GENERALIZATION OF THE NOTION OF 
WEIL BUNDLE

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Abstract. The manifold of near points such defined by André Weil 
always appears as a fiber of a certain fiber bundle. We generalize this fact 
by constructing a fibre bundle which fiber at any point is a manifold of near 
points.

1. Introduction

Given a smooth manifolf \( M, A \) a local algebra in the sense of André Weil 
(that is to say a real unitary commutative algebra of finite dimension having 
a unique maximum ideal of codimension 1 on \( \mathbb{R} \)), \( M^A \) denotes the manifold 
of near points to \( M \) of kind \( A \) (a Weil bundle of \( M \) of kind \( A \))[4].

When \( \mathbb{D} = \mathbb{R}[X]/(X^2) \) is the algebra of dual numbers, \( M\mathbb{D} = J_0^0(\mathbb{R}, M) \) 
is the space of jets in zero of order 1 of the differentiable applications of \( \mathbb{R} \) 
into \( M \)[4].

When \( k \geq 1 \) is an integer and when \( A = J_k^k(M, \mathbb{R}) \) is the space of the jets 
at a point \( x \in M \) of order \( k \) of the differentiable functions on \( M \), for any 
other smooth manifold \( N \), we have [3]
\[
N^A = J_k^k(M, N).
\]

Note that the module \( \mathfrak{X}(M^A) \) of the vector fields on \( M^A \) is simultaneously 
a \( C^\infty(M^A, A) \)-module and a Lie algebra on \( A \) [1].

The examples above show that the manifold of near points to \( M \) of kind 
\( A \) as defined by André Weil always appears as the fiber of a certain bundle. 
The aim of this paper is to construct a bundle in such a way that the fibers 
are manifolds of near points ( Weil bundles).

In all that follows \( M \) is a smooth manifold, \( A \) a local algebra in the sense 
of André Weil and \( C^\infty(M) \) the real algebra of smooth functions on \( M \). Fiber 
bundles considered are assumed to have global sections.

Keywords and phrases: Local algebra, Weil bundle, Sections 
(2010)Mathematics Subject Classification: 58A20, 58A32
Received: 20.06.2020. In revised form: 16.10.2020. Accepted: 05.10.2020.
2. GENERALIZED MANIFOLD OF NEAR POINTS

We consider a fibered space \((E, \pi, M)\) and \((A, p, M)\) a fibered space in local algebras of fiber-type local algebra \(A\). For \(x \in M\), \(E_x\) and \(A_x\) denote the respective fibers above \(x\).

The set \(E^A = \bigcup_{x \in M} E_x^A\) is a fibered manifold on \(M\) of dimension \(m + n \cdot \dim(A)\) where \(m\) is the dimension of \(M\) and \(n\) the dimension of each fiber \(E_x\). The fibers of this fibered manifold are manifolds of near points as defined by André Weil.

We say that the manifold \(E^A\) is the generalized Weil bundle.

When \(M\) and \(N\) are smooth manifolds, the set \(C^\infty(M, N)\) of differentiable maps of class \(C^\infty\) from \(M\) to \(N\) is canonically isomorphic to the set of sections of the trivial bundle \((M \times N, pr_1, M)\).

When \(E = M \times N\) and \(A = M \times A\), the bundle \(E^A\) is identified canonically with the trivial bundle \(M \times N^A\).

For \(x \in M\), we recall that \(C^\infty(E_x^A, A_x)\) is an \(A_x\)-unitary commutative real algebra and that \(x(E_x^A)\) is simultaneously a \(C^\infty(E_x^A, A_x)\)-module and an \(A_x\)-Lie algebra[1].

3. STRUCTURE OF \(\Gamma(A)\)-ALGEBRA ON \(\Gamma[C^\infty_M(E^A, A)]\)

The set \(\Gamma(A)\) of global sections of the bundle \((A, p, M)\) is a \(C^\infty(M)\)-module. For \(a, b \in \Gamma(A)\) and for \(\lambda \in \mathbb{R}\), the applications

\[
ab : M \to A, x \mapsto a(x) \cdot b(x) \\
\lambda a : M \to A, x \mapsto \lambda \cdot a(x)
\]

define a unitary commutative real algebra structure on \(\Gamma(A)\).

