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EQUILATERALS INSCRIBED IN CONICS

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Abstract. In this article we study the properties of equilateral triangles inscribed in conics and the ways one can obtain all equilaterals inscribed in a given conic as well as the peculiarities of all conics circumscribing a given equilateral. In the course of this discussion several relations are proved that connect the position and the dimensions of an equilateral to the concrete circumscribing it conic.

1 Introduction

There is an easy way to create theoretically an equilateral ABC inscribed in a given conic κ and having a given point A of the conic as a vertex. It suffices to consider the rotation f_A with center A and the angle of measure $60°$. Then, apply the rotation to the conic and generate the conic $\kappa' = f_A(\kappa)$. The two conics intersect at least at a second real point B and if $C \in \kappa$ is a point mapping to $B = f_A(C)$, then obviously ABC is an equilateral

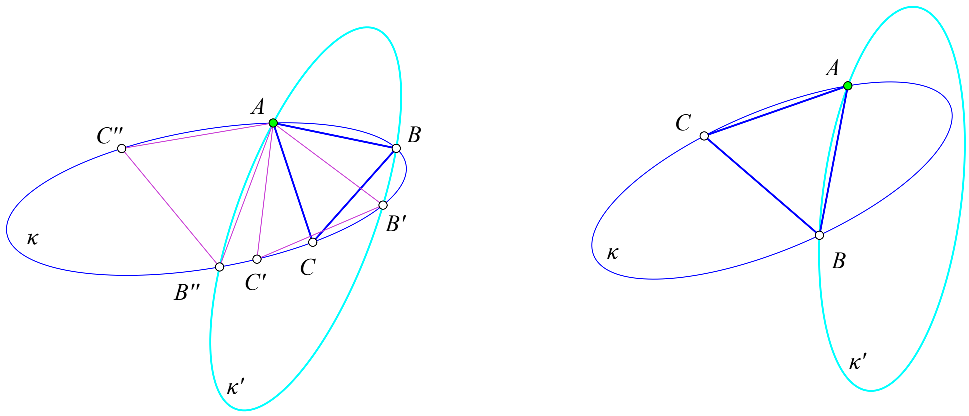

Figure 1: Rotating the conic at $60°$

inscribed in κ (see Figure 1). In general there can be three different equilaterals with a common vertex at A. As seen in the figure on the right, there are though locations for the

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point A on the conic κ, for which there are fewer than three inscribed equilaterals with a vertex at A.

The use of this method in order to investigate the relations between the equilateral and the circumscribing it conic leads however to quartic and cubic equations and excessive computations giving no insight into the geometry of the configurations involved. Therefore in this article we follow another method, relaying on the determination of the equilateral through the fourth intersection point of its circumcircle with the conic. This point is intimately connected with the generation of circumconics by the tripoles of lines revolving about a fixed point. This is the *"perspector"* of the circumconic and the subject becomes a problem of *"triangle conics"*. A concise exposition of this chapter of advanced euclidean geometry can be found in [7] and [18]. A clear, concise and rich in content exposition of the related *"geometry of the triangle"* is the one by Lalesco [10]. For more elementary notions and facts there are many good references as f.e. [5], [14], [6], [9], [1], [12]. Concerning the exposition, in section 2 we introduce a homography, helpful in the discussion to follow. In sections 3 and 4 we discuss the case of equilaterals inscribed in central conics. In section 5 we discuss the case of parabolas and in the last section 6, for the convenience of the reader, we review some more or less known and here repeatedly used properties of tripolars of equilaterals.

2 The homography g_Q

It turns out that conics having their perspector Q on a circle κ concentric to the circumcircle $\kappa_0(K)$ of the equilateral ABC possess some common traits. There are also important common traits for conics κ_Q resulting from points Q varying on some particular lines. For example, if Q varies on a fixed line through the center K of the circumcircle κ_0, then the corresponding conic κ_Q passes through a fixed fourth point L of the circumcircle, which is the tripole of the line KQ (see Figure 2). Next theorem is an immediate consequence of this and a well known property concerning the intersection points of a conic with a circle ([11, I, p.372]), according to which *"if a circle intersects a conic at four distinct points, then the lines joining disjoint pairs of vertices are equally inclined to the axes of the conic"*.

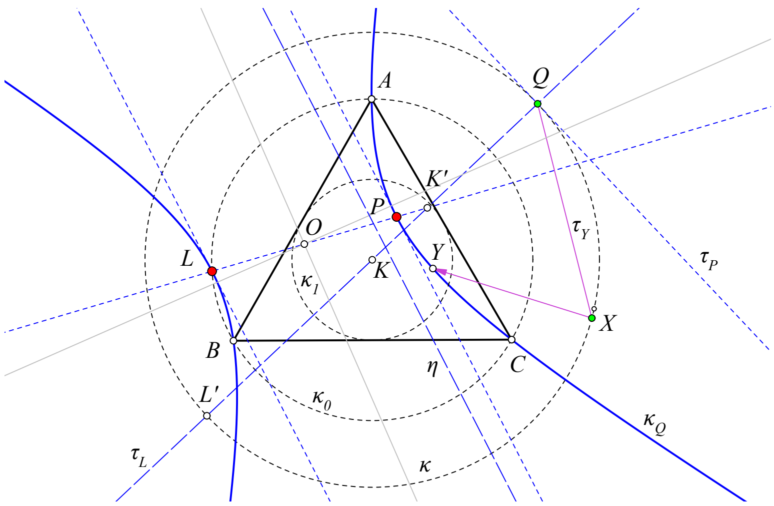

Figure 2: Hyperbola with finite perspector Q lying outside κ_1

Theorem 1. *For Q varying on a fixed line through the circumcenter K of the equilateral triangle ABC, the axes of the conic κ_Q with perspector Q point in two fixed directions.*

Proof. The proof results from the fact, noticed above, that as Q varies on a fixed line through K, the corresponding conic κ_Q passes through four fixed points $\{A, B, C, L\}$ of the circumcircle κ_0 of $\triangle ABC$. Thus, according to the preceding property the axes of the conic are equally inclined to the lines $\{AL, BC\}$, therefore they are parallel to the bisectors of their angle. □

In the sequel we study conics κ_Q resulting from perspectors Q lying on circles κ concentric to the circumcircle κ_0 of the equilateral with big radius and proceed to smaller radii, observing in each case their characteristics. The shape of the conic κ_Q depends on the distance of its perspector Q from the incircle $\kappa_1(K, R_1)$ of the equilateral. In fact, for $\{|KQ| > R_1\}$ we obtain hyperbolas, since there are two tangents to κ_1 from Q, whose tripoles define two points at infinity of Q (lemma 7). For the same reason for $|KQ| = R_1$ we obtain parabolas and for $|KQ| < R_1$ we obtain ellipses.

Figure 2 shows a general case in which the perspector Q is on a circle κ concentric and larger than the incircle κ_1 of the equilateral. The corresponding hyperbola κ_Q is generated by the tripoles $\{Y\}$ of lines QX through Q, for $X \in \kappa$. Point P is the tripole of the tangent to κ at Q. Next theorem clarifies the relation of the pair $\{X, Y\}$ and leads to a homography g_Q intimately related to the geometry of the conic.

Theorem 2. *With the conventions and notation of this section, the map $g_Q(X) = Y$ associating the tripole Y to the line QX for $X \in \kappa$ is a homography, mapping the circle κ onto the conic κ_Q, which passes through the vertices of the triangle.*

Proof. Though the proof does not depend essentially on the shape of the conic, we stick to the case of a circle κ larger than the incircle κ_1 of the triangle. As we noticed already, for $Q \in \kappa$ the conic κ_Q is then a hyperbola. To prove that the map g_Q is a homography of κ onto the hyperbola, we test the preservation of the cross ratios ([2, II, p.178]). This is a simple matter involving though tedious calculations, which I omit and content myself to describe the necessary steps. For this we work with barycentrics (u, v, w) w.r.t. the

