CIRCUMINVARIANTS OF 3-PERIODICS IN THE ELLIPTIC BILLIARD

DAN REZNIK and RONALDO GARCIA

Abstract. A Circumconic passes through a triangle’s vertices; an Inconic is tangent to the sidelines. We study the variable geometry of certain conics derived from the 1d family of 3-periodics in the Elliptic Billiard. Some display intriguing invariances such as aspect ratio and pairwise ratio of focal lengths. We also define the Circumbilliard, a circumellipse to a generic triangle for which the latter is a 3-periodic.

1. Introduction

Given a triangle, a circumconic passes through its three vertices and satisfies two additional constraints, e.g., center location. An Inconic touches each side and is centered at a specified location. Both these objects are associated (via isogonal or isotomic conjugation) with lines on the plane [28, Circumconic, Inconic] and therefore lend themselves to agile algebraic manipulation.

We study properties and invariants of such conics derived from a 1d family of triangles: 3-periodics in an Elliptic Billiard (EB): these are triangles whose bisectors coincide with normals to the boundary (bounces are elastic), see Figure 1.

Amongst all planar curves, the EB is uniquely integrable [15]. It can be regarded as a special case of Poncelet’s Porism [6]. These two properties imply two classic invariances: N-periodics have constant perimeter and envelop a confocal Caustic. The seminal work is [26] and more recent treatments include [18, 24].

We have shown 3-periodics also conserve the Inradius-to-Circumradius ratio which implies an invariant sum of cosines, and that their Mittenpunkt (where lines drawn from each Excenter thru sides’ midpoints meet) is stationary at the EB center [23]. Indeed many such invariants have been effectively generalized for N > 3 [1, 3].

We have also studied the loci of 3-periodic Triangle Centers over the family: out of the first 100 listed in [16], 29 sweep out ellipses (a remarkable fact on its own) with the remainder sweeping out higher-order curves [9]. Related is

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Figure 1. 3-periodics (blue) in the Elliptic Billiard (EB, black): normals to the boundary at vertices (black arrows) are bisectors. The family is constant-perimeter and envelops a confocal Caustic (brown). This family conserves the ratio inradius-to-circumradius and has a stationary Mittenpunkt at the EB center. Video: [21, PL#01].

the study of loci described by the Triangle Centers of the Poristic Triangle family [20].

Summary of the paper: We first describe the Circumbilliard: the circumellipse associated with a generic triangle for which the latter is a 3-periodic. We then analyze the dynamic of geometry of Circumbilliards for triangles derived from the 3-periodic family such as the Excentral, Anticomplementary, Medial, and Orthic, as well as the loci swept by their centers. We then describe invariants detected for Circumconics and Inconics, namely:

- Proposition 3.4 in Section 3 describes regions of the EB which produce acute, right-triangle, and obtuse 3-periodics.
- Theorem 3.1 in Section 3: The aspect ratio of Circumbilliards of the Poristic Triangle Family [7] is invariant. This is a family of triangle with fixed Incircle and Circumcircle.
- Theorem 4.1 in Section 4: The ratio of semi-axis of Circumellipses centered on the Incenter is invariant over the 3-periodic family. We conjecture this to be the case for a 1d-family of circumellipses.
- Theorem 5.1 in Section 5: The focal lengths of two special circum-hyperbola (Feuerbach and Excentral Jerabek) is constant over the 3-periodic family.
- Conjectures 6.1 and 6.2 in Section 6 claim the aspect ratios of two important Excentral Inconics are invariant. Candidate expressions are provided which match our experiments.

Appendices A, B, and C contain some longer derivations supporting the above. A reference table with all Triangle Centers, Lines, and Symbols appears in Appendix D. Videos illustrating some of the results appear on Table 2 in Section 7.

2. The Circumbilliard

Let the boundary of the EB satisfy:
(1) \[ f(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \]

Where \(a > b > 0\) denote the EB semi-axes throughout the paper. Below we use aspect ratio as the ratio of an ellipse’s semi-axes. When referring to Triangle Centers we adopt Kimberling \(X_i\) notation [16], e.g., \(X_1\) for the Incenter, \(X_2\) for the Barycenter, etc., see Table 3 in Appendix D.

The following five-parameter equation is assumed for all circumconics not passing through \((0, 0)\).

(2) \[ 1 + c_1x + c_2y + c_3xy + c_4x^2 + c_5y^2 = 0 \]

**Proposition 2.1.** Any triangle \(T = (P_1, P_2, P_3)\) is associated with a unique ellipse \(E_9\) for which \(T\) is a billiard 3-periodic. The center of \(E_9\) is \(T\)’s Mittenpunkt.

**Proof.** If \(T\) is a 3-periodic of \(E_9\), by Poncelet’s Porism, \(T\) is but an element of a 1d family of 3-periodics, all sharing the same confocal Caustic\(^1\). This family will share a common Mittenpunkt \(X_9\) located at the center of \(E_9\) [23]. Appendix A shows how to obtain the parameters for (2) such that it passes through \(P_1, P_2, P_3\) and is centered on \(X_9\): this yields a \(5 \times 5\) linear system. Solving it its obtained a quadratic equation with positive discriminant, hence the conic is an ellipse.

\(E_9\) is called the Circumbilliard (CB) of \(T\). Figure 2 shows examples of CBs for two sample triangles.

**Figure 2.** Two random triangles are shown as well as their Circumbilliards (CBs). Notice their axes in general are not horizontal/vertical. An algorithm for computing the CB is given in Appendix A. **Video:** [21, PL#02]

3. **Circumbilliards of Derived Triangles**

Figure 3 shows CBs for the Excentral, Anticomplementary (ACT), and Medial Triangles, derived from 3-periodics.

\(^1\) This turns out to be the Mandart Inellipse \(I_9\) of the family [28].
Figure 3. Draw in black in each picture is an $a/b \approx 1.618$, and a 3-periodic at $t = 7.0$ degrees. **Left:** the CB of the Excentral Triangle (solid green) centered on the latter’s Mittenpunkt is $X_{168}$ [16]. Its locus (red) is non-elliptic. Also shown (dashed green) is the elliptic locus of the Excenters (the MacBeath Circumellipse $E'_6$ of the Excentrals [28]), whose center is $X_9$ [9]. **Top Right:** the CB of the Anticomplementary Triangle (ACT) (blue), axis-aligned with the EB. Its center is the Gergonne Point $X_7$, whose locus (red) is elliptic and similar to the EB [9]. The locus of the ACT vertices is not elliptic (dashed blue). **Bottom Right:** the CB of the Medial Triangle (teal), also axis-aligned with the EB, is centered on $X_{142}$, whose locus (red) is also elliptic and similar to the EB, since it is the midpoint of $X_9X_7$ [16]. The locus of the medial vertices is a dumb-bell shaped curve (dashed teal). **Video:** [21, PL#03]

3.1. **Excentral Triangle.** The locus of the Excenters is shown in Figure 3 (left). It is an ellipse similar to the $90^\circ$-rotated locus of $X_1$ and its axes $a_e, b_e$ are given by [8, 9]:

$$a_e = \frac{b^2 + \delta}{a}, \quad b_e = \frac{a^2 + \delta}{b}$$

Where $\delta = \sqrt{\frac{a^4 - a^2b^2 + b^4}{a^2}}$.

**Proposition 3.1.** The locus of the Excenters the stationary MacBeath Circumellipse $E'_6$ [28] of the Excentral Triangles.

**Proof.** The center of $E'_6$ is the Symmedian Point $X_6$ [28, MacBeath Circumconic]. The Excentral Triangle’s $X_6$ coincides with the Mittenpunkt $X_9$ of the reference [16]. Since over the 3-periodics the vertices of the Excentral lie on an ellipse and its center is stationary, the result follows.

**Proposition 3.2.** The Excentral CB is centered on $X_{168}$, whose trilinears are irrational, and whose locus is non-elliptic.

**Proof.** $X_{168}$ is the Mittenpunkt of the Excentral Triangle [16] and its trilinears are irrational on the sidelengths. To determine if its locus is an

\[\text{No Triangle Center whose trilinears are irrational on sidelengths has yet been found whose locus under the 3-periodic family is an ellipse [9].}\]
ellipse we use the algebro-numeric techniques described in [9]. Namely, a least-squares fit of a zero-centered, axis-aligned ellipse to a sample of $X_{168}$ positions of the 3-periodic family produces finite error, therefore it cannot be an ellipse.