We notice

\(C^\infty_M(E^A, A) = \bigcup_{x \in M} C^\infty(E_x^A, A_x)\)

and \(\Gamma[C^\infty_M(E^A, A)]\) denotes the set of global sections of the bundle \(C^\infty_M(E^A, A) \to M\).

For \(a \in \Gamma(A)\) and \(s \in \Gamma[C^\infty_M(E^A, A)]\), we define \(as \in \Gamma[C^\infty_M(E^A, A)]\), with

\[as : M \to C^\infty_M(E^A, A), x \mapsto a(x) \cdot s(x)\]

and for \(s_1, s_2 \in \Gamma[C^\infty_M(E^A, A)]\),

\[s_1s_2 : M \to C^\infty_M(E^A, A), x \mapsto s_1(x) \cdot s_2(x)\]

We can thus state:
Proposition 3.1. The set $\Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right]$ is a unitary commutative $\Gamma(\mathcal{A})$-algebra.

We deduce the following corollary:

Corollary 3.2. The set

$$Der_{\Gamma(\mathcal{A})} \left( \Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right),$$

of derivations of $\Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right]$ which are $\Gamma(\mathcal{A})$-linear, is simultaneously a $\Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right]$-module and a Lie algebra on $\Gamma(\mathcal{A})$.

4. Other structure of $\Gamma(\mathcal{A})$-Lie algebra and of $\Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right]$-module

We notice

$$Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] = \bigcup_{x \in \mathcal{M}} Der_{\mathcal{A}_x} \left[ C^{\infty}(E^A_x, \mathcal{A}_x) \right]$$

where $Der_{\mathcal{A}_x} \left[ C^{\infty}(E^A_x, \mathcal{A}_x) \right]$ is the $C^{\infty}(E^A_x, \mathcal{A}_x)$-module of derivations of $C^{\infty}(E^A_x, \mathcal{A}_x)$ which is $\mathcal{A}_x$-linear, it is also the $C^{\infty}(E^A_x, \mathcal{A}_x)$-module, $\mathcal{X}(E^A_x)$ of vector fields on the closed submanifold $E^A_x$ of $E^A$.

For $s \in \Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right]$ and $\zeta \in \Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right)$, we define the section

$$s \cdot \zeta : M \rightarrow Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right], x \mapsto s(x) \cdot \zeta(x).$$

Proposition 4.1. The map

$$\Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \times \Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right) \rightarrow \Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right),$$

$$(s, \zeta) \mapsto s \cdot \zeta,$$

defines a structure of $\Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right]$-module on $\Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right)$.

Proof. Simple verification.

For $\zeta_1, \zeta_2 \in \Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right)$, we define the following section:

$$[\zeta_1, \zeta_2] : M \rightarrow Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right], x \mapsto \zeta_1(x) \circ \zeta_2(x) - \zeta_2(x) \circ \zeta_1(x).$$

We can state:

Proposition 4.2. The map

$$\left[ \Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right) \right]^2 \rightarrow \Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right), (\zeta_1, \zeta_2) \mapsto [\zeta_1, \zeta_2],$$

is $\Gamma(\mathcal{A})$-bilinear alternating. Moreover it defines a structure of $\Gamma(\mathcal{A})$-Lie algebra on $\Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right)$.

Proof. Simple verification.

The two preceding propositions mean that the set $\Gamma \left( Der_{\mathcal{A}} \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right] \right)$ is simultaneously provided with a structure of $\Gamma \left[ C_{\mathcal{M}}^{\infty} (E^A, \mathcal{A}) \right]$-module and of $\Gamma(\mathcal{A})$-Lie algebra.
5. COMPARISON OF THE TWO STRUCTURES

In what follows we compare the structures of $\Gamma \left[ C_M^\infty (E^A, A) \right]$-modules and of $\Gamma (A)$-Lie algebras defined above.

