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GENERALIZATION OF THE NOTION OF WEIL BUNDLE

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Abstract. The manifold of near points such defined by André Weil always appears as a fiber of a certain fiber bundle. We generalize this fact by constructing a fibre bundle which fiber at any point is a manifold of near points.

1. Introduction

Given a smooth manifolf M, A a local algebra in the sense of André Weil (that is to say a real unitary commutative algebra of finite dimension having a unique maximum ideal of codimension 1 on $\mathbb{R}$), M^A denotes the manifold of near points to M of kind A (a Weil bundle of M of kind A)[4].

When $\mathbb{D} = \mathbb{R}[X]/(X^2)$ is the algebra of dual numbers, $M^{\mathbb{D}} = J_0^1(\mathbb{R}, M)$ is the space of jets in zero of order 1 of the differentiable applications of $\mathbb{R}$ into M[4].

When $k \geq 1$ is an integer and when $A = J_x^k(M, \mathbb{R})$ is the space of the jets at a point $x \in M$ of order k of the differentiable functions on M, for any other smooth manifold N , we have [3]

$$N^A = J_x^k(M, N).$$

Note that the module $\mathfrak{X}(M^A)$ of the vector fields on M^A is simultaneously a $C^\infty(M^A, A)$-module and a Lie algebra on A [1].

The examples above show that the manifold of near points to M of kind A as defined by André Weil always appears as the fiber of a certain bundle. The aim of this paper is to construct a bundle in such a way that the fibers are manifolds of near points (Weil bundles).

In all that follows M is a smooth manifold, A a local algebra in the sense of André Weil and $C^\infty(M)$ the real algebra of smooth functions on M. Fiber bundles considered are assumed to have global sections.

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2. Generalized manifold of near points

We consider a fibered space (E, π, M) and $(\mathcal{A}, p, M)$ a fibered space in local algebras of fiber-type local algebra A. For $x \in M$, E_x and $\mathcal{A}_x$ denote the respective fibers above x.

The set

$$E^{\mathcal{A}} = \bigcup_{x \in M} E_x^{\mathcal{A}_x} \tag{1}$$

is a fibered manifold on M of dimension $m + n \cdot \dim(A)$ where m is the dimension of M and n the dimension of each fiber E_x. The fibers of this fibered manifold are manifolds of near points as defined by André Weil.

We say that the manifold $E^{\mathcal{A}}$ is the generalized Weil bundle.

When M and N are smooth manifolds, the set $C^\infty(M, N)$ of differentiable maps of class C^∞ from M to N is canonically isomorphic to the set of sections of the trivial bundle

$$(M \times N, pr_1, M).$$

When $E = M \times \ N$ and $\mathcal{A} = M \times \ A$, the bundle $E^{\mathcal{A}}$ is identified canonically with the trivial bundle $M \times N^{A}$.

For $x \in M$, we recall that $C^\infty(E_x^{\mathcal{A}_x}, \mathcal{A}_x)$ is an $\mathcal{A}_x$-unitary commutative real algebra and that $\mathfrak{X}(E_x^{\mathcal{A}_x})$ is simultaneously a $C^\infty(E_x^{\mathcal{A}_x}, \mathcal{A}_x)$-module and an $\mathcal{A}_x$-Lie algebra[1].

3. Structure of $\Gamma(\mathcal{A})$-algebra on $\Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]$

The set $\Gamma(\mathcal{A})$ of global sections of the bundle $(\mathcal{A}, p, M)$ is a $C^\infty(M)$-module. For $a, b \in \Gamma(\mathcal{A})$ and for $\lambda \in \mathbb{R}$, the applications

$$\begin{aligned} ab &: M \longrightarrow \mathcal{A}, x \longmapsto a(x) \cdot b(x) \\ \lambda a &: M \longrightarrow \mathcal{A}, x \longmapsto \lambda \cdot a(x) \end{aligned}$$

define a unitary commutative real algebra structure on $\Gamma(\mathcal{A})$.