This had been observed in [9] for several irrational centers such as $X_i$, $i = 13-18$, as well as many others. Notice a center may be rational but produce a non-elliptic locus, the emblematic case being $X_6$, whose locus is a convex quartic. Other examples include $X_j$, $j = 19, 22–27$, etc.

3.2. Anticomplementary Triangle (ACT). The ACT is shown in Figure 3 (top right). The locus of its vertices is clearly not an ellipse.

The ACT is perspective with the reference triangle (3-periodic) at $X_2$ and all of its triangle centers correspond to the anticomplement of corresponding reference ones [28]. The center of the CB of the ACT is therefore $X_7$, the anticomplement of $X_9$. We have shown the locus of $X_7$ to be an ellipse similar to the EB with axes [9]:

$$(a_7, b_7) = k (a, b), \text{ with: } k = \frac{2\delta - a^2 - b^2}{a^2 - b^2}$$

Remark 3.1. The axes of the ACT CB are parallel to the EB and of fixed length.

This stems from the fact the ACT is homothetic to the 3-periodic.

3.3. Medial Triangle. The locus of its vertices is the dumbbell-shaped curve, which at larger $a/b$ is self-intersecting, and therefore clearly not an ellipse, Figure 3 (bottom right).

Like the ACT, the Medial is perspective with the reference triangle (3-periodic) at $X_2$. All of its triangle centers correspond to the complement of corresponding reference ones [28]. The center of the CB of the Medial is therefore $X_{142}$, the complement of $X_9$. This point is known to sit midway between $X_9$ and $X_7$.

Remark 3.2. The locus of $X_{142}$ is an ellipse similar to the EB.

This stems from the fact $X_9$ is stationary and the locus of $X_7$ is an ellipse similar to the EB (above). Therefore its axes will be given by:

$$(a_{142}, b_{142}) = (a_7, b_7)/2$$

Stemming from homothety of 3-periodic and its Medial:

Remark 3.3. The axes of the Medial CB are parallel to the EB and of fixed length.

3.4. Superposition of ACT and Medial.

Proposition 3.3. The Intouchpoints of the ACT (resp. 3-periodic) are on the EB (resp. on the CB of the Medial)

Proof. The first part was proved in [22, Thm. 2]. Because the 3-periodic can be regarded as the ACT of the Medial, the result follows.

\footnote{Anticomplement: a 1:2 reflection about $X_2$.}

\footnote{Complement: a 2:1 reflection about $X_2$.}
This phenomenon is shown in Figure 4. Also shown is the fact that \(X_i, i = 7, 142, 2, 9, 144\) are all collinear and their intermediate intervals are related as \(3 : 1 : 2 : 6\). In [17] this line is known as \(L(X_2, X_7)\) or \(L_{663}\). \(X_{144}\) is the perspector of the ACT and its Intouch Triangle (not shown) [28]. Video: [21, PL#04,05]

3.5. Orthic Triangle. Let \(\alpha_4 = \sqrt{2} - 1 \simeq 1.352\). In [22, Thm. 1] we show that if \(a/b > \alpha_4\), the 3-periodic family will contain obtuse triangles.

**Proposition 3.4.** If \(a/b > \alpha_4\), the 3-periodic is a right triangle when one of its vertices is at four symmetric points \(P_i^\perp, i = 1, 2, 3, 4\) given by \((\pm x^\perp, \pm y^\perp)\) with:

\[
x^\perp = \frac{a^2 \sqrt{a^4 + 3b^4 - 4b^2 \delta}}{c^3}, \quad y^\perp = \frac{b^2 \sqrt{-b^4 - 3a^4 + 4a^2 \delta}}{c^3}
\]

**Proof.** Let the coordinates of the 3-periodic vertices be \(P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3)\) as derived in [8].
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Figure 5. Two 3-periodics are shown: one acute (solid blue) and one obtuse (dashed blue) inscribed into an $a/b = 1.618$ EB. Red arcs along the top and bottom halves of the EB indicate that when a 3-periodic vertex is there, the 3-periodic is obtuse. These only exist when $a/b > \alpha_4 \simeq 1.352$.

Computing the equation $(P_2 - P_1, P_3 - P_1) = 0$, after careful algebraic manipulations, it follows that $x_1$ satisfies the quartic equation
\[ c^8 x_1^4 - 2a^4 c^2 (a^4 + 3b^4) x_1^2 + a^8 (a^4 + 2a^2 b^2 - 7b^4) = 0. \]
For $a/b > \sqrt{2} - 1$ the only positive root in the interval $(0, a)$ is given by
\[ x = \frac{a^2 \sqrt{a^4 + 3b^4 - 4b^2 \delta}}{c^3}. \]

With $y$ obtainable from (1).

Equivalently, a 3-periodic will be obtuse iff one of its vertices lies on top or bottom halves of the EB between the $P_1^\perp$, see Figure 5.

Consider the elliptic arc along the EB between $(\pm x^\perp, y^\perp)$. When a vertex of the 3-periodic lies within (resp. outside) this interval, the 3-periodic is obtuse (resp. acute).

**Proposition 3.5.** When $a/b > \alpha_4$, the locus of the center of the Orthic CB has four pieces: 2 for when the 3-periodic is acute (equal to the $X_6$ locus), and 2 when it is obtuse (equal to the locus of $X_6$ of $T'' = P_2 P_3 X_4$).

**Proof.** It is well-known that [16] an acute triangle $T$ has an Orthic whose vertices lie on the sidelines. Furthermore the Orthic’s Mittenpunkt coincides with the Symmedian $X_6$ of $T$. Also known is the fact that:

**Remark 3.4.** Let triangle $T' = P_1 P_2 P_3$ be obtuse on $P_1$. Its Orthic has one vertex on $P_2 P_3$ and two others exterior to $T'$. Its Orthocenter $X_4$ is also exterior. Furthermore, the Orthic’s Mittenpunkt is the Symmedian Point $X_6$ of acute triangle $T'' = P_2 P_3 X_4$.

To see this, notice the Orthic of $T''$ is also $T'$. $T''$ must be acute since its Orthocenter is $P_2$.

The CB of the orthic is shown in Figures 6 for four 3-periodic configurations in an EB whose $a/b > \alpha_4$.

\[ ^5 \text{The anti-orthic pre-images of } T' \text{ are both the 3-periodic and } T''. \]
Proposition 3.6. The coordinates \((\pm x^*, \pm y^*)\) where the locus of the center of the Orthic’s CB transitions from one curve to the other are given by:

\[
x^* = \frac{x^\perp}{c^6} \left( a^6 + 2a^2b^4 - b^2\delta(3a^2 + b^2) + b^6 \right)
\]

\[
y^* = -\frac{y^\perp}{c^6} \left( b^6 + 2a^4b^2 - a^2\delta(3b^2 + a^2)\delta + a^6 \right)
\]

Proof. Let \(P_1 = (x_1, y_1)\) be the right-triangle vertex of a 3-periodic, given by \((x^+, y^+)\) as in (4). Using [8], obtain \(P_2 = (p_{2x}/q_2, p_{2y}/q_2)\) and \(P_3 = (p_{3x}/q_3, p_{3y}/q_3)\), with:

\[
p_{2x} = b^4c^2x^3_1 - 2a^4b^2x^2_1y_1 + a^4c^2x_1y^2_1 - 2a^6y^3_1
\]

\[
p_{2y} = 2b^6x^3_1 - b^4c^2x^2_1y_1 + 2a^2b^4x_1y^2_1 - a^4c^2y^3_1
\]

\[
q_2 = b^4(a^2 + b^2)x^2_1 - 2a^2b^2c^2x_1y_1 + a^4(a^2 + b^2)y^2_1
\]

\[
p_{3x} = b^4c^2x^3_1 + 2a^4b^2x_1y_1 + a^4c^2x^2y^2_1 + a^6y^3_1
\]

\[
p_{3y} = -2b^6x^3_1 - b^4c^2x^2_1y_1 - 2a^2b^4x_1y^2_1 - a^4c^2y^3_1
\]

\[
q_3 = b^4(a^2 + b^2)x^2_1 + 2a^2b^2c^2x_1y_1 + a^4(a^2 + b^2)y^2_1
\]

It can be shown the Symmedian point \(X_6\) of a right-triangle is the midpoint of its right-angle vertex altitude. Computing \(X_6\) using this property leads to the result.