For $\zeta \in \Gamma \left( \text{Der}_A \left[ C_M^\infty (E^A, A) \right] \right)$ and $s \in \Gamma \left[ C_M^\infty (E^A, A) \right]$, we define the section

$$\zeta (s) : M \longrightarrow \text{Der}_A \left[ C_M^\infty (E^A, A) \right], x \mapsto \zeta (x) [s(x)].$$

**Proposition 5.1.** The map

$$\Phi : \Gamma \left( \text{Der}_A \left[ C_M^\infty (E^A, A) \right] \right) \longrightarrow \text{Der}_{\Gamma (A)} \left( \Gamma \left[ C_M^\infty (E^A, A) \right] \right), \zeta \longmapsto \Phi (\zeta),$$

such that

$$\Phi (\zeta) : \Gamma \left[ C_M^\infty (E^A, A) \right] \longrightarrow \Gamma \left[ C_M^\infty (E^A, A) \right], s \longmapsto \zeta (s),$$

is a morphism of $\Gamma \left[ C_M^\infty (E^A, A) \right]$-modules and of de $\Gamma (A)$-Lie algebras.

**Proof.** When $\zeta_1, \zeta_2 \in \Gamma \left( \text{Der}_A \left[ C_M^\infty (E^A, A) \right] \right)$ and for all $s \in \Gamma \left[ C_M^\infty (E^A, A) \right]$, we have:

$$[\Phi (\zeta_1 + \zeta_2)] (s) = (\zeta_1 + \zeta_2)(s).$$

For all $x \in M$, we have

$$[(\zeta_1 + \zeta_2)(s)] (x) = (\zeta_1(x) + \zeta_2(x)) [s(x)]$$

$$= (\zeta_1(x)) [s(x)] + (\zeta_2(x)) [s(x)]$$

$$= [\zeta_1(s)] (x) + [\zeta_2(s)] (x)$$

$$= [\zeta_1(s) + \zeta_2(s)] (x).$$

Since $x$ is arbitrary, we deduce that

$$(\zeta_1 + \zeta_2)(s) = \zeta_1(s) + \zeta_2(s).$$

Which means that

$$\Phi (\zeta_1 + \zeta_2)(s) = \Phi (\zeta_1)(s) + \Phi (\zeta_2)(s)$$

$$= (\Phi (\zeta_1) + \Phi (\zeta_2)) (s).$$

Since $s$ is arbitrary, we have thus

$$\Phi (\zeta_1 + \zeta_2) = \Phi (\zeta_1) + \Phi (\zeta_2).$$

We consider $s \in \Gamma \left[ C_M^\infty (E^A, A) \right]$ and $\zeta \in \Gamma \left( \text{Der}_A \left[ C_M^\infty (E^A, A) \right] \right)$. For all $s' \in \Gamma \left[ C_M^\infty (E^A, A) \right]$, we have:

$$[\Phi (s \cdot \zeta)] (s') = (s \cdot \zeta)(s').$$

For $x \in M$, we have

$$[(s \cdot \zeta)(s')] (x) = (s \cdot \zeta)(x) [s'(x)]$$

$$= (s(x) \cdot \zeta(x)) [s'(x)]$$

$$= s(x) \cdot (\zeta(x) [s'(x)])$$

$$= s(x) \cdot [\zeta(s')] (x)$$

$$= [s \cdot \zeta(s')] (x).$$
Since $x$ is arbitrary, we thus have
\[
[(s \cdot \zeta)(s')] = [s \cdot \zeta(s')]
\]
Which means
\[
[\Phi(s \cdot \zeta)](s') = [s \cdot \Phi(\zeta)](s').
\]
Since $s'$ is arbitrary, hence
\[
(13) \quad \Phi(s \cdot \zeta) = s \cdot \Phi(\zeta).
\]
We deduce that (12) and (13) signify that $\Phi$ is a morphism of $\Gamma [C^\infty_M(E^A, A)]$-modules.