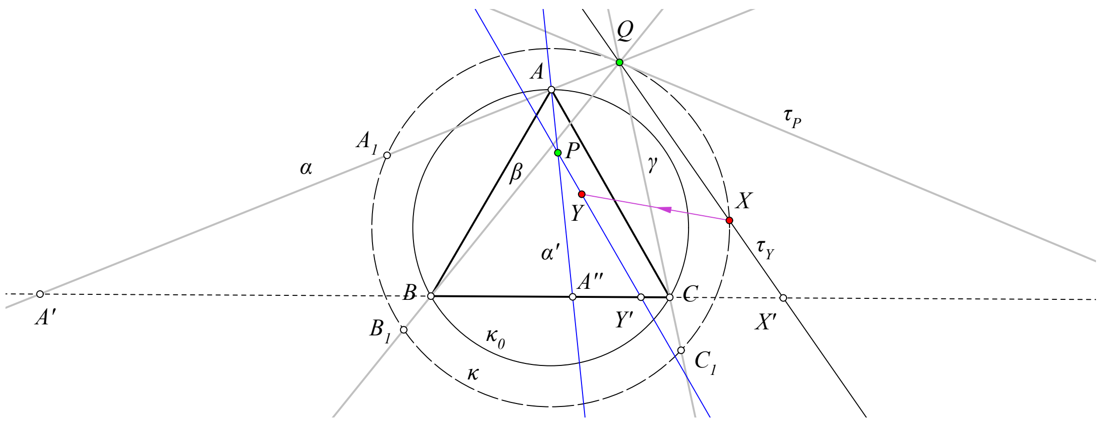

Figure 3: Testing cross ratios preservation

equilateral, by means of which the equation of the circles κ, concentric to the circumcircle κ_0, has the general form:

$$vw + wu + uv + \lambda(u + v + w)^2 = 0, \tag{1}$$

the constant λ depending on the radius of κ ([13], [18, p.69]). Figure 3 illustrates the test procedure for the cross ratios. The first $cr_1 = (B_1C_1, A_1X)$ for $X \in \kappa$ is a cross ratio of points on the circle κ and is equal to the cross ratio $Q(BC, AX)$ of the corresponding pencil of lines $Q(BCAX)$, which in turn, is equal to the cross ratio of its intersection points with line BC : $cr_1 = (BC, A'X')$. The second cross ratio cr_2 is defined for the corresponding under g_Q points on the hyperbola $cr_2 = (BC, AY)$. It is calculated by means of the pencil of lines $P(BCAY)$, where P is a fifth point on the hyperbola obtained as the tripole of the tangent τ_P of κ at the perspector Q. This is again equal to the cross ratio of its intersection points with line BC : $cr_2 = (BC, A''Y')$.

Setting $\{Q(u_0, v_0, w_0), X(u, v, w)\}$ and $\sigma = u_0 + v_0 + w_0$, the calculations involve the determination of the tangent τ_P at Q and its tripole P :

$$\tau_P \ : \ [(2\lambda+1)\sigma - u_0]u \ + \ [(2\lambda+1)\sigma - v_0]v \ + \ [(2\lambda+1)\sigma - w_0]w \ = \ 0\,, \tag{2}$$

$$P \ : \ \left(\frac{1}{(2\lambda+1)\sigma - u_0}\,,\ \frac{1}{(2\lambda+1)\sigma - v_0}\,,\ \frac{1}{(2\lambda+1)\sigma - w_0}\right)\,, \tag{3}$$

and those of the determination of line $\tau_Y = QX$ and its tripole Y :

$$Y \ : \ \left(\frac{1}{v_0w - vw_0}\,,\ \frac{1}{w_0u - wu_0}\,,\ \frac{1}{u_0v - uv_0}\right)\,. \tag{4}$$

The calculation of $cr_2 - cr_1$ is simplified by assuming *absolute* barycentrics satisfying $\sigma = 1$. It is then seen that $cr_2 - cr_1$ is an expression in $\{u_0, v_0, w_0, u, v, w, \lambda\}$ containing also the factor

$$u_0^2 + v_0^2 + w_0^2 \ - \ (2\lambda+1) \ = \ 0\,,$$

which completes the proof of the theorem. □

In the sequel, we identify g_Q with its extension on the whole plane ([2, II, p.178]), which, by the fundamental theorem of projective geometry ([17, I, p. 96]) can be defined by its action on the quadruple $\{g_Q : A_1 \mapsto A, B_1 \mapsto B, C_1 \mapsto C, Q \mapsto P\}$ (see Figure 3).

Theorem 3. *The tangents of the conic κ_Q at $P = g_Q(Q)$ and its fourth inersection L with the circumcircle κ_0 are parallel and the center of the conic, if any, is the middle O of the segment LP, whose direction is conjugate to that of the tangents w.r.t. κ_Q.*

Proof. Obviously the second claim is a consequence of the parallelity of the tangents. Regarding the first claim, the clue fact is, that the tangents to κ at $\{Q, L'\}$ are parallel lines and they map via g_Q to the tangents to κ_Q at $\{P, L\}$, which are also parallel lines (see Figure 2). In fact, the point at infinity of the tangent τ_P at $Q(u_0, v_0, w_0)$ is by equation (2): $(v_0 - w_0, w_0 - u_0, u_0 - v_0)$. It is then easily verified that the image point of this via g_Q is the point at infinity of the tangent to κ_Q at P the equation of κ_Q being

$$\kappa_Q \ : \ u_0 \cdot yz + v_0 \cdot zx + w_0 \cdot xy = 0\,. \tag{5}$$

□

3 Central conics circumscribing the equilateral

In this section we apply the homography g_Q to gain knowledge about the relations of the equilateral with its circumscribed conics in the case latter are *"central"*, i.e. they are hyperbolas or ellipses. The case of parabolas is discussed in the next section. In most proofs in this section we stick to hyperbolas. The case of ellipses is completely analogous and we state explicitly the minor changes if needed.

Theorem 4. *Let Q be a point not lying on the side-lines of the equilateral triangle ABC. Let also $\kappa(K,r)$ be a circle concentric to the circumcircle $\kappa_0(K)$ of the triangle passing through Q, τ_P be the tangent to κ at Q and P be its tripole w.r.t. $\triangle ABC$. Then, the following are valid properties (see Figure 4):*

1. *If L' is the diametral point of Q and $L = g_Q(L')$ and O is the middle of LP, then the lines $\{\alpha = KP, \beta = L'O\}$ are parallel.*
2. *The distance of Q from α is equal to the distance of the two parallels $\{\alpha,\beta\}$, which is equal to the distance of L from β.*
3. *The image $K' = g_Q(K)$ coincides with the intersection point $PL \cap QL'$.*
4. *The point $O' \in L'Q$ mapping onto $O = g_Q(O')$ defines equal segments $|O'L'| = |KK'|$.*

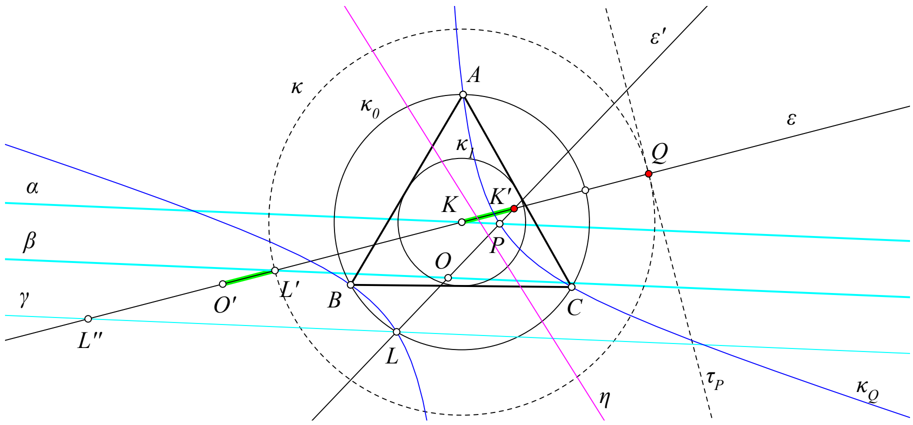

Figure 4: A property of the tripolars of the equilateral

Proof. Let K' denote the intersection point of lines $\{\varepsilon = L'Q\ ,\ \varepsilon' = g_Q(\varepsilon) = LP.\}$ Drawing the parallel γ to α from L, we form the trapezium $PKL''K$ of which line β is joining the middles of the non parallel sides, hence the proof of *nr-1* and *nr-2*.

For the location of K' it suffices to show that $g_Q(K)$ is collinear with $\{K,Q\}$. A proof of this can be given by an explicit description of the homography g_Q, involving a computation in barycentric coordinates w.r.t. $\triangle ABC$. The determination of g_Q results from the correspondence of the four points (see Figure 3)

$$A_1 \overset{g_Q}{\mapsto} A\ ,\quad B_1 \overset{g_Q}{\mapsto} B\ ,\quad C_1 \overset{g_Q}{\mapsto} C\ ,\quad L' \overset{g_Q}{\mapsto} L\ ,$$

where the points $\{A_1,B_1,C_1,L'\}$ of κ and $L = g_Q(L')$ are expressed in barycentrics by their coordinates:

$$A_1\begin{pmatrix} au-b \\ \lambda v \\ \lambda w\end{pmatrix},\ B_1\begin{pmatrix} \lambda u \\ av-b \\ \lambda w\end{pmatrix},\ C_1\begin{pmatrix} \lambda u \\ \lambda v \\ aw-b\end{pmatrix},\ L'\begin{pmatrix} 3u-2 \\ 3v-2 \\ 3w-2\end{pmatrix},\ L\begin{pmatrix} (w-v)^{-1} \\ (u-w)^{-1} \\ (v-u)^{-1}\end{pmatrix}.$$

Here $Q(u,v,w)$ and the constant λ is the one of equation (1) of the circle κ. We also have set $a = \lambda+1,\ \ b = 2\lambda+1$ and we assumed that $u+v+w = 1$. All this leads to the description of g_Q through (a non-zero scalar multiple of) the matrix:

$$g_Q \cong \begin{pmatrix} (2\lambda+1)(\lambda+u) & \lambda(2\lambda+1-w) & \lambda(2\lambda+1-v) \\ \lambda(2\lambda+1-w) & (2\lambda+1)(\lambda+v) & \lambda(2\lambda+1-u) \\ \lambda(2\lambda+1-v) & \lambda(2\lambda+1-u) & (2\lambda+1)(\lambda+w)\end{pmatrix}. \qquad (6)$$

The proof of *nr-3* is completed by showing the vanishing of the determinant of the matrix with rows $\{K(1,1,1), Q(u,v,w), g_Q(K)\}$.