We notice

$$C_M^\infty(E^{\mathcal{A}}, \mathcal{A}) = \bigcup_{x \in M} C^\infty(E_x^{\mathcal{A}_x}, \mathcal{A}_x) \tag{2}$$

and $\Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]$ denotes the set of global sections of the bundle

$$C_M^\infty(E^{\mathcal{A}}, \mathcal{A}) \longrightarrow M.$$

For $a \in \Gamma(\mathcal{A})$ and $s \in \Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]$, we define $as \in \Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]$, with

$$as : M \longrightarrow C_M^\infty(E^{\mathcal{A}}, \mathcal{A}), x \longmapsto a(x) \cdot s(x),$$

and for $s_1, s_2 \in \Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]$,

$$s_1 s_2 : M \longrightarrow C_M^\infty(E^{\mathcal{A}}, \mathcal{A}), x \longmapsto s_1(x) \cdot s_2(x).$$

We can thus state:

Proposition 3.1. *The set* $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$ *is a unitary commutative* $\Gamma(\mathcal{A})$*-algebra.*

We deduce the following corollary:

Corollary 3.2. *The set*

$$Der_{\Gamma(\mathcal{A})}\left(\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right), \tag{3}$$

of derivations of $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$ *which are* $\Gamma(\mathcal{A})$*-linear, is simultaneously a* $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$*-modulus and a Lie algebra on* $\Gamma(\mathcal{A})$*.*

4. Other structure of $\Gamma(\mathcal{A})$-Lie algebra and of $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$-module

We notice

$$Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]=\bigcup_{x\in M}Der_{\mathcal{A}_x}\left[C^\infty(E_x^{\mathcal{A}_x},\mathcal{A}_x)\right] \tag{4}$$

where $Der_{\mathcal{A}_x}\left[C^\infty(E_x^{\mathcal{A}_x},\mathcal{A}_x)\right]$ is the $C^\infty(E_x^{\mathcal{A}_x},\mathcal{A}_x)$- module of derivations of $C^\infty(E_x^{\mathcal{A}_x},\mathcal{A}_x)$ which are $\mathcal{A}_x$-linear, it is also the $C^\infty(E_x^{\mathcal{A}_x},\mathcal{A}_x)$-module, $\mathfrak{X}(E_x^{\mathcal{A}_x})$ of vector fields on the closed submanifold $E_x^{\mathcal{A}_x}$ of $E^{\mathcal{A}}$.

For $s\in\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$ and $\zeta\in\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$, we define the section

$$s\cdot\zeta: M\longrightarrow Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A}\right], x\longmapsto s(x)\cdot\zeta(x). \tag{5}$$

Proposition 4.1. *The map*

$$\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\times\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)\longrightarrow\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right), \tag{6}$$

$(s,\zeta)\longmapsto s\cdot\zeta,$
defines a structure of $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$*-module on* $\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$*.*

Proof. Simple verification.

For $\zeta_1,\zeta_2\in\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$, we define the following section:

$$[\zeta_1,\zeta_2]: M\longrightarrow Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right], x\longmapsto\zeta_1(x)\circ\zeta_2(x)-\zeta_2(x)\circ\zeta_1(x). \tag{7}$$

We can state:

Proposition 4.2. *The map*

(8)

$$\left[\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)\right]^2\longrightarrow\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right),(\zeta_1,\zeta_2)\longmapsto[\zeta_1,\zeta_2],$$

is $\Gamma(\mathcal{A})$*-bilinear alternating. Moreover it defines a structure of* $\Gamma(\mathcal{A})$ *-Lie algebra on* $\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$*.*

Proof. Simple verification.

The two preceding propositions mean that the set $\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$ is simultaneously provided with a structure of $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$-module and of $\Gamma(\mathcal{A})$-Lie algebra.

5. Comparison of the two structures

In what follows we compare the structures of $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$-modules and of $\Gamma(\mathcal{A})$-Lie algebras defined above.