Let $a \in \Gamma(A)$ and let $\zeta \in \Gamma \left( Der_A \left[ C^\infty_M(E^A, A) \right] \right)$. For all $s \in \Gamma \left[ C^\infty_M(E^A, A) \right]$, we have
\[
[\Phi(a \cdot \zeta)](s) = (a \cdot \zeta)(s).
\]
For all $x \in M$, we have
\[
[(a \cdot \zeta)(s)](x) = (a \cdot \zeta)(x)[s(x)] = a(x) \cdot \zeta(x)[s(x)] = a(x) \cdot [\zeta(s)](x) = (a \cdot \zeta(s))(x).
\]
Since $x$ is arbitrary, so
\[
(a \cdot \zeta)(s) = a \cdot \zeta(s).
\]
Which means
\[
[\Phi(a \cdot \zeta)](s) = [a \cdot \Phi(\zeta)](s).
\]
Since $s$ is also arbitrary, hence
\[
(14) \quad \Phi(a \cdot \zeta) = a \cdot \Phi(\zeta).
\]
The (14) means that $\Phi$ is $\Gamma(A)$-linear.

We consider $\zeta_1, \zeta_2 \in \Gamma \left( Der_A \left[ C^\infty_M(E^A, A) \right] \right)$. For all $s \in \Gamma \left[ C^\infty_M(E^A, A) \right]$, we have
\[
(\Phi[\zeta_1, \zeta_2])(s) = [\zeta_1, \zeta_2](s).
\]
For all $x \in M$, we have
\[
([\zeta_1, \zeta_2](s))(x) = [\zeta_1, \zeta_2](x)(s(x)) = \zeta_1(x)[\zeta_2(s)](x) - \zeta_2(x)[\zeta_1(s)](x) = (\zeta_1[\zeta_2(s)])(x) - (\zeta_2[\zeta_1(s)])(x)
\]
Since $x$ is arbitrary, we therefore have
\[
[\zeta_1, \zeta_2](s) = \zeta_1[\zeta_2(s)] - \zeta_2[\zeta_1(s)].
\]
Which means
\[
[\Phi([\zeta_1, \zeta_2])](s) = (\Phi(\zeta_1) \circ \Phi(\zeta_2))(s) - (\Phi(\zeta_2) \circ \Phi(\zeta_1))(s) = [\Phi(\zeta_1), \Phi(\zeta_2)](s).
\]
As $s$ is arbitrary, hence
\[
(15) \quad \Phi([\zeta_1, \zeta_2]) = [\Phi(\zeta_1), \Phi(\zeta_2)].
\]

The (15) signifies that $\Phi$ is a morphism of $\Gamma(A)$-Lie algebras.
Hence the assertion.
At each point \( x \in M \), we now assume that
\[
C^\infty_x(\mathcal{A}) = \{ s(x) / s \in \Gamma \left[ C^\infty_M(\mathcal{E}_A, \mathcal{A}) \right] \}.
\]

**Theorem 5.2.** Morphism
\[
\Phi: \Gamma(\text{Der}_A \left[ C^\infty_M(\mathcal{E}_A, \mathcal{A}) \right]) \longrightarrow \text{Der}_\Gamma(\mathcal{A}) \left( \Gamma \left[ C^\infty_M(\mathcal{E}_A, \mathcal{A}) \right] \right), \zeta \longmapsto \Phi(\zeta),
\]
is an isomorphism of \( \Gamma \left[ C^\infty_M(\mathcal{E}_A, \mathcal{A}) \right] \)-modules and of \( \Gamma(\mathcal{A}) \)-Lie algebras.

**Proof.** - The application \( \Phi \) is injective: Let \( \zeta \in \Gamma(\text{Der}_A \left[ C^\infty_M(\mathcal{E}_A, \mathcal{A}) \right]) \) such that \( \Phi(\zeta) = 0 \). So for \( x \in M \) and for \( s \in \Gamma \left[ C^\infty_M(\mathcal{E}_A, \mathcal{A}) \right] \) we have \( \zeta(x)(s(x)) = 0 \). Given (16), we therefore have \( \zeta(x) = 0 \). Since \( x \) is arbitrary, we deduce that \( \zeta = 0 \). The application \( \Phi \) is thus injective.

- Application \( \Phi \) is also surjective. Indeed for \( D \in \text{Der}_\Gamma(\mathcal{A}) \left( \Gamma \left[ C^\infty_M(\mathcal{E}_A, \mathcal{A}) \right] \right) \), \( \zeta \) is perfectly determined by the relationship \( \zeta(x)[s(x)] = D(s)(x) \) for all \( s \) and for all \( x \).

**References**