Let \(\alpha_{eq} = \sqrt{4\sqrt{3} - 3} \approx 1.982\) be the only positive root of \(x^4 + 6x^2 - 39\). It can be shown, see Figure 7:

Proposition 3.7. At \(a/b = \alpha_{eq}\), the locus of the Orthic CB is tangent to EB’s top and bottom vertices. If a 3-periodic vertex is there, the Orthic is equilateral.

Proof. Let \(T\) be an equilateral with side \(s_{eq}\) and center \(C\). Let \(h\) be the distance from any vertex of \(T\) to \(C\). It can be easily shown that \(h/s_{eq} = \sqrt{3}/3\). Let \(T’\) be the Excentral Triangle of \(T\): its sides are \(2s_{eq}\). Now consider the upside down equilateral in Figure 7, which is the Orthic of an upright isosceles 3-periodic. \(h\) is clearly the 3-periodic’s height and \(2s_{eq}\) is its base. The height and width of the upright isosceles are obtained from explicit expressions for the vertices [8]:

\[
s_{eq} = \frac{\alpha^2}{\alpha^2 - 1} \sqrt{2\delta - \alpha^2 - 1}, \quad h = \frac{\alpha^2 + \delta + 1}{\alpha^2 + \delta}
\]

where \(\alpha = a/b\). Setting \(h/s_{eq} = \sqrt{3}/3\) and solving for \(\alpha\) yields the required result for \(\alpha_{eq}\).

3.6. Summary. Table 1 summarizes the CBs discussed above, their centers, and their loci.
Figure 6. Orthic CB for an EB with $a/b = 1.5 > \alpha_4$, i.e., containing obtuse 3-periodics, which occur when a 3-periodic vertex lies on the top or bottom areas of the EB between the $P^\perp$. **Top left:** 3-periodic is sideways isosceles and acute (vertices outside $P^\perp$, so 3 orthic vertices lie on sidelines. The Orthic CB centers is simply the mittenpunkt of the Orthic, i.e, $X_6$ of the 3-periodic (blue curve: a convex quartic [9]). **Top right:** The position when a vertex is at a $P^\perp$ and the 3-periodic is a right triangle: its Orthic and CB degenerate to a segment. Here the CB center is at a first (of four) transition points shown in the other insets as $Q_i$, $i = 1, 2, 3, 4$. **Bottom left:** The 3-periodic is obtuse, the Orthic has two exterior vertices, and the center of the CB switches to the Symmedian of $T'' = P_1P_2X_4$ (red portion of locus). **Bottom right:** The 3-periodic is an upright isosceles, still obtuse, the center of the Orthic CB reaches its highest point along its locus (red). **Video:** [21, PL#06].

<table>
<thead>
<tr>
<th>Triangle</th>
<th>Center</th>
<th>Elliptic Locus</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Periodic</td>
<td>$X_9$</td>
<td>n/a</td>
</tr>
<tr>
<td>Excentral</td>
<td>$X_{168}$</td>
<td>No</td>
</tr>
<tr>
<td>ACT</td>
<td>$X_7$</td>
<td>Yes</td>
</tr>
<tr>
<td>Medial</td>
<td>$X_{142}$</td>
<td>Yes</td>
</tr>
<tr>
<td>Orthic</td>
<td>$X_6^*$</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 1. CBs mentioned in this Section, their Centers and loci types.

3.7. **Circumbilliard of the Poristic Family.** The Poristic Triangle Family is a set of triangles (blue) with fixed Incircle and Circumcircle [7]. It is a cousin of the 3-periodic family in that by definition its Inradius-to-Circumradius $r/R$ ratio is constant.

Weaver [27] proved the Antiorthic Axis$^6$ of this family is stationary. Odehnal showed the locus of the Excenters is a circle centered on $X_{10}$.

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$^6$The line passing through the intersections of reference and Excentral sidelines [28].
and of radius $2R$ [20]. He also showed that over the family, the locus of the Mittenpunkt $X_9$ is a circle whose radius is $2d^2(4R + r)$ and center is $X_1 + (X_1 - X_3)(2R - r)/(4R + r)$, where $d = |X_1X_3| = \sqrt{R(R - 2r)}$ [20, page 17].

Let $\rho = r/R$ and $a_9, b_9$ be the semi-axis lengths of the Circumbilliard a poristic triangle. As shown in Figure 8:

**Theorem 3.1.** The ratio $a_9/b_9$ is invariant over the family and is given by:

$$\frac{a_9}{b_9} = \sqrt{\frac{\rho^2 + 2(\rho + 1)\sqrt{1 - 2\rho} + 2}{\rho(\rho + 4)}}$$

where $\rho = r/R$.

**Proof.** The following expression for $r/R$ has been derived for the 3-periodic family of an $a, b$ EB [10, Equation 7]:

$$\rho = \frac{r}{R} = \frac{2(\delta - b^2)(a^2 - \delta)}{(a^2 - b^2)^2}$$

Solving the above for $a/b$ yields the result.
4. Invariants in Circumellipses

The Medial Triangle divides the plane in 7 regions, see Figure 9. The following is a known fact [2, 13]:

**Remark 4.1.** If the center of a Circumconic lies within 4 of these (resp. the remainder 3), the conic will be an Ellipse (resp. Hyperbola).

Centers $X_1$, $X_2$, and $X_9$ are always interior to the Medial Triangle [16], so the Circumconics $E_i$, $i = 1, 2, 9$ centered on them will ellipses, Figure 10. $E_2$ is the Steiner Circumellipse, least-area over all possible Circumellipses [28], and $E_9$ is $T$’s CB, see Section 2.

It is known that $E_1$ intersects the EB and the Circumcircle at $X_{100}$, the Anticomplement of the Feuerbach Point. Also that $E_2$ intersects $E_9$ at $X_{190}$, the Yff Parabolic Point [14, 16]. These two ellipses intersect at $X_{664}$ [19].

Given a generic triangle $T$:

**Proposition 4.1.** The axes of $E_1$ are parallel to $E_9$’s.

The proof is in Appendix B.

**Theorem 4.1.** Let $\eta_1$ and $\eta_1'$ be the lengths of minor and major semi-axes of $E_1$, respectively. The ratio of their lengths is constant over the 3-periodic family and given by:

$$\frac{\eta_1'}{\eta_1} = \frac{\sqrt{2b^2 + 2(a^2 - b^2)\delta - a^2b^2}}{b^2} > 1$$

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This is the isogonal conjugate of $X_{663}$, i.e., $L_{663}$ mentioned before is coincidentally its Trilinear Polar [28].
Figure 9. A reference triangle is shown (blue) as well as its Medial (red). The latter sides divide the plane into 7 regions, including the Medial’s interior. When a Circumconic center lies on any of the shaded regions (resp. unshaded) it is an Ellipse (resp. Hyperbola). Parabolas have centers at infinity. For illustration, the $X_1$ and $X_9$-centered Circumellipses and the $X_{11}$-centered Feuerbach Hyperbola are shown. Note that over the family of 3-periodics, a given Circumconic may alternate between Ellipse and Hyperbola, e.g., when centered on $X_4, X_5, X_6$, etc.

Proof. Calculate the ratio using vertex locations (see [10]) for an isosceles orbit, and then verify with a Computer Algebra System (CAS) the expression holds over the entire family.

Note: experimentally $\eta'_1$ is maximal (resp. minimal) when the 3-periodic is an isosceles with axis of symmetry parallel to the EB’s minor (resp. major) axis.

Proposition 4.2. The axes of $E_2$ are only parallel to $E_0$ if $T$ is isosceles.

See Appendix B.

4.1. Parallel-Axis Pencil. The Feuerbach Circumhyperbola of a Triangle is a rectangular hyperbola\(^8\) centered on $X_{11}$ [28]. Peter Moses has contributed a stronger generalization [19]:

Remark 4.2. The pencil of Circumconics whose centers lie on the Feuerbach Circumhyperbola $F_{med}$ of the Medial Triangle have mutually-parallel axes.

