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CYCLIC AVERAGES OF REGULAR POLYGONAL DISTANCES

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Abstract. We consider a regular plane polygon with n vertices and an arbitrary point in the plane. Let R be the circumscribed radius of the polygon and L a distance from the point to the centroid of the polygon. Then the averages of the $(2m)$-th powers of distances from the point to the polygon vertices satisfy the relations

$$S_n^{(2)} = R^2 + L^2,$$

$$S_n^{(2m)} = (R^2+L^2)^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{2k}{k}(R^2+L^2)^{m-2k}(RL)^{2k},$$

where $m = 2, \ldots, n-1$.

1. Introduction

In his book **Mathematical Circus**, Martin Gardner wrote (p. 65): *"There is a beautifully symmetric equation for finding the side of an equilateral triangle when given the distances of a point from its three corners:*

$$3(a^4+b^4+c^4+d^4) = (a^2+b^2+c^2+d^2)^2.\text{"}$$

This result was generalized by J. Bentin [1] from an equilateral triangle to a regular polygon. Consider a regular plane polygon with n vertices and an arbitrary point in the plane. Denote by s^2 and q^4 respectively the averages of the squares and the averages of the fourth powers of the distances from the point to the vertices of the polygon

$$s^2 = \frac{1}{n}\sum_{i=1}^{n} d_i^2 \quad \text{and} \quad q^4 = \frac{1}{n}\sum_{i=1}^{n} d_i^4.$$

Then

$$q^4 + 3R^4 = (s^2+R^2)^2$$

is satisfied, where R is the circumscribed radius of the polygon.

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This result was generalized to regular simplicial and regular polytopic distances in [2] and [3], respectively.

In the above-mentioned papers the distances are considered to the second and fourth powers only. Naturally, we are interested to know what happens if we consider the distances to higher (more than 4) powers. In the present paper, for a regular polygon we introduce a special kind of averages of the distances to the even powers – the cyclic averages and by using their properties establish the metrical relations for regular polygons.

2. General case

Let us consider a regular plane n-sided polygon $A_1A_2\cdots A_n$ with the circumscribed radius R and an arbitrary point P in the plane. Denote the distances between P and the centroid O of the polygon by L, and the distances between P and the i-vertex by d_i (Fig. 1).

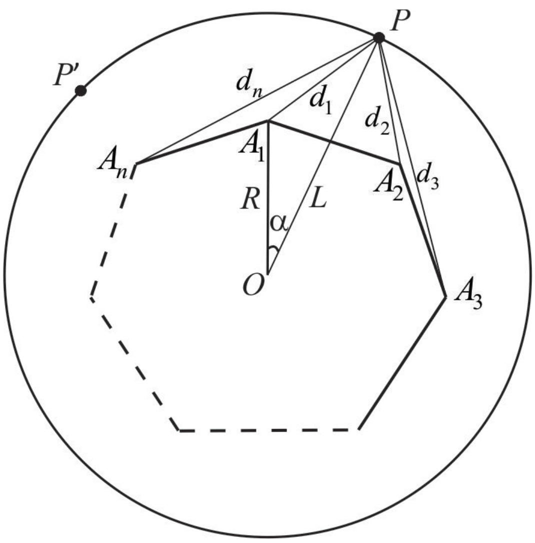

Figure 1

We use the following notation for the average of the $(2m)$-th powers of the distances:

$$S_n^{(2m)} = \frac{1}{n}\sum_{i=1}^{n} d_i^{2m}.$$

Theorem 2.1. *For a regular polygon with n vertices and an arbitrary point in the plane, let $S_n^{(2m)}$ be the averages of the $(2m)$-th powers of the distances from the point to the vertices. If R is the circumscribed radius and L the distance between the point and the centroid, then*

$$S_n^{(2)} = R^2 + L^2,$$

$$S_n^{(2m)} = (R^2+L^2)^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{2k}{k}(R^2+L^2)^{m-2k}(RL)^{2k},$$

where $m = 2, \dots, n-1$.