For $\zeta \in \Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$ and $s \in \Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$, we define the section

$$\zeta(s) : M \longrightarrow Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right], x \longmapsto \zeta(x)\left[s(x)\right]. \tag{9}$$

Proposition 5.1. *The map*

(10)

$$\Phi : \Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right) \longrightarrow Der_{\Gamma(\mathcal{A})}\left(\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right), \zeta \longmapsto \Phi(\zeta),$$

such that

$$\Phi(\zeta) : \Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right] \longrightarrow \Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right], s \longmapsto \zeta(s), \tag{11}$$

is a morphism of $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$-modules and of de $\Gamma(\mathcal{A})$-Lie algebras.

Proof. When $\zeta_1,\zeta_2 \in \Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$ and for all $s \in \Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$, we have:

$$\left[\Phi(\zeta_1+\zeta_2)\right](s) = (\zeta_1+\zeta_2)(s).$$

For all $x \in M$, we have

$$\begin{aligned} \left[(\zeta_1+\zeta_2)(s)\right](x) &= (\zeta_1(x)+\zeta_2(x))\left[s(x)\right] \\ &= (\zeta_1(x))\left[s(x)\right] + (\zeta_2(x))\left[s(x)\right] \\ &= \left[\zeta_1(s)\right](x) + \left[\zeta_2(s)\right](x) \\ &= \left[\zeta_1(s)+\zeta_2(s)\right](x). \end{aligned}$$

Since x is arbitrary, we deduce that

$$(\zeta_1+\zeta_2)(s) = \zeta_1(s)+\zeta_2(s).$$

Which means that

$$\begin{aligned} \Phi(\zeta_1+\zeta_2)(s) &= \Phi(\zeta_1)(s)+\Phi(\zeta_2)(s) \\ &= \left(\Phi(\zeta_1)+\Phi(\zeta_2)\right)(s). \end{aligned}$$

Since s is arbitrary, we have thus

$$\Phi(\zeta_1+\zeta_2) = \Phi(\zeta_1)+\Phi(\zeta_2). \tag{12}$$

We consider $s \in \Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$ and $\zeta \in \Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$. For all $s' \in \Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$, we have:

$$\left[\Phi(s\cdot\zeta)\right](s') = (s\cdot\zeta)(s').$$

For $x \in M$, we have

$$\begin{aligned} \left[(s\cdot\zeta)(s')\right](x) &= (s\cdot\zeta)(x)\left[s'(x)\right] \\ &= (s(x)\cdot\zeta(x))\left[s'(x)\right] \\ &= s(x)\cdot\left(\zeta(x)\left[s'(x)\right]\right) \\ &= s(x)\cdot\left[\zeta(s')\right](x) \\ &= \left[s\cdot\zeta(s')\right](x). \end{aligned}$$

Since x is arbitrary, we thus have

$$\left[(s\cdot\zeta)(s')\right]=\left[s\cdot\zeta(s')\right]$$

Which means

$$\left[\Phi(s\cdot\zeta)\right](s')=\left[s\cdot\Phi(\zeta)\right](s').$$

Since s' is arbitrary, hence

$$\Phi(s\cdot\zeta)=s\cdot\Phi(\zeta). \tag{13}$$

We deduce that (12) and (13) signify that Φ is a morphism of $\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$-modules.

Let $a\in\Gamma(\mathcal{A})$ and let $\zeta\in\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$. For all $s\in\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$, we have

$$\left[\Phi(a\cdot\zeta)\right](s)=(a\cdot\zeta)(s).$$

For all $x\in M$, we have

$$\begin{aligned}\left[(a\cdot\zeta)(s)\right](x) &= (a\cdot\zeta)(x)\left[s(x)\right]\\ &= a(x)\cdot\zeta(x)\left[s(x)\right]\\ &= a(x)\cdot\left[\zeta(s)\right](x)\\ &= (a\cdot\zeta(s))(x).\end{aligned}$$

Since x is arbitrary, so

$$(a\cdot\zeta)(s)=a\cdot\zeta(s).$$

Which means

$$\left[\Phi(a\cdot\zeta)\right](s)=\left[a\cdot\Phi(\zeta)\right](s).$$

Since s is also arbitrary, hence

$$\Phi(a\cdot\zeta)=a\cdot\Phi(\zeta). \tag{14}$$

The (14) means that Φ is $\Gamma(\mathcal{A})$-linear.