The complement\(^9\) of $X_{11}$ is $X_{3035}$ [16] so $F_{med}$ is centered there, see Figure 11. The following is a list of Circumellipses whose centers lie on $F_{med}$ [19]: $X_i, i=1, 3, 9, 10^{10}$, 119, 142, 214, 442, 600, 1145, 2092, 3126, 3307, 3308, 3647, 5507, 6184, 6260, 6594, 6600, 10427, 10472, 11517, 11530, 12631, 12639, 12640, 12641, 13089, 15346, 15347, 15348, 17057, 17060, 18258, 18642, 19557, 19584, 22754, 34261, 35204.

\(^8\)Since it passes through the Orthocenter $X_4$ [28].
\(^9\)The 2:1 reflection of a point about $X_2$.
\(^{10}\)Notice $X_{10}$ is the Incenter of the Medial. Interestingly, $X_8$, the Incenter of the ACT, does not belong to this select group.
Proposition 4.3. A circumellipse has center on $F_{\text{med}}$ iff it passes through $X_{100}$.

A proof appears in Appendix C. The following has been observed experimentally:

Conjecture 4.1. Over the family of 3-periodics, all Circumellipses in Moses’ pencil conserve the ratio of their axes.

5. A Special Pair of Circumhyperbolae

Here we study invariants of two well-known Circumhyperbolae: the Feuerbach and Jerabek Hyperbolas $F$ and $J$ [28, Jerabek Hyperbola]. Both are rectangular since they contain $X_4$ [28]. The former is centered on $X_{11}$ and the latter on $X_{125}$. With respect to 3-periodics no invariants have been detected for $J$. However, the Jerabek $J_{\text{exc}}$ of the Excentral Triangle, which passes through the Excenters and is centered on $X_{100}^{12}$, does produce interesting invariants. $F$ is known to pass through $X_1$ and $X_9$ of its reference triangle. Interestingly $J_{\text{exc}}$ also passes through $X_1$ and $X_9$. This stems from the fact that $J$ passes through $X_4$ and $X_6$. Since the Excentral Triangle is always acute [5], its $X_4$ is $X_1$. Likewise, the excentral $X_6$ is $X_9$.

The Isogonal Conjugate of a Circumconic is a line [28, Circumconic]. Remarkably:

Remark 5.1. The Isogonal conjugate of $F$ with respect to a reference triangle and that of $J_{\text{exc}}$ with respect to the Excentral one is line $X_1X_3 = L_{650}$.

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11Its centers lie in the unshaded regions in Figure 9.
12The Excentral’s $X_{125}$ [28].
13The Feuerbach Hyperbola $F_{\text{exc}}$ has not yet yielded any detectable invariants over the 3-periodic family.
Figure 11. An $a/b = 1.5$ EB is shown (black) centered on $X_a$ as well as a sample 3-periodic (blue). Also shown are Circumellipses centered on $X_i, i = 1, 3, 10, 142$, whose centers lie on the Feuerbach Circumhyperbola of the Medial Triangle (both shown red), centered on $X_{3035}$, the complement of $X_{11}$. Notice all conics drawn (including the Circumhyp) have axes parallel to the EB and all Circumellipses pass through $X_{100}$. Note: the Circumellipse centered on $X_3$ is the Circumcircle, its axes, drawn diagonally, are immaterial.

The first part is well-known [28, Feuerbach Hyperbola]. For the second part, consider that $J$ is the Isogonal Conjugate of the Euler Line [28, Jerabek Hyperbola]. The Euler Line of the Excentral Triangle passes through its $X_4$ and $X_5$ which are $X_1$ and $X_3$ in the reference 3-periodic.

Referring to Figure 12:

**Proposition 5.1.** $J_{exc}$ intersects $E_9$ in exactly two locations.

**Proof.** Let $s_i, i = 1, 2, 3$ refer to 3-periodic sidelengths. The perspector of $J_{exc}$ is $X_{649} = s_1(s_2 - s_3) ::$ (cyclical) [16]. Therefore the trilinears $x : y : z$ of $J_{exc}$ satisfy [29]:

$$J_{exc} : s_1(s_2 - s_3)x^2 + s_2(s_3 - s_1)y^2 + s_3(s_1 - s_2)z^2 = 0.$$ 

Notice the above is satisfied for the Excenters $[1 : 1 : -1], [1 : -1 : 1]$ and $[-1 : 1 : 1]$. As $X_1 = [1 : 1 : 1]$ and $X_9 = s_2 + s_3 - s_1 ::$ (cyclical) it follows that $J_{exc}(X_1) = J_{exc}(X_9) = 0$.

Eliminating variable $x$, the intersection of $J_{exc} = 0$ and $E_9 = 0$ is given by the quartic:

$$s_2(s_1 - s_3)k_1y^4 + 2s_2(s_1 - s_3)k_1k_2y^3z + 2s_3(s_1 - s_2)k_1k_2yz^3 + s_3(s_1 - s_2)k_1z^4 = 0$$
With \( k_1 = (s_1 + s_2 - s_3)^2 \) and \( k_2 = s_1 + s_3 - s_2 \). The discriminant of the above equation is:

\[-432[(s_2-s_3)(s_1-s_3)(s_1-s_2)(s_1+s_3-s_2)^2(s_1-s_2-s_3)^2(s_1+s_2-s_3)^2(s_1s_2s_3)^2]\]

Since it is negative, there will be two real and two complex solutions [4].

**Proposition 5.2.** \( F \) intersects the \( X_9 \)-centered Circumellipse at \( X_{1156} \).

**Proof.** The perspector of \( X_9 \) is \( X_1 \) and that of \( X_{11} \) is \( X_{650} = (s_3 - s_3)(s_3 + s_3 - s_1) :: (\text{cyclic}) \). Therefore, the trilinears \( x : y : z \) of \( F \) and \( E_9 \) satisfy:

\[
F : (s_2 - s_3)(s_2 + s_3 - s_1)/x + \\
(s_3 - s_1)(s_3 + s_1 - s_2)/y + \\
(s_1 - s_2)(s_1 + s_2 - s_3)/z = 0
\]

\( E_9 : 1/x + 1/y + 1/z = 0. \)

\( X_{1156} \) is given by \( 1/[(s_2 - s_3)^2 + s_1(s_2 + s_3 - 2s_1)] :: (\text{cyclic}) \). This point can be readily checked to satisfy both of the above.

Given a generic triangle \( T \), the following two claims are known:

**Proposition 5.3.** The asymptotes of both \( F \) and \( J_{\text{exc}} \) are parallel to the \( X_9 \)-centered circumconic, i.e., \( c_4 \) and \( c_5 \) in (2) vanish.

**Proof.** To see the first part, consider that since the Caustic is centered on \( X_9 \) and tangent to the \( 3 \)-periodics, it is the (stationary) Mandart Inellipse \( I_9 \) of the family [28]. This inconic is known to have axes parallel to the asymptotes of \( F \) [11]. Since the Caustic is confocal with the EB, \( F \) asymptotes must be parallel to the EB axes.

Secondly, \( 3 \)-periodics are the Orthic Triangles of the Excentrals, therefore the EB is the (stationary) Excentral’s Orthic Inconic [28]. The latter’s axes are known to be parallel to the asymptotes of the Jerabek hyperbola. [28, Orthic Inconic].

An alternate, algebraic proof appears in Appendix B.

Let \( \lambda \) (resp. \( \lambda' \)) be the focal length of \( F \) (resp. \( J_{\text{exc}} \)).

**Remark 5.2.** Isosceles \( 3 \)-periodics have \( \lambda' = \lambda = 0. \)

To see this consider the sideways isosceles \( 3 \)-periodic with \( P_1 = (a, 0) \).

\( P_2 \) and \( P_3 \) will lie on the 2nd and 3rd quadrants at \((-a_c, \pm y')\), where \( a_c = a(\delta - b^2)/(a^2 - b^2) \) is the length of the Caustic major semi-axis [9]. \( X_1 \) and \( X_4 \) will lie along the \( 3 \)-periodic’s axis of symmetry, i.e., the x-axis. To pass through all 5 points, \( F \) degenerates to a pair of orthogonal lines: the x-axis and the vertical line \( x = -a_c \). The foci will collapse to the point \((-a_c, 0)\).