First we need to prove two lemmas.

Lemma 2.1. *For arbitrary positive integers* m *and* n*, such that* $m < n$*, the following condition*

$$\sum_{k=1}^{n} \cos\left(m\Big(\alpha - (k-1)\,\frac{2\pi}{n}\Big)\right) = 0$$

is satisfied, where α *is an arbitrary angle.*

Denote

$$T = e^{im\alpha} + e^{im(\alpha - \frac{2\pi}{n})} + e^{im(\alpha - 2\,\frac{2\pi}{n})} + \dots + e^{im(\alpha-(n-1)\,\frac{2\pi}{n})}.$$

The real part of T is

$$\mathrm{Re}(T) = \sum_{k=1}^{n} \cos\left(m\Big(\alpha - (k-1)\,\frac{2\pi}{n}\Big)\right).$$

The formula of the sum of geometric progression gives

$$T = e^{im\alpha}\bigg(1 + e^{-im\,\frac{2\pi}{n}} + \big(e^{-im\,\frac{2\pi}{n}}\big)^2 + \dots + \big(e^{-im\,\frac{2\pi}{n}}\big)^{n-1}\bigg) =$$
$$= e^{im\alpha}\,\frac{1 - (e^{-im\,\frac{2\pi}{n}})^n}{1 - e^{-im\,\frac{2\pi}{n}}}\,,$$

$$e^{-im\,2\pi} = \cos(-2\pi m) + i\sin(-2\pi m) = 1.$$

Since $m < n$, $e^{-im\,\frac{2\pi}{n}} \neq 1$. So $T = 0$, i.e. $\mathrm{Re}(T) = 0$, which proves Lemma 2.1.

Remark 2.1. *If* $m \geq n$*, the sum always contains* α*.*

Lemma 2.2. *For arbitrary positive integers* m *and* n*, such that* $m < n$ *and for an arbitrary angle* α *the following conditions are satisfied:*

if m *is odd*

$$\sum_{k=1}^{n} \cos^m\Big(\alpha - (k-1)\,\frac{2\pi}{n}\Big) = 0;$$

if m *is even*

$$\sum_{k=1}^{n} \cos^m\Big(\alpha - (k-1)\,\frac{2\pi}{n}\Big) = n\,\frac{\binom{m}{\frac{m}{2}}}{2^m}\,.$$

When m is odd, using the power-reduction formula for cosine

$$\cos^m\theta = \frac{2}{2^m}\sum_{k=0}^{\frac{m-1}{2}} \binom{m}{k} \cos\big((m-2k)\theta\big),$$

we obtain

$$\sum_{k=1}^{n} \cos^m \left(\alpha - (k-1)\frac{2\pi}{n}\right) =$$

$$= \cos^m \alpha + \cos^m \left(\alpha - \frac{2\pi}{n}\right) + \cdots +$$

$$+ \cos^m \left(\alpha - (n-1)\frac{2\pi}{n}\right) =$$

$$= \frac{2}{2^m}\Bigg[\binom{m}{0}\cos m\alpha + \binom{m}{1}\cos(m-2)\alpha + \cdots + \binom{m}{\frac{m-1}{2}}\cos\alpha +$$

$$+ \binom{m}{0}\cos m\Big(\alpha - \frac{2\pi}{n}\Big) + \binom{m}{1}\cos(m-2)\Big(\alpha - \frac{2\pi}{n}\Big) + \cdots +$$

$$+ \binom{m}{\frac{m-1}{2}}\cos\left(\alpha - \frac{2\pi}{n}\right) + \cdots +$$

$$+ \binom{m}{0}\cos m\Big(\alpha - (n-1)\frac{2\pi}{n}\Big) + \binom{m}{1}\cos(m-2)\Big(\alpha - (n-1)\frac{2\pi}{n}\Big) +$$

$$+ \cdots + \binom{m}{\frac{m-1}{2}}\cos\left(\alpha - (n-1)\frac{2\pi}{n}\right)\Bigg] =$$