Nr-4 follows from a cross ratio computation. In fact, the two cross ratios of points related by g_Q are equal: $(O'K, L'Q) = (OK', LP)$. Setting $\{x = O'L', y = LO, z = PK'\}$ this implies the equalities

$$\frac{L'O'}{L'K} : \frac{QO'}{QK} = \frac{LO}{LK'} : \frac{PO}{PK'} \quad \Leftrightarrow \quad \frac{x}{r} : \frac{x+2r}{r} = \frac{y}{z+2y} : \frac{y}{z} \quad \Rightarrow \quad \frac{x}{r} = \frac{z}{y} \; .$$

The last quotient because of the parallels is $z/y = PK'/OP = KK'/r$, which implies the equality $O'L' = KK'$. □

Theorem 5. *The distances $\{|K'K|, |O'K|\}$ remain constant for points Q varying on the circle $\kappa(K,r)$.*

Proof. The constancy of the second distance follows from that of $|KK'|$ and the relation $|O'K| = |KK'| + r$, the radius r taken negative in the case of the ellipse and zero in the case of the parabola. The distance function of two points in barycentrics ([13]), adapted to the case of the equilateral with side-length d is expressed through a quadratic form with matrix J :

$$|XY|^2 = -\frac{d^2}{2\sigma_X^2\sigma_Y^2}(\sigma_Y X - \sigma_X Y)^T \cdot J \cdot (\sigma_Y X - \sigma_X Y) \; , \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} . \tag{7}$$

Here the point X is identified with the column-vector of its coordinates and X^T denotes the corresponding (transposed) row-vector. Also $\sigma_X = X_u + X_v + X_w$ denotes the sum of the coordinates, which for *absolute barycentrics* is per definition $\sigma_X = 1$ and for points at infinity $\sigma_X = 0$. The distance $|KK'|$ results from equation (1), which defines the constant λ and the barycentrics of K' :

$$K' = QK \cap LP \;\cong (\; 2\lambda + u \; , \; 2\lambda + v \; , \; 2\lambda + w \;) \quad \text{with} \quad \sigma_{K'} = 6\lambda + 1 \quad \Rightarrow \tag{8}$$

$$|KK'|^2 \;=\; \frac{d^2}{3} \cdot \frac{3\lambda + 1}{(6\lambda + 1)^2} \; . \qquad \square \tag{9}$$

Corollary 1. *The polar η of the point $K' = LP \cap QL'$ w.r.t. the conic κ_Q is the image line via g_Q of the line at infinity. Also η is parallel to the tangents of κ_Q at the points $\{L, P\}$.*

Proof. Referring to figure 2, the line at infinity is the polar w.r.t. κ of K and this maps via g_Q to the polar of K'. This shows the first part of the theorem. The second part follows from the fact (theorem 3), that the direction of the line LP and that of the tangents at $\{P, L\}$ are conjugate, hence the polars of points like K', lying on LP, are parallel to these tangents. □

There is a particular case of hyperbolas arising when the perspector Q is a point at infinity. In this case the pencil of lines through Q coincides with a pencil of parallel lines and Q can be identified with their common *direction*. The tripoles of these parallel lines generate a hyperbola, since we have again two parallels $\{\delta, \delta'\}$ passing through Q and tangent to the incircle κ_1 at the diametral points $\{D, D'\}$, their tripoles being then two points at infinity (see Figure 5). We examine here this case in short for its own sake and also because we need some of its properties in the sequel.These properties are consequences of those exposed in the review on tripolars of equilaterals in section 6. The hyperbola is rectangular since the tripole of the line at infinity is K, hence the hyperbola

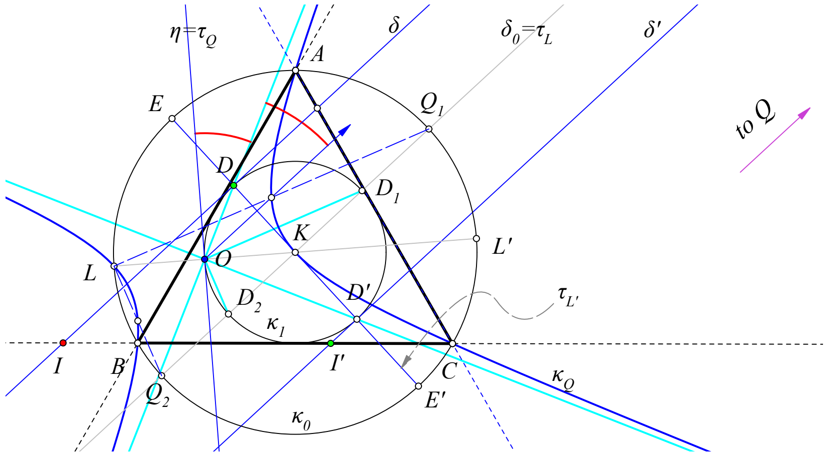

Figure 5: Rectangular hyperbola of the equilateral with perspector Q at infinity

passes through its orthocenter, a property characteristic of rectangular hyperbolas ([4, p.290]). The tripoles of $\{\delta, \delta'\}$ are determined by the directions of two WS-lines w.r.t. two diametral points $\{E, E'\}$ of the circumcircle (lemma 9). The asymptotes, which are the tangents at the points at infinity coincide with the WS-lines of $\{E, E'\}$ which pass through the diametral points $\{D, D'\}$ of the incircle κ_1 (lemma 6). Since the two asymptotes are orthogonal (lemma 3) and pass through diametral points $\{D, D'\}$ of the incircle, their intersection K is a point of the incircle and lines $\{LE, LE'\}$ are parallel to the asymptotes. Since K is the symmetry center of the hyperbola, the symmetric L of P w.r.t. K is the fourth intersection point of the hyperbola with the circumcircle. It follows that the intersections $\{D_1, D_2\}$ of κ_1 with δ_0 define the axes $\{OD_1, OD_2\}$ of the hyperbola, since these lines bisect the angles of the asymptotes. Also the axes are parallel to the WS-lines of the intersections $\{Q_1, Q_2\}$ of D_1D_2 with κ_0 (lemma 9), which in turn are parallel to the lines $\{LQ_1, LQ_2\}$. This fact is used in the proof of the next theorem.

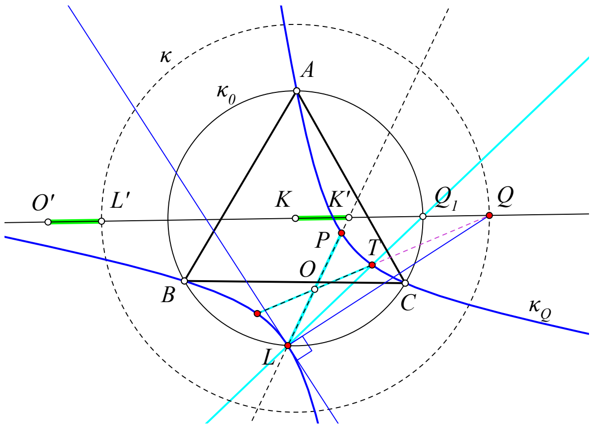

Figure 6: Hyperbola κ_Q with perspector w.r.t. ABC

Theorem 6. *The intersection point Q_1 of the half-line KQ with the circumcircle κ_0 defines the line Q_1L which is parallel to an axis of the conic. Line Q_1L intersects the conic a second time at the point T, which is collinear with $\{Q, O\}$. Thus, the segments $\{OQ, OL\}$ define two diameters of the conic which are symmetric w.r.t. its axes (see Figure 6).*