A similar degeneracy occurs for the upright isosceles, i.e., when \( P_1 = (0, b) \), namely, the foci collapse to \((0, -a_c)\), where \( b_c = b(a^2 - \delta)/(a^2 - b^2) \) is the Caustic minor semi-axis length.

**Theorem 5.1.** For all non-isosceles \( 3 \)-periodics, \( \lambda'/\lambda \) is invariant and given by:
(6) \[
\frac{\lambda'}{\lambda} = \frac{\sqrt{\delta^2 + (a^2 + b^2) \delta + a^2b^2}}{ab} = \sqrt{2/\rho} > 2
\]

**Proof.** Assume the EB is in the form of (1). Let the 3-periodic be given by
\[ P_i = (x_i, y_i), i = 1, 2, 3. \] F passes through the \( P_i, X_1 \) and \( X_9 = (0, 0) \). The asymptotes of \( F \) are parallel to the EB axes, therefore this hyperbola is given by
\[ c_1 x + c_2 y + c_3 xy = 0 \] and \( \lambda^2 = |8c_1c_2/c_3^2| \), where:
\[ c_1 = y_2y_3(x_2 - x_3)x_1^2 + (x_2^2y_3 - x_3^2y_2 - y_2^2y_3 + y_2y_3^2)x_1y_1 + y_2y_3(x_2 - x_3)y_1^2 - (x_2y_3 - x_3y_2)(x_2x_3 + y_2y_3)y_1 \]
\[ c_2 = x_2x_3(y_2 - y_3)x_1^2 + (x_2x_3^2 - x_2^2x_3 - x_2y_3^2 + x_3y_3^2)x_1y_1 + (x_2y_3 - x_3y_2)(x_2x_3 + y_2y_3)x_1 - x_2x_3(y_2 - y_3)y_1^2 \]
\[ c_3 = (x_2y_3 - x_3y_2)x_1^2 + (x_2^2y_3 - x_2x_3^2 + y_2^2y_3 - y_2y_3^2)x_1 + (x_2y_3 - x_3y_2)y_1^2 + (x_2^2x_3 - x_2x_3^2 + x_2y_3^2 - x_3y_3^2)y_1 \]

Let \( P'_i = (x'_i, y'_i), i = 1, 2, 3 \) be the Excenters. They are given by
\[
\begin{align*}
P'_1 &= \left(-\frac{x_1 s_1 + x_2 s_2 + x_3 s_3}{s_2 + s_3 - s_1}, -\frac{y_1 s_1 + y_2 s_2 + y_3 s_3}{s_2 + s_3 - s_1} \right) \\
P'_2 &= \left(\frac{x_1 s_2 + x_2 s_2 + x_3 s_3}{s_3 + s_1 - s_2}, \frac{y_1 s_2 + y_2 s_2 + y_3 s_3}{s_3 + s_1 - s_2} \right) \\
P'_3 &= \left(\frac{x_1 s_1 + x_2 s_2 + x_3 s_3}{s_1 + s_2 - s_3}, \frac{y_1 s_1 + y_2 s_2 + y_3 s_3}{s_1 + s_2 - s_3} \right)
\end{align*}
\]

Here, \( s_1 = |P_2 - P_3|, s_2 = |P_1 - P_3| \) and \( s_3 = |P_1 - P_2| \).

Since \( J_{exc} \) is also centered on the origin and has horizontal/vertical asymptotes, \( J_{exc} \) is given by \( c'_1 x + c'_2 y + c'_3 xy = 0 \), and \( (\lambda')^2 = |8c'_1c'_2/c'_3^2| \), where \( c'_i \) are constructed as (7) replacing \( (x_i, y_i) \) with \( (x'_i, y'_i) \).

Consider a right-triangle\(^1\) 3-periodic, e.g., with \( P_1 \) at \( (x^1, y^1) \) given in (3) and \( P_2 \) and \( P_3 \) obtained explicitly [8]. From these obtain \( c_1 \) using (7). Using (8) obtain \( P'_i \) and the \( c'_i \). Finally, obtain a symbolic expression for \( \lambda'/\lambda \). After some manipulation and simplification with a Computer Algebra System (CAS), we obtain (6) which we call a candidate.

Parametrize the 3-periodic family with \( P_1(t) = (a \cos t, b \sin t) \) and using the sequence above arrive at an expression for \( \lambda'/\lambda \) in terms of \( t \). Subtract that from the right-triangle candidate. After some algebraic manipulation and CAS simplification verify the subtraction vanishes, i.e., \( \lambda'/\lambda \) is independent of \( t \).

5.1. **Focal Length Extrema.** Let \( P_1(t) = (a \cos t, b \sin t) \). While their ratio is constant, \( \lambda \) and \( \lambda' \) undergo three simultaneous maxima in \( t \in (0, \pi/2) \), see Figure 13. In fact, the following additional properties occur at configurations of maximal focal length (we omit the rather long algebraic proofs), see Figure 14:

- \( F' \) is tangent to the Caustic at \( \pm X_{11} \).
- \( J'_{exc} \) is tangent to the EB at \( \pm X_{100} \), i.e., at \( \mp X_{1156} \) (see below).

\(^1\)We found this to best simplify the algebra.
Figure 12. An $a/b = 1$ EB is shown (black) as well as a sample 3-periodic (blue), the confocal Caustic (brown), and the Excentral Triangle (green). The 3-periodic’s Feuerbach Circumhyperbola $F$ (orange) passes through its three vertices as well as $X_1$, $X_9$, and $X_4$. The Excentral’s Jerabek Circumhyperbola $J_{exc}$ (purple) passes through the three Excenters, as well as $X_1$, $X_9$ and $X_{40}$ (not shown). Two invariants have been detected over the orbit family: (i) the asymptotes (dashed) of both $F$ and $J_{exc}$ stay parallel to the EB axes, (ii) the ratio of focal lengths is constant (focal axis appears dashed). $F$ intersects the Billiard at $X_{1156}$.

Remark 5.3. Like $F$, $J_{exc}'$ intersects the EB at $X_{1156}$.

This happens because $X_{1156}$ is the reflection of $X_{100}$ about $X_9$. If the latter is placed on the origin, then $X_{1156} = -X_{100}$, and $J_{exc}'$ passes through $\pm X_{1156}$.

Let $F'$ and $J_{exc}'$ be copies of $F$ and $J_{exc}$ translated by $-X_{11}$ and $-X_{100}$ respectively, i.e., they become concentric with the EB (focal lengths are unchanged). Since their asymptotes are parallel to the EB axes and centered on the origin, their equations will be of the form:

$$F': xy = k'_F, \quad J_{exc}': xy = k'_j$$

Remark 5.4. $\lambda = 2\sqrt{2k'_F}$, $\lambda' = 2\sqrt{2k'_j}$, $\lambda' / \lambda = \sqrt{k'_j / k'_F} = \sqrt{2 / \rho}$.

6. Inconic Invariants

A triangle’s Inconic touches its three sides while satisfying two other constraints, e.g., the location of its center. Similar to Circumconics, if the latter is interior to the 4 shaded regions in Figure 9 it is an ellipse, else it is
Figure 13. Focal lengths $\lambda, \lambda'$ of $F, J_{exc}$ vs the parameter $t$ in $P(t) = (a \cos t, b \sin t)$ are shown red and green. The solid (resp. dashed) curves correspond to $a/b = 1.5$ (resp. $a/b = 1.3$). In the first quadrant there are 3 maxima. $\lambda'/\lambda$ (blue) remain constant for the whole interval.