$$= \frac{2}{2^m}\Bigg[\binom{m}{0}\Bigg(\cos m\alpha + \cos m\Big(\alpha - \frac{2\pi}{n}\Big) + \cdots +$$

$$+ \cos m\Big(\alpha - (n-1)\frac{2\pi}{n}\Big)\Bigg) +$$

$$+ \binom{m}{1}\Bigg(\cos(m-2)\alpha + \cos(m-2)\Big(\alpha - \frac{2\pi}{n}\Big) + \cdots +$$

$$+ \cos(m-2)\Big(\alpha - (n-1)\frac{2\pi}{n}\Big)\Bigg) +$$

$$\vdots$$

$$+ \binom{m}{\frac{m-1}{2}}\Bigg(\cos\alpha + \cos\left(\alpha - \frac{2\pi}{n}\right) + \cdots +$$

$$+ \cos\left(\alpha - (n-1)\frac{2\pi}{n}\right)\Bigg)\Bigg].$$

Since $m < n$, from Lemma 2.1 it follows that each sum equals zero, which proves the first part of Lemma 2.2.

When m is even, the power-reduction formula for cosine is

$$\cos^m \theta = \frac{1}{2^m}\binom{m}{\frac{m}{2}} + \frac{2}{2^m}\sum_{k=0}^{\frac{m}{2}-1}\binom{m}{k}\cos\big((m-2k)\theta\big).$$

Analogously to the case with odd m, the sum of the second addenda vanishes, and since the number of the first addenda is n, the total sum equals

$$n\binom{m}{\frac{m}{2}}2^m,$$

which proves Lemma 2.2.

Proof of the theorem. We introduce the new notations

$$A = R^2 + L^2 \text{ and } B = 2RL.$$

Then

$$\begin{aligned} n\,S_n^{(2m)} = (A - B\cos\alpha)^m &+ \left(A - B\cos\left(\frac{2\pi}{n} - \alpha\right)\right)^m + \\ &+ \left(A - B\cos\left(2\cdot\frac{2\pi}{n} - \alpha\right)\right)^m + \cdots + \\ &+ \left(A - B\cos\left((n-1)\,\frac{2\pi}{n} - \alpha\right)\right)^m. \end{aligned}$$

If $m = 1$, by Lemma 2.1we have

$$\begin{aligned} n\,S_n^{(2)} = (A - B\cos\alpha) + \left(A - B\cos\left(\frac{2\pi}{n} - \alpha\right)\right) + \cdots + \\ + \left(A - B\cos\left((n-1)\cdot\frac{2\pi}{n} - \alpha\right)\right) = nA. \end{aligned}$$

Therefore

$$S_n^{(2)} = R^2 + L^2.$$

If $m > 1$, we have

$$\begin{aligned} n\,S_n^{(2m)} = nA^m - \binom{m}{1}A^{m-1}B\bigg(&\cos\alpha + \cos\left(\frac{2\pi}{n} - \alpha\right) + \cdots + \\ &+ \cos\left((n-1)\,\frac{2\pi}{n} - \alpha\right)\bigg) + \\ + \binom{m}{2}A^{m-2}B^2\bigg(&\cos^2\alpha + \cos^2\left(\frac{2\pi}{n} - \alpha\right) + \cdots + \\ &+ \cos^2\left((n-1)\,\frac{2\pi}{n} - \alpha\right)\bigg) - \\ - \binom{m}{3}A^{m-3}B^3\bigg(&\cos^3\alpha + \cos^3\left(\frac{2\pi}{n} - \alpha\right) + \cdots + \\ &+ \cos^3\left((n-1)\,\frac{2\pi}{n} - \alpha\right)\bigg) + \\ &\vdots \\ \pm \binom{m}{m}B^m\bigg(&\cos^m\alpha + \cos^m\left(\frac{2\pi}{n} - \alpha\right) + \cdots + \\ &+ \cos^m\left((n-1)\,\frac{2\pi}{n} - \alpha\right)\bigg). \end{aligned}$$

According to Lemma 2.2, all sums with the negative sign "–" vanishes because they contain odd powers and there remain only the sums with even powers.