We consider $\zeta_1,\zeta_2\in\Gamma\left(Der_{\mathcal{A}}\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]\right)$. For all $s\in\Gamma\left[C^\infty_M(E^{\mathcal{A}},\mathcal{A})\right]$, we have

$$\left(\Phi\left[\zeta_1,\zeta_2\right]\right)(s)=\left[\zeta_1,\zeta_2\right](s).$$

For all $x\in M$, we have

$$\begin{aligned}\left(\left[\zeta_1,\zeta_2\right](s)\right)(x) &= \left[\zeta_1,\zeta_2\right](x)(s(x))\\ &= \zeta_1(x)\left[\zeta_2(x)(s(x))\right]-\zeta_2(x)\left[\zeta_1(x)(s(x))\right]\\ &= \zeta_1(x)\left[\zeta_2(s)\right](x)-\zeta_2(x)\left[\zeta_1(s)\right](x)\\ &= \left(\zeta_1\left[\zeta_2(s)\right]\right)(x)-\left(\zeta_2\left[\zeta_1(s)\right]\right)(x)\end{aligned}$$

Since x is arbitrary, we therefore have

$$\left[\zeta_1,\zeta_2\right](s)=\zeta_1\left[\zeta_2(s)\right]-\zeta_2\left[\zeta_1(s)\right].$$

Which means

$$\begin{aligned}\left[\Phi(\left[\zeta_1,\zeta_2\right])\right](s) &= \left(\Phi(\zeta_1)\circ\Phi(\zeta_2)\right)(s)-\left(\Phi(\zeta_2)\circ\Phi(\zeta_1)\right)(s)\\ &= \left[\Phi(\zeta_1),\Phi(\zeta_2)\right](s).\end{aligned}$$

As s is arbitrary, hence

$$\Phi(\left[\zeta_1,\zeta_2\right])=\left[\Phi(\zeta_1),\Phi(\zeta_2)\right]. \tag{15}$$

The (15) signifies that Φ is a morphism of $\Gamma(\mathcal{A})$-Lie algebras.

Hence the assertion.

At each point $x \in M$, we now assume that

$$C^\infty(E_x^{\mathcal{A}_x}, \mathcal{A}_x) = \left\{ s(x)/s \in \Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right] \right\}. \tag{16}$$

Theorem 5.2. *Morphism*

(17)

$$\Phi : \Gamma\left(Der_{\mathcal{A}}\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]\right) \longrightarrow Der_{\Gamma(\mathcal{A})}\left(\Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]\right), \zeta \longmapsto \Phi(\zeta),$$

is an isomorphism of $\Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]$*-modules and of* $\Gamma(\mathcal{A})$*-Lie algebras.*

Proof. - The application Φ is injective: Let $\zeta \in \Gamma\left(Der_{\mathcal{A}}\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]\right)$ such that $\Phi(\zeta) = 0$. So for $x \in M$ and for pour $s \in \Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]$ we have $\zeta(x)(s(x)) = 0$. Given (16), we therefore have $\zeta(x) = 0$. Since x is arbitrary, we deduce that $\zeta = 0$. The application Φ is thus injective.

- Application Φ is also surjective. Indeed for $D \in Der_{\Gamma(\mathcal{A})}\left(\Gamma\left[C_M^\infty(E^{\mathcal{A}}, \mathcal{A})\right]\right)$, ζ is perfectly determined by the relationship $\zeta(x)\left[s(x)\right] = D(s)(x)$ for all s and for all x.

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