Figure 14. Two snapshots of $J$ and $F_{exc}$ drawn solid blue and solid green, respectively, for $a/b = 1.5$. Also shown (dashed) are copies $F'$ and $J'_{exc}$ of both hyperbolas translated so they are dynamically concentric with the EB (translate $J$ by $-X_{11}$ and $F_{exc}$ by $-X_{100}$). Their focal lengths $\lambda, \lambda'$ are identical to the original ones; their focal axes are collinear and shown the dashed diagonal through the EB center. Notice that like $F$, $J'$ also intersects the EB at $X_{1156}$. Left: $t = 10.1^\circ$, showing an intermediate value of ether focal length. Right: $t = 6.2^\circ$, focal lengths are at a maximum. When this happens, the translated copy of $F$ (resp. $J_{exc}$) is tangent to the Caustic (resp. EB) at $X_{11}$ (resp. $X_{1156}$). Video: [21, PL#09,10]

a hyperbola. Lines drawn from each vertex to the Inconic tangency points concur at the perspector or Brianchon Point $B$ [28].

Let the Inconic center $C$ be specified by Barycentrics $g(s_1, s_2, s_3)$ (cyclic), then $B$ is given by $1/[(g(s_2, s_3, s_1) + g(s_3, s_1, s_2) - g(s_1, s_2, s_3))$ (cyclic) [25]. For example, consider the Inconic centered on $X_1$ (the Incircle), i.e., $g = s_2 s_3$ (cyclic). Then $B = 1/(s_1 s_3 + s_1 s_2 - s_2 s_3)$. Dividing by the product $s_1 s_2 s_3$ obtain $B = 1/(s_2 + s_3 - s_1)$ (cyclic), confirming that the perspector of the Incircle is the Gergonne Point $X_7$ [28, Perspector]. The contact points are the vertices of the Cevian triangle through $B$.

15Barycentrics $g$ can be easily converted to Trilinears $f$ via: $f(s_1, s_2, s_3) = g(s_1, s_2, s_3)/s_1$ (cyclic) [29].
Above we identified the confocal Caustic with the Mandart Inellipse $I_9$ [28] of the 3-periodic family, i.e., it is a stationary ellipse centered on $X_9$ and axis-aligned with the EB, Figure 1. Its semi-axes $a_c, b_c$ are given by [8]:

$$a_c = \frac{a (\delta - b^2)}{a^2 - b^2}, \quad b_c = \frac{b (a^2 - \delta)}{a^2 - b^2}.$$ 

Similarly, the $X_9$-centered Inconic of the family of Excentral Triangles is the stationary EB, i.e., the EB is the Orthic Inconic [28] of the Excenters.

6.1. Excentral $X_3$-Centered Inconic. No particular invariants have yet been found for any of the Inconics to 3-periodics with centers in $X_i, i = 3, 4, \ldots, 12$. Let $I_3$ denote the $X_3$-centered inconic. Its Brianchon Point is $X_{69}$ and its foci aren’t named centers [19].

**Remark 6.1.** $I_3$ is always an ellipse.

To see this consider the Circumcenter of an acute, right, or obtuse triangle lies inside, on a vertex, or opposite to the interior of the Medial, respectively, i.e., within one of the 4 shaded regions in Figure 9.

Let $I'_3$ denote the $X_3$-centered Inconic of the Excentral Triangle. Its Brianchon Point is $X_{69}$ of the Excentral, i.e., $X_{2951}$ of the reference 3-periodic [19], Figure 15(left).

**Remark 6.2.** $I'_3$ is axis-aligned with the EB and intersects it at $X_{100}$.

Let $\mu'_3, \mu_3$ denote $I'_3$ major and minor semi-axes.

**Conjecture 6.1.** $\mu'_3/\mu_3$ is invariant over all 3-periodics and given by:

$$\frac{\mu'_3}{\mu_3} = \frac{R + d}{R - d} = 1 + \frac{\sqrt{1 - 2\rho}}{\rho} - 1$$

with $d = \sqrt{R(R - 2r)}$, and $\rho = r/R$. The above has been derived for an isosceles 3-periodic. It matches the ratio numerically for any combination of $a, b$.

6.2. Excentral $X_5$-Centered (MacBeath) Inconic. Above the locus of the Excenters is identified with the Excentral MacBeath Circumconic $E'_6$. The MacBeath Inconic $I_5$ of a triangle is centered on $X_5$ and has foci on $X_4$ and $X_3$. Its Brianchon Point is $X_{264}$, and it can be both an ellipse or a hyperbola [28, MacBeath Inconic]. No invariants have yet been found for $I_5$.

Consider $I'_5$, the MacBeath Inconic of the Excentral Triangle, Figure 15(right). The center and foci of $I'_5$ with respect to the reference triangle are $X_3$, $X_1$, and $X_{40}$, respectively, and its Brianchon is $X_{1742}$ [19]. Unlike $I'_3$, the axes of $I'_5$ are askew with respect to the EB.

**Remark 6.3.** $I'_5$ is always an ellipse.

This is due to the fact that the Excentral is acute as is its homothetic Medial. Since $X_5$ is the latter’s Circumcenter, it must lie inside it.

Let $\mu'_5, \mu_5$ denote $I'_5$ major and minor semi-axes.
Conjecture 6.2. $\mu'_5/\mu_5$ is invariant over all 3-periodics and given by:

$$\frac{\mu'_5}{\mu_5} = \frac{R}{\sqrt{R^2 - d^2}} = \frac{1}{\sqrt{2}\rho}$$

The above was derived for an isosceles 3-periodic and shown to work for any $a, b$.

7. Conclusion

Videos mentioned above have been placed on a playlist [21]. Table 2 contains quick-reference links to all videos mentioned, with column “PL#” providing video number within the playlist.

Additionally to Conjecture 4.1 we submit the following questions to the reader:

- Can alternate proofs be found for Theorems 4.1 and 5.1 with tools of algebraic and/or projective geometry?
- Are there other notable circumconic pairs which exhibit interesting invariants?
- Can any of the invariants cited above be generalized to $N$-periodics?
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Table 2. Videos mentioned in the paper. Column “PL#” indicates the entry within the playlist [21].

- Can the invariant aspect ratios for the Excentral Inconics mentioned in Section 6 be proven?
- Are there other Inconic invariants over 3-periodics and/or their derived triangles?
- Are there interesting properties for the loci of the foci of Feuerbach and/or Jerabek Circumhyperbolas?
- The Yff Circumparabola whose focus is on $X_{190}$ is shown in [21, PL#12] over 3-periodics. Are there interesting invariants?
- Does any Triangle Circumcubic display interesting invariants over 3-periodics? We found none for the Thomson Cubic shown in [21, PL#13]. How about the Darboux, Neuberg, Lucas, and myriad others catalogued in [12].
- The ratio of focal lengths of $F$ by $J_{exc}$ is numerically invariant for the Poristic family. Can this be proved?
- Like 3-periodics, poristic triangles conserve $r/R$. Can a continuous map be specified between the two families? Are there any interesting properties?

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Appendix A. Computing a Circumconic

Let a Circumconic have center \( M = (x_m, y_m) \). Equation (2) is subject to the following 5 constraints\(^{16}\): it must be satisfied for vertices \( P_1, P_2, P_3 \), and its gradient must vanish at \( M \):

\[
\begin{align*}
 f(P_i) &= 0, \quad i = 1, 2, 3 \\
 \frac{dg}{dx}(x_m, y_m) &= c_1 + c_3 y_m + 2 c_4 x_m = 0 \\
 \frac{dg}{dy}(x_m, y_m) &= c_2 + c_3 x_m + 2 c_5 y_m = 0 \\
 \end{align*}
\]

Written as a linear system:

\[
\begin{bmatrix}
 x_1 & y_1 & x_1^2 & y_1^2 \\
 x_2 & y_2 & x_2^2 & y_2^2 \\
 x_3 & y_3 & x_3^2 & y_3^2 \\
 1 & 0 & x_m & 2x_m \\
 0 & 1 & x_m & 0 & 2y_m
\end{bmatrix}
\begin{bmatrix}
 c_1 \\
 c_2 \\
 c_3 \\
 c_4 \\
 c_5
\end{bmatrix}
= \begin{bmatrix}
 -1 \\
 -1 \\
 -1 \\
 0 \\
 0
\end{bmatrix}
\]

Given sidelengths \( s_1, s_2, s_3 \), the coordinates of \( X_9 = (x_m, y_m) \) can be obtained by converting its Trilinears \((s_2 + s_3 - s_1 :: \ldots)\) to Cartesians [16].