If m is even, we have

$$
\begin{aligned}
n\,S_n^{(2m)} &= nA^m + \\
&+\binom{m}{2}A^{m-2}B^2\left(\cos^2\alpha+\cos^2\left(\frac{2\pi}{n}-\alpha\right)+\cdots+\right.\\
&\qquad\left.+\cos^2\left((n-1)\,\frac{2\pi}{n}-\alpha\right)\right)+\\
&\quad\vdots\\
&+\binom{m}{m}B^m\left(\cos^m\alpha+\cos^m\left(\frac{2\pi}{n}-\alpha\right)+\cdots+\right.\\
&\qquad\left.+\cos^m\left((n-1)\,\frac{2\pi}{n}-\alpha\right)\right)=\\
&= n\left(A^m+\sum_{k=1}^{\frac{m}{2}}\binom{m}{2k}A^{m-2k}B^{2k}\,\frac{1}{2^{2k}}\binom{2k}{k}\right).
\end{aligned}
$$

If m is odd, we write

$$
\begin{aligned}
n\,S_n^{(2m)} &= nA^m + \\
&+\binom{m}{2}A^{m-2}B^2\left(\cos^2\alpha+\cos^2\left(\frac{2\pi}{n}-\alpha\right)+\cdots+\right.\\
&\qquad\left.+\cos^2\left((n-1)\,\frac{2\pi}{n}-\alpha\right)\right)+\cdots+\\
&+\binom{m}{m-1}AB^{m-1}\left(\cos^{m-1}\alpha+\cos^{m-1}\left(\frac{2\pi}{n}-\alpha\right)+\right.\\
&\qquad\left.+\cdots+\cos^{m-1}\left((n-1)\,\frac{2\pi}{n}-\alpha\right)\right)=\\
&= n\left(A^m+\sum_{k=1}^{\frac{m-1}{2}}\binom{m}{2k}A^{m-2k}B^{2k}\,\frac{1}{2^{2k}}\binom{2k}{k}\right).
\end{aligned}
$$

Using the floor function (the integer part), the obtained results can be combined into a single formula as follows

$$
S_n^{(2m)} = A^m + \sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}A^{m-2k}B^{2k}\,\frac{1}{2^{2k}}\binom{2k}{k},
$$

which proves the theorem. ∎

The values of the averages $S_n^{(2)}, S_n^{(4)}, \ldots, S_n^{(2n-2)}$ remain constant when the point P moves on the circle $C(O,L)$, i.e. if we consider any point on the circle P' (Fig. 1), these averages will retain the same values. So we can formulate

Definition 2.1. *The cyclic average of a regular polygon is the average of the power of the distances from the point to the vertices, the value of which is constant for any point on the circle $C(O, L)$, where O is the centroid of the polygon and L is the distance between the point and the centroid.*

The properties of the cyclic average are as follows:

Property 1. *Each regular n-gon has an $n-1$ number of cyclic averages*

$$S_n^{(2)}, S_n^{(4)}, \dots, S_n^{(2n-2)}.$$

Property 2. *Cyclic averages can be expressed only in terms of the circumscribed radius R and the distance L.*

Property 3. *The expressions of the non-cyclic averages contain α, i.e. depend on the direction OP (Fig. 1).*

Property 4. *For fixed R and L, the cyclic averages of equal powers of different regular n-gons are the same:*

$$S_3^{(2)} = S_4^{(2)} = S_5^{(2)} = S_6^{(2)} = \cdots ,$$
$$S_3^{(4)} = S_4^{(4)} = S_5^{(4)} = S_6^{(4)} = \cdots ,$$
$$S_4^{(6)} = S_5^{(6)} = S_6^{(6)} = \cdots , \quad S_5^{(8)} = S_6^{(8)} = \cdots .$$