Proof. In fact, by theorem 1 the axes of the conic, as well as, all the conics κ_Q for Q varying on a fixed line through K are parallel to a fixed direction. By the preceding remarks this direction is determined by Q_1L in the case of the rectangular hyperbola $\kappa_{Q'}$ with Q' the point at infinity of line KQ. From this follows also the last claim about the diameters of the conic. To show that points $\{T, Q, O\}$ are collinear, we consider T as the intersection $OQ \cap Q_1L$ and show that $T \in \kappa_Q$. For this we compute first the barycentrics of the involved points $\{Q_1, O, T\}$. Next list gives the results of these calculations, R denoting the circumradius of $\triangle ABC$ and $r = |KQ|$ denoting the radius of the circle κ. The symbol " $\cong$ " denotes equality up to a non-zero multiplicative constant.

$$K \cong (1, 1, 1), \quad Q = (u, v, w) \quad \text{with} \quad \sigma_Q = u + v + w = 1,$$
$$L \cong ((w-v)^{-1}, (u-w)^{-1}, (v-u)^{-1}),$$
$$Q_1 = \frac{1}{3}\left(1 - \frac{R}{r}\right)K + \frac{R}{r}Q, \quad \Rightarrow \quad Q_1 \cong (s+u, s+v, s+w), \quad s = \frac{r-R}{3R}, \tag{10}$$
$$O \cong (u(1-2u), v(1-2v), w(1-2w)) \quad \text{with} \quad \sigma_O = -4\lambda - 1, \tag{11}$$
$$Q_1(u', v', w') \in \kappa_Q \quad \Leftrightarrow \quad \lambda = -(v'w' + w'u' + u'v') = 2s + 3s^2 = \frac{r^2 - R^2}{3R^2}, \tag{12}$$
$$T = OQ \cap Q_1L \cong (u(s(2-3u) - \lambda), v(s(2-3u) - \lambda), w(s(2-3w) - \lambda)). \tag{13}$$

The first two lines display information already seen in theorem 4. The expression for Q_1 results from $KQ_1/KQ = R/r$. The expression for O is found by considering it as the pole w.r.t. κ_Q of the line at infinity. The relation between $\{\lambda, s\}$ results from the condition $Q_1 \in \kappa_Q$. The barycentrics of T are computed using the vector product of the barycentrics vectors $(O \times Q) \times (Q_1 \times L)$. The claim about the collinearity follows by verifying that T satisfies the equation of κ_Q :

$$f_Q(x, y, z) = u \cdot yz + v \cdot zx + w \cdot xy = 0. \tag{14}$$

This, given the conditions (1) satisfied by Q and the consequence of it expressed by the equation

$$u^2 + v^2 + v^2 = 2\lambda + 1,$$

lead finally to the vanishing (by (12)) expression

$$f_Q(T) = 3(uvw)\lambda(\lambda - 2s - 3s^2) = 0. \qquad \square$$

Theorem 7. *The line LQ is orthogonal to the parallel tangents $\{t_P, t_L\}$ of the conic κ_Q at the diametral points $\{P, L\}$. Thus, the perspector Q of the conic coincides with the intersection point of the reflected diameter line OT of LP w.r.t. to the axes of the conic with the normal n_L at L of κ_Q (see Figure 6).*

Proof. The proof uses again barycentrics to compute the directions of the tangent t_L of κ_Q at L and the line LQ. These are the intersections of the respective lines with the line at infinity ε_∞ and are determined by the vector product of $(1, 1, 1)$ with the respective coefficient vectors of the lines. It turns out that the direction of the line LQ is given by the barycentrics vector:

$$v_1 \cong ((u-w)(v-u) + u(1+3\lambda), \ldots),$$

the dots, as usual, denoting the other coordinates resulting from the first by cyclically permuting the letters. The direction of the tangent t_L is analogously found to be:

$$v_2 \cong ((v-w)(\lambda + u), \ldots).$$

The condition of orthogonality is, in the case of the *equilateral*, equivalent to the vanishing of the usual inner product of the barycentrics vectors:

$$\langle v_1, v_2\rangle \;=\; [(u-w)(v-u)+u(1+3\lambda)][(v-w)(\lambda+u)] \;+\; \ldots\;,$$

proved indeed to be zero. □

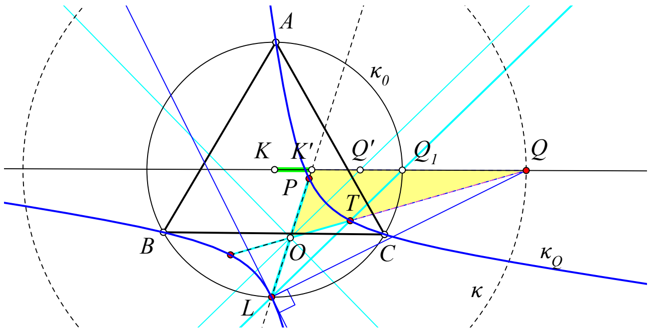

Figure 7: Line OQ' bisecting the angle $\widehat{O}$ of $\triangle K'OQ$

Theorem 8. *The harmonic conjugate $Q' = Q(KQ_1)$ is a point on the axis of the conic, to which is parallel the line LQ_1 (see Figure 7).*

Proof. For Q_1 expressed in absolute barycentrics as on the left side of equation (10) we deduce:

$$Q' \;=\; Q(KQ_1) \;=\; \frac{r}{R}Q_1 - \frac{R-r}{3R}K \;\cong\; (2s+u\;,\; 2s+v\;,\; 2s+w)\;. \tag{15}$$

To prove that this is on the axis parallel to LT it suffices to show that the intersection point of the lines $\{OQ', LT\}$ is on the line at infinity ε_∞. This amounts to proving that the determinant of the vectors of the coefficients of the corresponding lines, which are $\{O\times Q'\;,\; L\times T\;,\; (1,1,1)\}$, vanishes. The corresponding calculation shows indeed that this determinant is equal to

$$(2\lambda+1)(3s^2+2s-\lambda) \;=\; 0. \qquad \square$$

Theorem 9. *The ratio OK'/OQ, where $K' = LP\cap KQ$, remains constant if Q varies on the circle $\kappa(K,r)$.*

Proof. In fact, the triangle $K'OQ$ (see Figure 7), has the axes of the hyperbola as bisectors of its angle $\widehat{O}$. Hence this ratio is equal to $Q'K'/Q'Q$. Using the barycentrics of K' from equation (8) and the barycentrics (15) of $Q' \cong (2s+u,\ldots)$, we deduce its representation:

$$Q' \;\cong\; \frac{s}{\lambda}K' \;+\; \frac{\lambda-s}{\lambda}Q \quad\Rightarrow\quad \frac{OK'}{OQ} \;=\; \frac{Q'K'}{Q'Q} \;=\; \frac{s-\lambda}{s(6\lambda+1)}\;. \tag{16}$$

The theorem follows from the fact that the last expression is constant for $Q\in\kappa$. □

Theorem 10. *The conics κ_Q, for Q varying on the concentric to κ_0 circle $\kappa(K,r)$, are pairwise similar.*

Proof. The square of the ratio b^2/a^2 of the minor to the major axis of the ellipse resp. hyperbola is related to its eccentricity $e^2 = 1-b^2/a^2$ resp. $e^2 = 1+b^2/a^2$ and is also expressed ([15, p.160]) through the oriented ratio LU/LV of two segments on the normal LQ at an arbitrary point L of the conic, $\{U,V\}$ being the intersections of the normal with

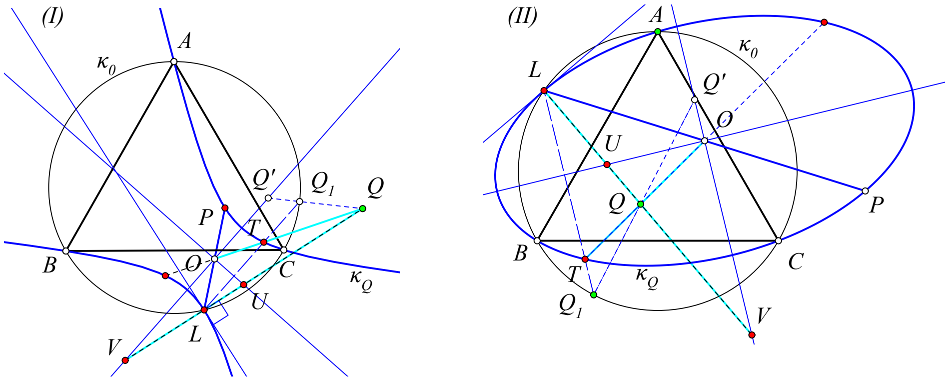

Figure 8: I: hyperbola $LV/LU = -b^2/a^2$ II: ellipse $LU/LV = b^2/a^2$

the axes of the central conic (see Figure 8). Thus, considering the line LQ, which by theorem 7 is the normal of the conic at L , this ratio is related to

$$\frac{UL}{UV} = \frac{LT/2}{VO} = \frac{1}{2}\frac{QT}{QO} = k \quad \text{by}$$
$$\frac{b^2}{a^2} = e^2 - 1 = -\frac{LV}{LU} = \frac{1-k}{k} \quad \text{(hyperbola),}$$
$$\frac{b^2}{a^2} = \frac{LU}{LV} = \frac{k}{k-1} = 1 - e^2 \text{ (ellipse)}$$