Principal axes’ directions are given by the eigenvectors of the Hessian matrix \( H \) (the jacobian of the gradient), whose entries only depend on \( c_3, c_4, \) and \( c_5 \):

\[
H = J(\nabla g) = \begin{bmatrix}
 2c_4 & c_3 \\
 c_3 & 2c_5
\end{bmatrix}
\]

The ratio of semiaxes’ lengths is given by the square root of the ratio of \( H \)’s eigenvalues:

\[
a/b = \sqrt{\lambda_2/\lambda_1}
\]

Let \( U = (x_u, y_u) \) be an eigenvector of \( H \). The length of the semi-axis along \( u \) is given by the distance \( t \) which satisfies:

\[
g(M + tU) = 0
\]

This yields a two-parameter quadratic \( d_0 + d_2 t^2 \), where:

\[
\begin{align*}
 d_0 &= 1 + c_1 x_m + c_4 x_m^2 + c_2 y_m + c_3 x_m y_m + c_5 y_m^2 \\
 d_2 &= c_4 x_u^2 + c_3 x_u y_u + c_5 y_u^2
\end{align*}
\]

The length of the semi-axis associated with \( U \) is then \( t = \sqrt{-d_0/d_2} \). The other axis can be computed via (10).

\(^{16}\)If \( M \) is set to \( X_9 \) one obtains the Circumbilliard.
The eigenvectors (axes of the conic) of $H$ are given by the zeros of the quadratic form

$$q(x, y) = c_3(y^2 - x^2) + 2(c_2 - c_5)xy$$

**Appendix B. Circumellipses of Elementary Triangle**

Let a triangle $T$ have vertices $P_1 = (0, 0)$, $P_2 = (1, 0)$ and side lengths $s_1, s_2, s_3$. Using the linear system in Appendix A, one can obtain implicit equations for the circumellipses $E_9, E_1, E_2$ centered on $T$’s Mittenpunkt $X_9$, Incenter $X_1$, and Barycenter $X_2$, respectively:

$$E_9(x, y) = v^2 x^2 - v(s_1 - s_2 - 1 + 2u)xy + ((s_1 - s_2 - 1)u + u^2 + s_2)y^2$$
$$-v^2 x + ((s_1 - s_2 - 1)u + u^2 + s_2)y^2 + v(u - s_2)y = 0$$

$$E_1(x, y) = (L - 2) v^2 x^2 + (L - 2 s_2 - 2u)(L - 2) vxy$$
$$+ (-L^2u + (2u + 1) Ls_2 + (u^2 + 2u) L - 2 s_2^2 - 4u s_2 - 2 u^2) y^2$$
$$- (L - 2) v^2 x - v(Ls_2 - uL - 2 s_2^2 + 2u) y$$

$$E_2(x, y) = v^2 x^2 + v(1 - 2u)xy + (u^2 - u + 1)y^2 - v^2 x + v(u - 1)y = 0$$

$s_1 = \sqrt{(u - 1)^2 + v^2}$, $s_2 = \sqrt{u^2 + v^2}$, $L = s_1 + s_2 + 1$

Consider the quadratic forms

$$q_9(x, y) = v(s_1 - s_2 + 2u - 1)x^2 + 2((s_2 - s_1)u - s_2 - u^2 + u + v^2) xy$$
$$+v(1 - 2u - s_1 + s_2) y^2$$

$$q_1(x, y) = -v(L - 2) (L - 2 s_2 - 2u)x^2 + v(L - 2) (L - 2 s_2 - 2u) y^2$$
$$+2(L^2u - (2u + 1)Ls_2 + (v^2 - u^2 - 2u)L + 4s_2 u + 4u^2)xy$$

$$q_2(x, y) = v(2u + 1)x^2 + 2(-(u^2 + v^2 + u - 1)xy + v(1 - 2u) y^2$$

The axes of $E_9$ (resp. $E_1$) are defined by the zeros of $q_9$ (resp. $q_1$). Using the above equations it is straightforward to show that the axes of $E_1$ and $E_9$ are parallel.

The axes of $E_2$ and $E_9$ are parallel if and only if $(u - 1)^2 + v^2 = 1$ or $u^2 + v^2 = 1$; this means that the triangle is isosceles.

The implicit equations of the circumhyperbolas $F$ passing through the vertices of the orbit centered on $X_{11}$ and $J_{exc}$ passing through the vertices of
the excentral triangle and centered on $X_{100}$ are:

$$F(x, y) = v^3(2u - 1)(x^2 - y^2) + v^3(1 - 2u)x$$
$$+ [((s_3^2 + (u - 1)s_2^2 - us_2)s_1 + (2u - 1)s_2^2 - u(s^2_2 - s_2) - 4u^2v^2 + v^4 + 2uv^2)xy$$
$$+ s_1^2u^2s_2 + (us_2 - u(u - 1)s_2^2 + s_2u^2)s_1 + us_2^2 - v^2s_2^2 - u^3(2u - 1)y = 0$$

$$J_{exc}(x, y) = 4v^3(2u - 1)(x^2 - y^2)$$
$$+ [(4s_2^2 + 4u - 1)s_2^2 - 4us_2) s_1 - 4us_2s_2 - 4s_2^2$$
$$- 4u^4 - 16u^2v^2 + 4v^4 + 8u^3 + 16uv^2]xy$$
$$+ [(2(s_3^2 + (1 - s_1)u - v^2 + s_2) + us_1(s_1 + 1)s_2 + u(u - 2)s_1^2)]v|x$$
$$+ [(4 - 4u)s_2((u - (1 - s_1)u + (1/2)s_1 + 1/2)s_2 + 1/2us_1(s_1 + 1) - 1/2s_2^2(s_2 + s_1))]y$$
$$- us_1^2v_2 + (vs_2^2 + (-1 + u)uv_2^2 - us_2)s_1 + vs_2^2 - (2u^2 - 2u + 1)uv_2^2 = 0$$

Using the above equations it is straightforward to show that the axes of $E_9$ and asymptotes of $F$ and $J_{exc}$ are parallel.

**APPENDIX C. CIRCUMELLIPSES WITH PARALLEL AXES**

Consider a triangle with vertices $A = (u, v), B = (-1, 0)$ and $C = (1, 0)$.

Let $s_2 = |A - B|$ and $s_3 = |B - C|$. The equation of $F_{med}$ is given by:

$$F_{med}(x, y) = 4uv^3(x^2 - y^2)(2u - 1)s_3s_2 - (2(u - 1)s_3^2 - 2(u^2 + v^2 - 1)s_3)s_2$$
$$2(u^2 - 1)^2 + 2v^2(4u^2 - v^2)]xy$$

(11) $$+ (-(u^2 + v^2 - 1)s_3 + s_3^2(u - 1))vs_2 + s_3^2(u + 1)vs_3$$
$$- vu^3 + (u^2 - 1)^2)x$$
$$+ (2u^2 + 2v^2 - 2)uv^2 y = 0$$

A one parameter family of circumellipses passing through $A, B, C$ and $X_{100}$ is given by:

$$E_b(x, y) = uv^3(bxy - 4x^2)$$
$$+ ((b(u^2 + v^2 - 1)s_3 + b(u - 1)s_2^2)s_2 - b(u + 1)s_2^2s_3$$

(12) $$- v^2(4u^2 - v^2) + 4uv) - b(u^2 - 1)^2)xy$$
$$- v((b(u^2 + v^2 - 1)s_2 - b(u + 1)s_2^2)s_3 + b(u - 1)s_3^2s_2$$
$$- b(u^2 + v^2 - 1)(u^2 - v^2 + uv - 1))y + 4uv^3 = 0$$

It is straightforward to verify that the family $E_b$ is axis aligned (independent of b). Denoting the center of $E_b$ by $(x_c, y_c)$ it follows, using CAS, that $F_{med}(x_c, y_c) = 0$.

The reciprocal follows similarly.
### Appendix D. Table of Symbols

Tables 3 and 4 list most Triangle Centers and symbols mentioned in the paper.

