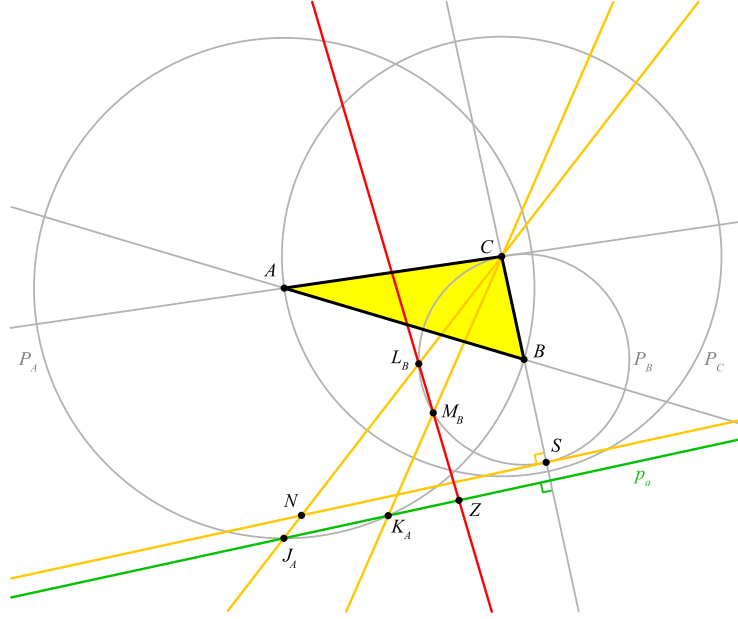


FIGURE 6. Construction for the proof of Theorem 3.1, in which P_B and P_C do not intersect.

or

$$(10) \quad ZM_B \cdot ZL_B = ZK_A \cdot ZJ_A .$$

The power ρ of point Z with respect to circle P_B is

$$(11) \quad \rho = ZM_B \cdot ZL_B ,$$

so from Equations (10) and (11) we have

$$\rho = ZM_B \cdot ZL_B = ZK_A \cdot ZJ_A .$$

Since $ZK_A \cdot ZJ_A$ is the power of Z with respect to circle P_A , we conclude that Z lies on the radical axis of P_A and P_B , and hence is the radical center of circles P_A , P_B , and P_C , i.e., Z is the major periambic point P .

By remarks following Proposition 5 in [2] (concerning the radial symmetry of the major periambic points P and Q around the midpoint of the segment joining the orthocenter and circumcenter), there is a corresponding result for Q . Identical to Theorem 3.1, but using Q instead of P and reversing the roles of B and C , we state it here for completeness.

Theorem 3.2. *Let q_a intersect Q_A at J_A (on the same side of BC as A) and at K_A . Let BJ_A and BK_A intersect Q_C at L_C and M_C , respectively. Then L_C , Q , and M_C are collinear.*

REFERENCES

- [1] Getz, M., personal communication, March 2003.
- [2] Jones, D. J., The periambic constellation: altitudes, perpendicular bisectors, and other radical axes in a triangle, *Forum Geometricorum*, **17** (2017) 383–399.

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