The ratio QT/QO is found from the expression of the corresponding vectors of barycentric coordinates

$$T \cong (3s)O + (s - 2\lambda)Q \quad \Rightarrow \quad \hat{T} = \frac{6s+1}{2(3s+1)}\hat{O} + \frac{1}{2(3s+1)}Q\,, \tag{17}$$

the equation on the right being exact and $\hat{X}$ denoting the vector of absolute barycentrics of the corresponding point. It follows

$$t = \frac{TO}{TQ} = -\frac{1}{6s+1} = \frac{R}{R-2r} \quad \Rightarrow \quad 2k = \frac{QT}{QO} = \frac{1}{1-t} = \frac{2r-R}{2r} \quad \Rightarrow \tag{18}$$
$$\text{(hyperbola)}\ \frac{b^2}{a^2} = e^2 - 1 = -\frac{LV}{LU} = \frac{2r+R}{2r-R} \quad \Rightarrow \quad e^2 = \frac{4r}{2r-R}\,, \tag{19}$$
$$\text{(ellipse)}\ \frac{b^2}{a^2} = 1 - e^2 = \frac{LU}{LV} = \frac{R-2r}{R+2r} \quad \Rightarrow \quad e^2 = \frac{4r}{2r+R}\,. \tag{20}$$

showing that the eccentricity remains constant along the circle $\kappa(K, r)$ and the conics $\{\kappa_Q : Q \in \kappa\}$ are of the same similarity type. □

4 Equilaterals inscribed in central conics

In this section we use the relations found in section 3 in order to explore the properties of all equilaterals inscribed in a given central conic. Motivated by the discussion in that section we start with an arbitrary point L on the conic. This will play the role of the fourth intersection point of the conic and the circumcircle $\kappa_0(K, R)$ of the inscribed equilateral under construction (see Figure 9). Point L defines the diameter LP and its symmetric TT' w.r.t. the axes of the conic. It defines also the intersection point Q of the normal n_L

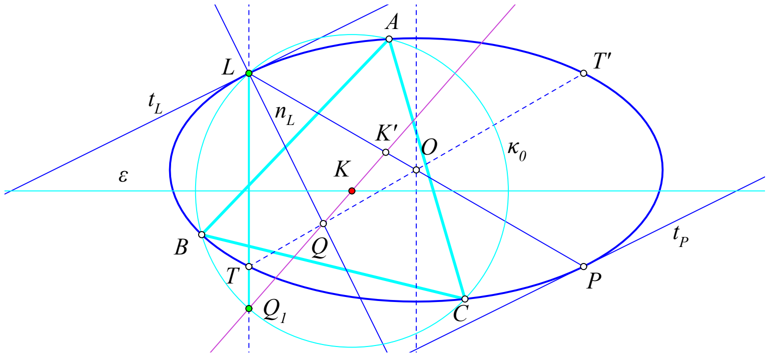

Figure 9: Inscribed equilateral determined by point L

at L with the diameter TT'. Point Q will play the role of the *perspector* of the conic w.r.t. the equilateral under construction. From equation (18) we know that the ratio QT/QO is equal to

$$\mu = \frac{QT}{QO} = \frac{2r-R}{2r} = \frac{2(r/R)-1}{2(r/R)} \quad \Rightarrow \quad \frac{R}{r} = 2(1-\mu). \tag{21}$$

Thus, the position of L uniquely determines the ratio R/r of the circumradius R of the equilateral to the distance $r = |KQ|$ of Q from the circumcenter K of the equilateral under construction. From equation (16) we know also that

$$\frac{OK'}{OQ} = \frac{s-\lambda}{s(6\lambda+1)} = \frac{Rr}{R^2-2r^2} = \frac{1}{(R/r)-2(r/R)} = \frac{1-\mu}{2\mu^2-4\mu+1}. \tag{22}$$

Thus, the location of the intersection $K' = OL \cap QK$ is also determined from the position of L and this defines line QK' carrying the center of the circumcircle κ_0 of the equilateral under construction. By theorem 6 the intersection point $Q_1 = QK' \cap LT$ is on the circumcircle of the equilateral, hence the center K of the equilateral is on the medial line ε of the segment LQ_1. This leads to the definition of K as intersection of the lines $K = \varepsilon \cap QK'$ and consequently the determination of the circle $\kappa_0(K,R)$ circumscribing the equilateral under construction. It remains to show that the three other than L intersection points of κ_0 with the ellipse define an equilateral triangle ABC. We formulate this as a theorem.

Theorem 11. *Every equilateral triangle ABC, inscribed in a cenral conic, is obtained by the procedure described above.*

Proof. The discussion in section 3 implies that every central conic circumscribing an equilateral, leads to the relations between the ellipse and the equilateral contained in the preceding lines.

Conversely, if we define the circle $\kappa_0(K,R)$ (see Figure 9) through the procedure described above, then the other than $L(x_0,y_0)$ intersection points of κ_0 with the central conic define an equilateral triangle ABC. To show this we refer the conic to the cartesian coordinate system defined by its axes. We work out the case of ellipse, the case of hyperbolas being completely analogous. The following elements are easily found:

$$\text{ellipse: } \frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = 0, \text{ normal at } (x_0,y_0): -\frac{y_0}{b^2}x+\frac{x_0}{a^2}y+\frac{a^2-b^2}{a^2b^2}x_0y_0 = 0\,,$$

$$Q = \frac{a^2-b^2}{a^2+b^2}(x_0,-y_0)\ ,\ \mu = \frac{2b^2}{b^2-a^2}\ ,\ \nu = \frac{1-\mu}{2\mu^2-4\mu+1}\ ,\ |OK'| = \nu|OQ|\,,$$

$$|OQ| = \frac{a^2-b^2}{a^2+b^2}\sqrt{x_0^2+y_0^2}\ , \quad |OK'| = \frac{(a^2-b^2)^2}{(a^2+b^2)^2+4a^2b^2}\sqrt{x_0^2+y_0^2}\ ,$$
$$Q_1 = \left(x_0\ ,\ \frac{b^2-5a^2}{b^2+3a^2}y_0\right)\ ,\quad K = \left(\frac{a^2-b^2}{a^2+3b^2}x_0\ ,\ -\frac{a^2-b^2}{b^2+3a^2}y_0\right)\ ,$$
$$|KL|^2 = \frac{16}{(a^2+3b^2)^2(b^2+3a^2)^2}(b^4(b^2+3a^2)^2x_0^2+a^4(a^2+3b^2)^2y_0^2)\ .$$

Having that, the equation of the circle $\kappa_0(K,|KL|)$ is found to be

$$[(b^2+3a^2)(a^2+3b^2)](x^2+y^2)$$
$$+\,2[(b^2-a^2)(b^2+3a^2)x_0]x - 2[(b^2-a^2)(a^2+3b^2)y_0]y + c = 0\ ,$$

where c is a constant depending on $\{a,b,x_0,y_0\}$ but not important for the sequel. In order to find the intersection points of the ellipse with the circle κ_0 we can use a variable line through L in the parametric form $X(t) = L+te$, where $e(e_1,e_2)$ is a unit vector. The other than L intersection point of the line with the ellipse κ_Q is determined for the value

$$t_1 = -2\frac{a^2e_2y_0+b^2e_1x_0}{(a^2e_2^2+b^2e_1^2)}\ .$$

The other than L intersection point of the line with the circle κ is determined for the value

$$t_2 = -8\frac{(b^2(b^2+3a^2)x_0e_1+a^2(a^2+3b^2)y_0e_2)}{(b^2+3a^3)(a^2+3b^2)(e_1^2+e_2^2)}.$$

At the intersection points the two values must coincide, thus leading to $t_2-t_1=0$, which is found equivalent to

$$[a^2(a^2+3b^2)y_0]e_2^3 + [3b^2(b^2+3a^2)x_0]e_2^2e_1$$
$$-\,[3a^2(a^2+3b^2)y_0]e_2e_1^2 - [b^2(b^2+3a^3)x_0]e_1^3 = 0\ .$$

This leads to a cubic equation w.r.t. to the tangens $t = e_2/e_1$ of the polar angle determining, through the unit vector e, the directions of the line joining the point L with the other intersection points of the circle κ_0 with the ellipse (see Figure 10).