<table>
<thead>
<tr>
<th>Center</th>
<th>Meaning</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>Incenter</td>
<td>Locus is Ellipse</td>
</tr>
<tr>
<td>$X_2$</td>
<td>Barycenter</td>
<td>Perspector of Steiner Circum/Inellipses</td>
</tr>
<tr>
<td>$X_3$</td>
<td>Circumcenter</td>
<td>Locus is Ellipse, Perspector of $M$</td>
</tr>
<tr>
<td>$X_4$</td>
<td>Orthocenter</td>
<td>Exterior to EB when 3-periodic is obtuse</td>
</tr>
<tr>
<td>$X_5$</td>
<td>Center of the 9-Point Circle</td>
<td></td>
</tr>
<tr>
<td>$X_6$</td>
<td>Symmedian Point</td>
<td>Locus is Quartic [9]</td>
</tr>
<tr>
<td>$X_6^*$</td>
<td>$X_9$ of Orthic</td>
<td>Detached from $X_6$ locus for obtuse triangles</td>
</tr>
<tr>
<td>$X_7$</td>
<td>Gergonne Point</td>
<td>Perspector of Incircle</td>
</tr>
<tr>
<td>$X_8$</td>
<td>Nagel Point</td>
<td>Perspector of $I_9$, $X_1$ of ACT Incircle</td>
</tr>
<tr>
<td>$X_9$</td>
<td>Mittenpunkt</td>
<td>Center of (Circum)billiard</td>
</tr>
<tr>
<td>$X_{10}$</td>
<td>Spieker Point</td>
<td>Incenter of Medial</td>
</tr>
<tr>
<td>$X_{11}$</td>
<td>Feuerbach Point</td>
<td>on confocal Caustic</td>
</tr>
<tr>
<td>$X_{40}$</td>
<td>Bevan Point</td>
<td>$X_3$ of Excentral</td>
</tr>
<tr>
<td>$X_{69}$</td>
<td>$X_6$ of the ACT</td>
<td>Perspector of $I_3$</td>
</tr>
<tr>
<td>$X_{100}$</td>
<td>Anticomplement of $X_{11}$</td>
<td>On Circumcircle and EB, $J_{exc}$ center</td>
</tr>
<tr>
<td>$X_{125}$</td>
<td>Center of Jerabek Hyperbola $J$</td>
<td></td>
</tr>
<tr>
<td>$X_{142}$</td>
<td>$X_9$ of Medial</td>
<td>Midpoint of $X_9X_7$, lies on $L(2, 7)$</td>
</tr>
<tr>
<td>$X_{144}$</td>
<td>Anticomplement of $X_7$</td>
<td>Perspector of ACT and its Intouch Triangle</td>
</tr>
<tr>
<td>$X_{168}$</td>
<td>$X_9$ of the Excentral Triangle</td>
<td>Non-elliptic Locus</td>
</tr>
<tr>
<td>$X_{190}$</td>
<td>Focus of the Yff Parabola</td>
<td>Intersection of $E_2$ and the EB</td>
</tr>
<tr>
<td>$X_{264}$</td>
<td>Isotomic Conjugate of $X_3$</td>
<td>Perspector of $I_5$</td>
</tr>
<tr>
<td>$X_{649}$</td>
<td>Cross-difference of $X_1, X_2$</td>
<td>Perspector of $J_{exc}$</td>
</tr>
<tr>
<td>$X_{664}$</td>
<td>Trilinear Pole of $L(2, 7)$</td>
<td>Intersection of $E_1$ and $E_2$ [19]</td>
</tr>
<tr>
<td>$X_{650}$</td>
<td>Cross-difference of $X_1, X_3$</td>
<td>Perspector of $F$</td>
</tr>
<tr>
<td>$X_{1156}$</td>
<td>Isogonal Conjugate of Schröder Point $X_{1155}$</td>
<td>Intersection of $F$ with EB</td>
</tr>
<tr>
<td>$X_{1742}$</td>
<td>Mimosa Transform of $X_{212}$</td>
<td>Perspector of $I'_5$</td>
</tr>
<tr>
<td>$X_{2951}$</td>
<td>Excentral-Isogonal Conjugate of $X_{57}$</td>
<td>Perspector of $I'_3$</td>
</tr>
<tr>
<td>$X_{3035}$</td>
<td>Complement of $X_{11}$</td>
<td>Center of $F_{med}$</td>
</tr>
<tr>
<td>$X_{3659}$</td>
<td>$X_{11}$ of Excentral Triangle</td>
<td>Center of $F_{exc}$</td>
</tr>
<tr>
<td>$L(2, 7)$</td>
<td>ACT-Medial Mittenpunkt Axis</td>
<td>Line $L_{063}$ [17]</td>
</tr>
<tr>
<td>$L(1, 3)$</td>
<td>Isogonal Conjugate of $F$ and $J_{exc}$</td>
<td>Line $L_{050}$ [17]</td>
</tr>
</tbody>
</table>

*Table 3. Kimberling Centers and Central Lines mentioned in paper*
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a, b$</td>
<td>EB semi-axes</td>
<td>$a &gt; b &gt; 0$</td>
</tr>
<tr>
<td>$P_i, s_i$</td>
<td>Vertices and sidelengths of 3-periodic invariant</td>
<td>$\sum s_i$</td>
</tr>
<tr>
<td>$P'_i$</td>
<td>Vertices of the Excentral Triangle</td>
<td></td>
</tr>
<tr>
<td>$a_c, b_c$</td>
<td>Semi-axes of confocal Caustic</td>
<td></td>
</tr>
<tr>
<td>$a_9, b_9$</td>
<td>Semi-axes of Poristic Circumbilliard</td>
<td></td>
</tr>
<tr>
<td>$r, R, \rho$</td>
<td>Inradius, Circumradius, $\rho$ is invariant</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>Oft-used constant</td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>Distance $</td>
<td>X_1X_3</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>EB aspect ratio</td>
<td></td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>$a/b$ threshold for obtuse 3-Periodics</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{eq}$</td>
<td>$a/b$ for equilateral Orthic</td>
<td></td>
</tr>
<tr>
<td>$P_{\perp}$</td>
<td>Obtuse 3-periodic limits on EB</td>
<td></td>
</tr>
<tr>
<td>$x^<em>, y^</em>$</td>
<td>where $X_{4i}^*$ detaches from $X_4$ locus</td>
<td></td>
</tr>
<tr>
<td>$F, J$</td>
<td>Feuerbach, Jerabek Hyperbola</td>
<td></td>
</tr>
<tr>
<td>$F_{exc}$</td>
<td>$F$ of Excentral Triangle</td>
<td></td>
</tr>
<tr>
<td>$J_{exc}$</td>
<td>$J$ of Excentral Triangle</td>
<td></td>
</tr>
<tr>
<td>$F', J'_{exc}$</td>
<td>$F, J_{exc}$ translated by $-X_{11}, -X_{100}$</td>
<td></td>
</tr>
<tr>
<td>$F_{med}$</td>
<td>$F$ of Medial</td>
<td></td>
</tr>
<tr>
<td>$E_i$</td>
<td>Circumellipse centered on $X_i$</td>
<td></td>
</tr>
<tr>
<td>$E'_i$</td>
<td>Excentral MacBeath Circumellipse</td>
<td></td>
</tr>
<tr>
<td>$I_{3, 5}$</td>
<td>Inellipses Centered on $X_3, X_5$</td>
<td></td>
</tr>
<tr>
<td>$I_{3, 5}$</td>
<td>Excentral Inellipse</td>
<td></td>
</tr>
<tr>
<td>$I'_5$</td>
<td>Excentral $I_3$</td>
<td></td>
</tr>
<tr>
<td>$I''_5$</td>
<td>Excentral $I_5$</td>
<td></td>
</tr>
<tr>
<td>$\eta'_{i, i}, \eta_i$</td>
<td>Major and minor semiaxis of $E_i$</td>
<td></td>
</tr>
<tr>
<td>$\mu'_{i, i}, \mu_i$</td>
<td>Major and minor semiaxis of $I'_i$</td>
<td></td>
</tr>
<tr>
<td>$\lambda', \lambda$</td>
<td>Focal lengths of $J_{exc}, F$ (and $J'_{exc}, F'\dots$)</td>
<td></td>
</tr>
</tbody>
</table>

| Table 4. Symbols used in paper |

REFERENCES


Circuminvariants of 3-Periodics in the Elliptic Billiard