Property 5. *Any relations in terms of the cyclic averages $S_{n_1}^{(2m)}$, the circumscibed radius R and the distance L, which are satisfied for a regular n_1-gon, are at the same time satisfied for any regular n_2-gon, where $n_1 \le n_2$, i.e. $S_{n_1}^{(2m)}$ can be replaced by $S_{n_2}^{(2m)}$.*

3. Special cases

Equilateral triangle

There are 2 cyclic averages:

$$S_3^{(2)} = \frac{1}{3}\,(d_1^2 + d_2^2 + d_3^2) = R^2 + L^2,$$
$$S_3^{(4)} = \frac{1}{3}\,(d_1^4 + d_2^4 + d_3^4) = (R^2 + L^2)^2 + 2R^2L^2.$$

By eliminating L, we obtain the formula introduced by Garden

$$\frac{d_1^4 + d_2^4 + d_3^4}{3} + 3R^4 = \Big(\frac{d_1^2 + d_2^2 + d_3^2}{3} + R^2\Big)^2.$$

In terms of the cyclic averages

$$S_3^{(4)} + 3R^4 = (S_3^{(2)} + R^2)^2.$$

By Property 5, for any $S_n^{(4)}$ and $S_n^{(2)}$, where $n \ge 3$ (Bentin's result) we have

$$S_n^{(4)} + 3R^4 = (S_n^{(2)} + R^2)^2.$$

Square

There are 3 cyclic averages:

$$S_4^{(2)} = \frac{1}{4}\,(d_1^2 + d_2^2 + d_3^2 + d_4^2) = R^2 + L^2,$$
$$S_4^{(4)} = \frac{1}{4}\,(d_1^4 + d_2^4 + d_3^4 + d_4^4) = (R^2 + L^2)^2 + 2R^2L^2,$$
$$S_4^{(6)} = \frac{1}{4}\,(d_1^6 + d_2^6 + d_3^6 + d_4^6) = (R^2 + L^2)^3 + 6R^2L^2(R^2 + L^2).$$

Eliminating L from the cyclic averages $S_4^{(2)}$ and $S_4^{(6)}$ we obtain the second relation between the distances and the circumscribed radius

Proposition 3.1. *For any regular n-gon, where $n \geq 4$, we have*
$$S_n^{(6)} = S_n^{(2)}\big((S_n^{(2)} + 3R^2)^2 - 15R^4\big).$$

Substituting
$$R^2 + L^2 = S_4^{(2)} \text{ and } 2R^2L^2 = S_4^{(4)} - (S_4^{(2)})^2$$
into $S_4^{(6)}$, we establish the direct correspondence between the distances

Proposition 3.2. *For any regular n-gon, where $n \geq 4$,*
$$S_n^{(6)} = S_n^{(2)}\big(3S_n^{(4)} - 2(S_n^{(2)})^2\big).$$

For the square, from Proposition 3.2 it follows that
$$8(d_1^6+d_2^6+d_3^6+d_4^6)+(d_1^2+d_2^2+d_3^2+d_4^2)^3 = 6(d_1^2+d_2^2+d_3^2+d_4^2)(d_1^4+d_2^4+d_3^4+d_4^4),$$
which is equivalent to
$$(d_1^2 + d_2^2 - d_3^2 - d_4^2)(d_1^2 + d_3^2 - d_2^2 - d_4^2)(d_1^2 + d_4^2 - d_2^2 - d_3^2) = 0.$$
Enumerate the vertices of the square: $A_1A_2A_3A_4$. Then only
$$d_1^2 + d_3^2 = d_2^2 + d_4^2$$
holds, which together with the cyclic averages $S_4^{(2)}$ and $S_4^{(4)}$ implies
$$d_1^2 + d_3^2 = d_2^2 + d_4^2 = 2(R^2 + L^2),$$
$$d_1^2d_3^2 + d_2^2d_4^2 = 2(R^4 + L^4).$$

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