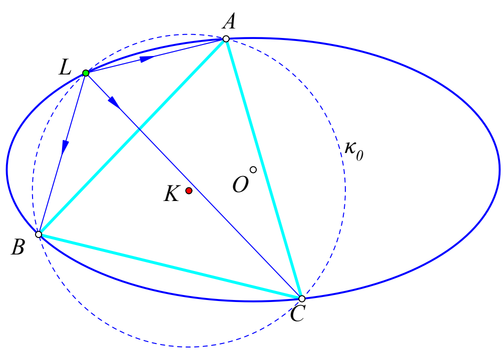

Figure 10: The directions of lines $\{LA, LB, LC\}$

$$At^3 + Bt^2 + Ct + D = 0 \quad \Leftrightarrow$$
$$[a^2(a^2+3b^2)y_0]t^3 + [3b^2(b^2+3a^2)x_0]t^2 - [3a^2(a^2+3b^2)y_0]t - [b^2(b^2+3a^3)x_0] = 0\ .$$

The proof is completed if we can show that the polar angles between the three directions are of the form

$$\phi - 60^\circ \ , \ \phi \ , \ \phi + 60^\circ \ .$$

This, for the respective tangens $\{ \ t_1 = \tan(\phi - 60^\circ) \ , \ t_2 = \tan(\phi) \ , \ t_3 = \tan(\phi + 60^\circ) \ \}$, is equivalent ([8, p.58]) to the condition $\{t_1t_2 + t_2t_3 + t_3t_1 = -3\}$, which can be verified using Vieta's relations ([3, p.37]) for the roots of the cubic equation:

$$t_1t_2 \ + \ t_2t_3 \ + \ t_3t_1 \ = \ \frac{C}{A} \ = \ \frac{-3a^2(a^2+3b^2)y_0}{a^2(a^2+3b^2)y_0} \ = \ -3 \ .$$

The discriminant of the cubic equation for $y_0 \neq 0$ is found to be negative

$$-\frac{(b^4(b^2+3a^2)^2)x_0^2 + (a^4(3b^2+a^2)^2)y_0^2)^2}{a^8(3b^2+a^2)^4y_0^4} \ < \ 0 \ ,$$

which means ([16, p.120]) that the corresponding cubic has three real roots, which in turn means that the equilateral is well defined for every position of $L \in \kappa$ with $y_0 \neq 0$. The case $y_0 = 0$ corresponds to the vertices of the conic and leads trivially to a circle κ_0 and a corresponding equilateral, κ_0 being tangent to the conic and symmetric w.r.t. its major axis. □

5 Equilaterals inscribed in parabolas

Several relations between the equilateral and the circumscribing it conic valid for central conics (section 4) are valid also in the case of parabolas. There is also a clue property which plays in this case a particular role and I formulate it as a lemma.

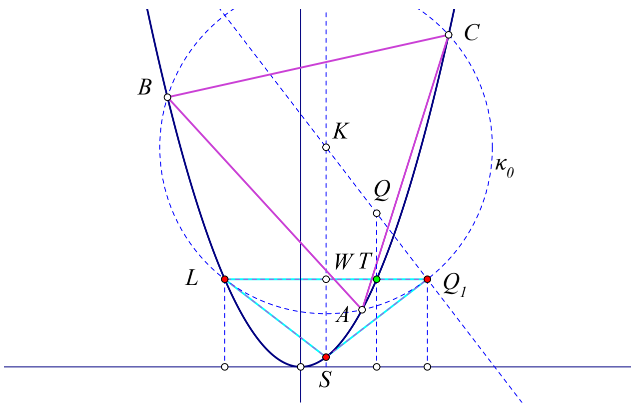

Figure 11: A property of the parabola

Lemma 1. *Let LT be a chord of the parabola orthogonal to its axis and extend it by the length $TQ_1 = LT/3$. Then the orthogonal bisector line WK of the segment LQ_1 intersects the parabola at a point S such that SQ_1 is tangent to the parabola at S (see Figure 11). Conversely, if the isosceles LQ_1S has its base LQ_1 orthogonal to the axis, the vertices $\{L, S\}$ on the parabola and SQ_1 tangent to the parabola ta S, then the second intersection point T of the parabola with line LQ_1 satisfies $LT/TQ_1 = 3$. Further, the center K of the circle κ_0 which is tangent to the legs of the isosceles at $\{L, Q_1\}$ lies on the parallel to the axis of the parabola through S and its distance $|KW|$ from the chord is constant and independent from the location of the chord LT.*

Proof. Using a representation of the parabola in the form $y = ax^2$ and denoting the coordinates of T by $(x, y = ax^2)$, we have the coordinate representations of the other points:

$$L(x_L = -x\ ,\ y_L = y), \qquad Q_1(x' = \frac{5}{3}x\ ,\ y' = y),$$
$$S(x_S = \frac{1}{3}x\ ,\ y_S = \frac{1}{9}y), \qquad K(x_K = \frac{1}{3}x\ ,\ y_K = y + \frac{2}{a}).$$

From these we deduce easily the first part of the theorem. The converse is equally easy, the last statement resulting from the expression for y_K. □

Theorem 12. *Continuing with the notation of the preceding lemma, the three other than L intersection points of the circle κ_0 and the parabola define an equilateral triangle ABC. All equilaterals inscribed in a parabola result by the construction procedure of κ_0 of the preceding theorem.*

Proof. Starting with a circle κ_0, constructed from an isosceles LST as in the preceding lemma, we must show that the three, other than L, intersection points of this circle and the parabola define an equilateral triangle. This can be seen by proving that the centroid of the created triangle ABC coincides with the center K of its circumcircle. For this, it suffices to see that the coordinates of K can be expressed through the coordinates of $\{A, B, C\}$ in the form

$$K(x_K, y_K)\ , \quad \text{with} \quad x_K = \frac{1}{3}(x_A + x_B + x_C)\ ,\ y_K = \frac{1}{3}(y_A + y_B + y_C)\ .$$

The coordinates of the points $\{A, B, C\}$ are found by considering the intersections of the parabola with the circle κ_0. Using the equations of the preceding theorem the equation of κ_0 is found to be

$$3a(u^2 + v^2) - (2ax)u - 6(a^2x^2 + 2)v + ax^2(3a^2x^2 + 7) = 0\ . \tag{23}$$

Setting $v = au^2$, this reduces to the equation w.r.t. u :

$$a(x + u)((3a^2)u^3 - (3a^2x)u^2 - 3(a^2x^2 + 3)u + x(3a^2x^2 + 7)) \ = \ 0\ . \tag{24}$$

The root $u = -x$ corresponds to point L and the vertices of the triangle correspond to the roots of the third degree factor of the general form $pu^3 + qu^2 + ru + s = 0$, with

$$p = 3a^2\ , \quad q = -3a^2x\ , \quad r = -3(a^2x^2 + 3)\ , \quad s = x(3a^2x^2 + 7)\ .$$

The discriminant ([16, p.120]) of this equation is seen to be negative

$$-(4(ax)^2 + 9)^2/(81a^6) \ < \ 0\ ,$$

which guarantees the existence of three real roots $\{u_1, u_2, u_3\}$, corresponding to the vertices of the triangle ABC. From Vieta's relations of the coefficients with the symmetric functions of roots we have ([3, p.37]):

$$\frac{1}{3}(u_1 + u_2 + u_3) \ = \ -\frac{q}{3p} \ = \ \frac{1}{3}x \ = \ x_K\ .$$

For the corresponding values of $\{v_i = au_i^2\}$ we use the other Vieta relations

$$\frac{1}{3}(v_1 + v_2 + v_3) \ = \ \frac{a}{3}(u_1^2 + u_2^2 + u_3^2) \ = \frac{a}{3}[(u_1 + u_2 + u_3)^2 - 2(u_1u_2 + u_2u_3 + u_3u_1)]$$
$$= \ \frac{a}{3}[(q/p)^2 - 2(r/p)] \ = \ \frac{a^2x^2 + 2}{a} \ = \ y_K\ .$$

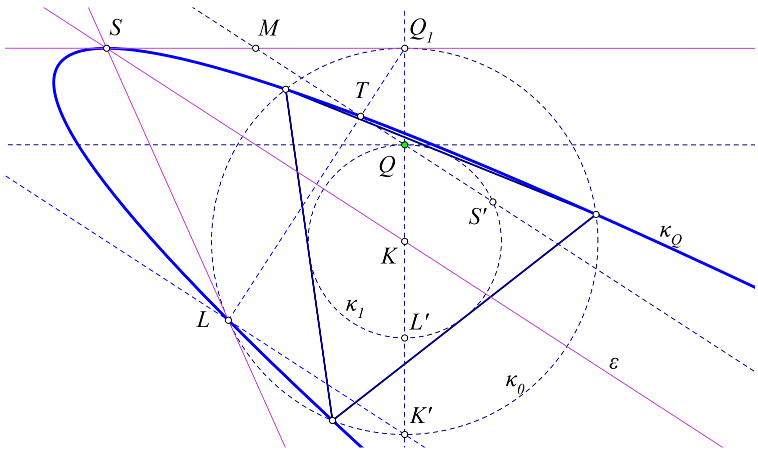

Figure 12: A property of the parabola

To see that all equilaterals inscribed in the parabola are obtained in this way, we consider an equilateral inscribed in the parabola and show the existence of a configuration like the preceding one. From the general theory we know that the parabola κ_Q occurs when the perspector Q lies on the incircle $\kappa_1(K, R_1)$ of the equilateral (see Figure 12). The definition procedure of the homography g_Q is still valid and has the following properties:

1. $P = g_Q(Q)$ is the point at infinity of the parabola, which determines also the direction of its axis.

2. The tripolar ε of Q_1 is parallel to the axis, passes through K and intersects the parabola at a point S.

3. The tripolar τ_S of S is the WS-line of Q_1, is parallel to KS and intersects SQ_1 at its middle M (lemma 6).

4. The fourth intersection point L of κ_Q with the circumcircle κ_0 of $\triangle ABC$ defines line LQ_1 intersecting the parabola at its point T with the property $LT/LQ_1 = 3/4$.

It follows that the triangle LSQ_1 is isosceles and from lemma 1 follows that the parabola is tangent to SQ_1 at S and the configuration satisfies the claimed conditions. □

6 Tripolars of the equilateral

The simplest *triangle conic* is the circumcircle $\kappa_0(K)$ of the *equilateral* triangle. It will be seen below that the *perspector* Q in this case, generating κ_0 by the tripoles of all lines through it, is the center K of the equilateral, its tripolar τ_Q being the line at infinity of the plane. In general, the tripolars τ_Q of the equilateral are related, in various ways depending on the position of the tripole Q, to the *Wallace-Simson lines (WS-lines)* and their parallels through the orthocenter, the *Steiner lines* of the triangle. For convenience I list a few of the relevant definitions and properties formulated as lemmata. I supply only the proofs of those seemingly not to be widely known. The proofs of the other lemmata can be found in [10, p.8], [7, p.24], [5, p.140], [1, p.54].

Lemma 2. *The projections $\{P_1, P_2, P_3\}$ on the sides of the triangle ABC of a point P on the circumcircle κ_0 of the triangle lie on a line s_P, called WS-line of the point P w.r.t. the triangle ABC (see Figure 13). The direction of the WS-line is determined by extending one of the projecting*

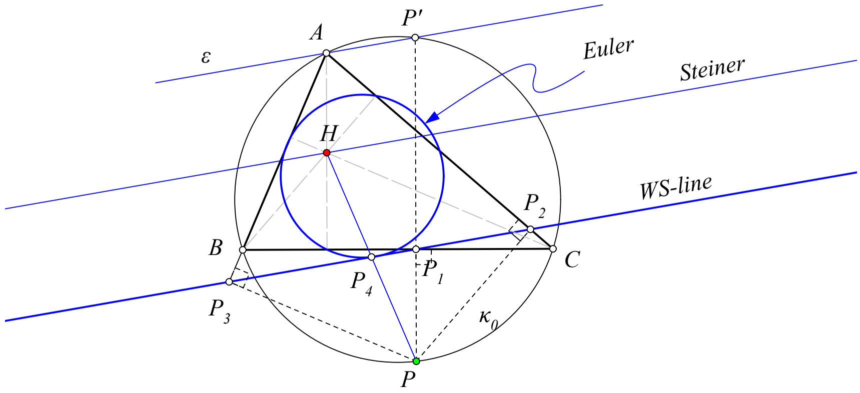

Figure 13: WS-line and Steiner line

lines, PP_1 say, until to cut a second time the circumcircle κ_0 at a point P'. The line $\varepsilon = AP'$ is parallel to the WS-line of P. The middle P_4 of the segment PH, joining P to the orthocener H of the triangle, is on its Euler circle, passing through the feet of its altitudes. The Steiner line is defined to be the parallel to the WS-line passing through the orthocenter H of the triangle.

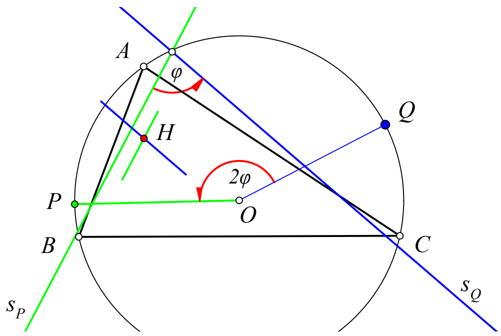

Figure 14: Angle of two WS-lines

Lemma 3. *The oriented angle ϕ of two WS-lines $\{s_P, s_Q\}$ and their parallel Steiner lines is half of the central angle of the points $\{P, Q\}$ defining them and has the opposite orientation of that of ϕ (see Figure 14). In particular, diametral points of the circumcircle define orthogonal WS-lines and corresponding orthogonal Steiner lines.*

Lemma 4. *The tripolar τ_P of a point P on the circumcircle $\kappa_0(O)$ of the equilateral passes through the center O of the circumcircle and coincides with the Steiner line of P.*

Proof. Let $\{P_1, P_2, P_3\}$ be the intersections of the WS-line with the sides of the triangle and $\{A_1 = PA \cap BC\ ,\ A_2 = \tau_P \cap BC\}$ (see Figure 15). It suffices to show that $\{A_1, A_2\}$ are harmonic conjugate to $\{B, C\}$ and also show the analogous properties for the other sides and their intersections with the other cevians through P and τ_P. By lemma 2, the direction of the WS-line is that of AP_4. This has the consequence of the angle equalities noticed in the figure. Further, from the same lemma follows that $\triangle DA_2P \cong A''A_2O$ is isosceles and PA_2 is perpendicular to PA and passes through the diametral A'' of A. For the intersection $D = PP_4 \cap \tau_P$ follows then, that A_1D is perpendicular to τ_P. It is then seen that $\{PS, PT\}$ are bisectors of the the angle $\widehat{A_2PD}$, hence $\{A_2, D\}$ are harmonic conjugate to $\{S, T\}$. This implies that DA_1 is the polar of A_2 and consequently $\{A_1, A_2\}$ are harmonic conjugate to $\{B, C\}$. □

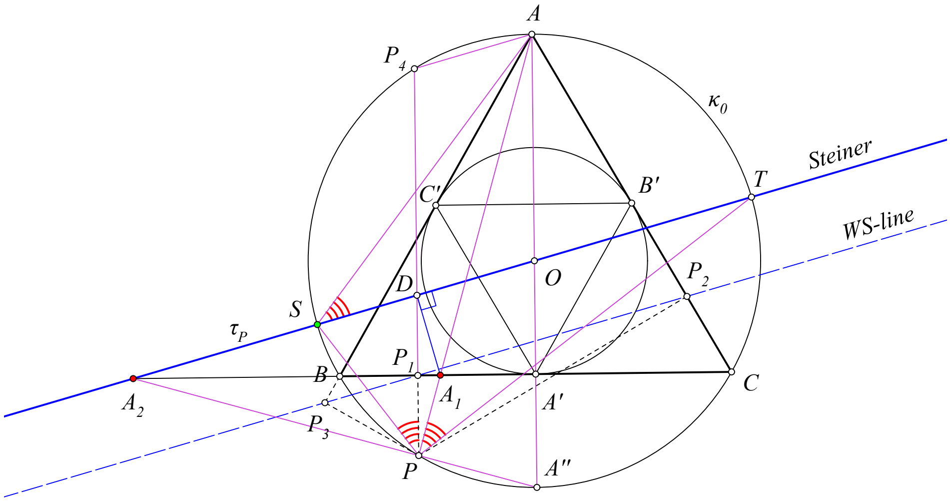

Figure 15: Coincidence of the tripolar τ_P with the Steiner line

Lemma 5. *Let $\{X, Y\}$ be the intersections of the tangent t_A to κ_0 at A correspondingly with the tangent t_T to κ_0 at T and the Steiner line τ_T of the point T on the circumcircle $\kappa_0(O)$ of the equilateral ABC. Then A is the middle of XY (see figure 16).*

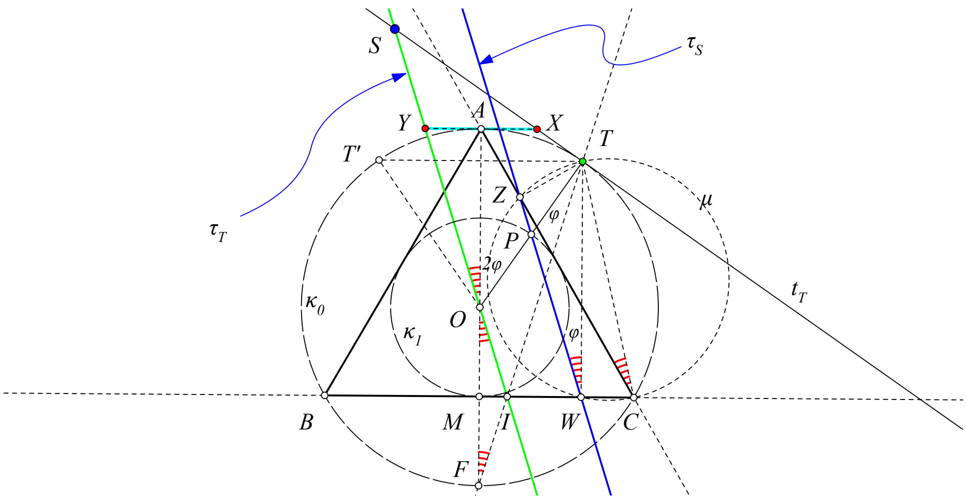

Figure 16: The middle of TT'

Proof. Project T on the sides of ABC to $\{Z \in AC, W \in BC\}$. The quadrangle $TZWC$ is cyclic and $\phi = \widehat{ZWT} = \widehat{ZCT}$. By the parallelity of the lines it is also $\phi = \widehat{YOA}$. This implies that $TX = XA = AY$. □

Lemma 6. *Let S be the intersection point of the tangent to the circumcircle $\kappa_0(O)$ of the equilateral at the point T and the tripolar τ_T. Then, the tripolar τ_S of S is the WS-line of T.*

Proof. Let $D = SA \cap BC$ and W be the projection of T on BC (see Figure 17). It suffices to show that $\{W, D\}$ are harmonic conjugate w.r.t. $\{B, C\}$ and the analogous properties for $\{SB, SC\}$. We do it for SA. The harmonicity property is equivalent to the orthogonality of the circles with diameters respectively $\{BC, WD\}$. To prove this we use the proved in

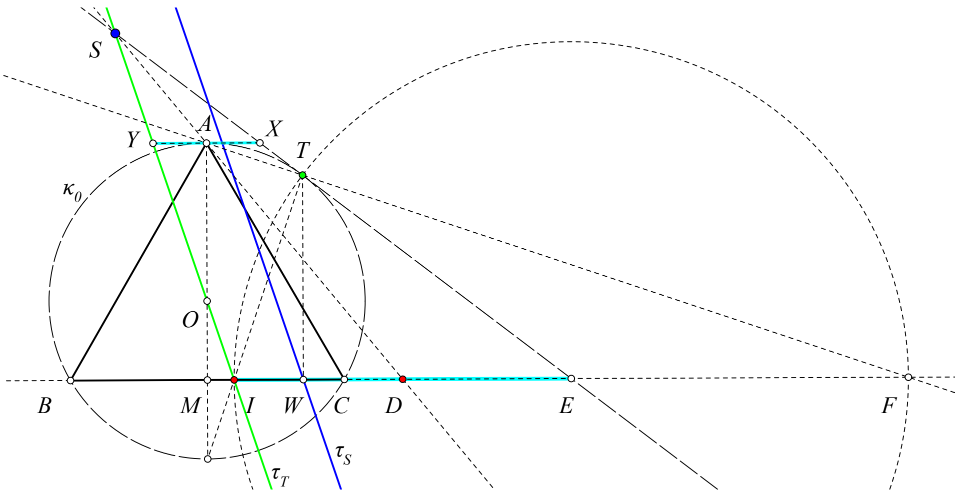

Figure 17: Related tripolars

lemma 4 harmonicity of $\{I, F\}$ to $\{B, C\}$, where $I = \tau_T \cap BC$ and $F = AT \cap BC$, which is equivalent to the orthogonality of the circles with diameters $\{BC, IF\}$. Using the notation for the lengths

$$a = BC\ ,\ x = MI\ ,\ y = MF\ ,\ z = ME\ ,\ d = MD\ ,\ w = MW,$$

we deduce easily the following relations, the last one proving the claim:

$$x \cdot y = \frac{a^2}{4}\ ,\ z = \frac{x+y}{2}\ ,\ d = \frac{3x+y}{4}\ ,\ w = \frac{y(4xy+3a^2)}{4y^2+3a^2} \quad \Rightarrow \quad d \cdot w = \frac{a^2}{4}\ . \tag{25}$$

□

Figure 18 shows the tripolar w.r.t. to the equilateral of *a point at infinity* Q identified with a *direction of parallel lines*. The parallel lines shown are the *cevians* of Q intersecting the sides at the points $\{A', B', C'\}$. The tripolar t_Q of Q carries the conjugates $\{A'', B'', C''\}$ of these points w.r.t. corresponding side-ends. Thus, $\{A'', A'\}$ are harmonic conjugate w.r. to $\{B, C\}$, points $\{B'', B'\}$ are conjugate w.r.t. $\{C, A\}$ etc. Next lemma verifies the suggestion made by this figure.

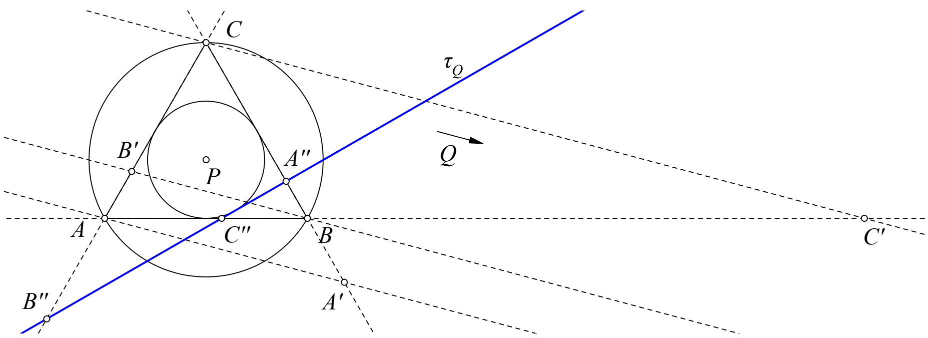

Figure 18: The tripolar τ_Q of a point at infinity

Lemma 7. *The tripolar* τ_Q *of a point* Q *at infinity w.r.t the equilateral* ABC *is tangent to its incircle.*

Proof. This is an easy test in barycentrics, for $Q(u,v,w)$ at infinity has $u+v+w=0$ and its tripolar has coefficients $(p=1/u, q=1/v, r=1/w)$ seen to be tangent to the incircle, whose equation is $u^2+v^2+w^2-2(vw+wu+uv)=0$ ([13]) with corresponding *tangential equation* $pq+qr+rp=0$ ([11, II,p.62]). □

Next lemma is again a property of the equilateral supplying additional information on the determination of the tripole of a tangent τ to the incircle of the equilateral, which, according to lemma 7 lies at infinity.

Lemma 8. *Let the tangent τ at the point D of the incircle $\kappa_1(K)$ of the equilateral ABC intersect BC at the point I and E be the symmetric of K w.r.t. τ. Let also L be the second intersection with the circumcircle κ_0 of the perpendicular from E to BC. Then the point $S = BC \cap AL$ is harmonic conjugate to I w.r.t. $\{B,C\}$.*

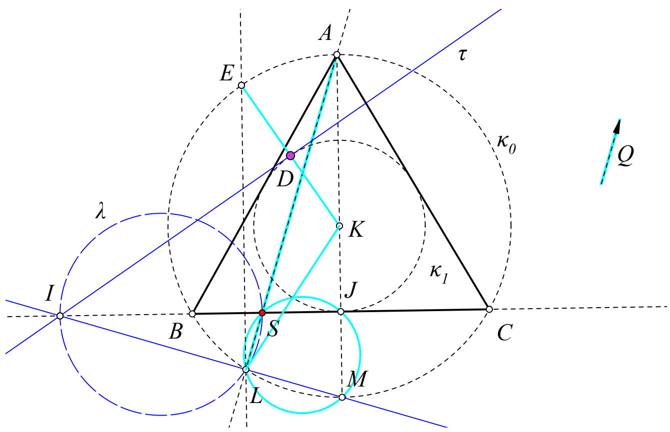

Figure 19: Determining the tripole Q at infinity of the tangent τ to κ_1

Proof. Consider the symmetric M of A w.r.t. K (see figure 19) and define L as the other intersection point of IM with κ_0 and $S = AL \cap BC$. The points $\{S,L,M,J\}$, where J the middle of BC are concyclic. This implies $IB \cdot IC = IS \cdot IJ$, which is easily seen to be equivalent with the harmonicity property $(BC,IS) = -1$. Define E by drawing the parallel from L to AM. Using the isosceles trapezium $AMLE$ the points $\{E,L,S\}$ are easily identified with those of the statement. □

Lemma 9. *The tripole Q of a a tangent τ to the incircle $\kappa_1(O)$ of the equilateral at its point D is the point at infinity determined by the parallels to the WS-line of the point E on the circumcircle, which is the symmetric w.r.t. τ of the center O.*

Proof. Referring to figure 19 and lemma 8, the direction of the parallel lines, determining the tripole of τ as a point at infinity, is given by the line AS joining the vertex A of the triangle with the harmonic conjugate $S = I(BC)$. But this line determines also the direction of the WS-line of the point E. □

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