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Revisiting linear Weingarten spacelike hypersurfaces immersed in the de Sitter space

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Abstract. We study closed linear Weingarten spacelike hypersurfaces immersed in the de Sitter space $\mathbb{S}_1^{n+1}$, which means that these spacelike hypersurfaces have mean and scalar curvatures linearly related. Using a modified Cheng-Yau operator, we obtain a suitable integral inequality involving the norm of the total umbilicity tensor of these spacelike hypersurfaces and we apply it to characterize totally umbilical round spheres of $\mathbb{S}_1^{n+1}$.

1. Introduction and statement of the main results

Let $\mathbb{R}_1^{n+2}$ be an $(n+2)$-dimensional real vector space endowed with an inner product of index 1 given by

$$\langle x, y\rangle = \sum_{j=1}^{n+1} x_j y_j - x_{n+2} y_{n+2},$$

where $x = (x_1, x_2, \ldots, x_{n+2})$ is the natural coordinate of $\mathbb{R}_1^{n+2}$.

$\mathbb{R}_1^{n+2} = \mathbb{L}^{n+2}$ is called the $(n+2)$-dimensional *Lorentz-Minkowski space.* We define the $(n+1)$-dimensional *de Sitter space* $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \left\{(x_1, x_2, \ldots x_{n+2}) \in \mathbb{R}_1^{n+2}; \langle x, x\rangle = 1\right\}.$$

The induced metric $\langle\,,\rangle$ makes $\mathbb{S}_1^{n+1}$ a Lorentzian manifold with constant sectional curvature 1.

Let M^n be an n-dimensional hypersurface immersed in $\mathbb{S}_1^{n+1}$. We recall that M^n is said to be *spacelike* if the induced metric on M^n from the metric of $\mathbb{S}_1^{n+1}$ is positive definite. In 1977, Goddard [11] conjectured that every complete spacelike hypersurface with constant mean curvature H in $\mathbb{S}_1^{n+1}$ should be totally umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses.

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For instance, Ramanathan proved Goddard's conjecture in [15] for $\mathbb{S}^3_1$ and $0 \leq H \leq 1$. Moreover, if $H > 1$ he showed that the conjecture is false as can be seen from an example due to Dajczer and Nomizu in [9]. In [1], Akutagawa proved that Goddard's conjecture is true when $n = 2$ and $H^2 \leq 1$ or when $n \geq 3$ and $H^2 < \frac{4(n-1)}{n^2}$. He also constructed complete spacelike rotation surfaces in $\mathbb{S}^3_1$ with constant H satisfying $H > 1$ and which are not totally umbilical. Next, Montiel [14] proved that Goddard's conjecture is true provided that M^n is compact. Furthermore, he exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}^{n+1}_1$ with constant mean curvature H satisfying $H^2 \geq \frac{4(n-1)}{n^2}$ and being non totally umbilical, the so-called *hyperbolic cylinders*, which are isometric to the Riemannian product $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ of a hyperbolic line of radius $r > 0$ and an $(n-1)$-dimensional Euclidean sphere of radius $\sqrt{1+r^2}$.

Concerning the study of spacelike hypersurfaces with constant scalar curvature in the de Sitter space, Zheng [16] proved that a compact spacelike hypersurface in $\mathbb{S}^{n+1}_1$ with constant normalized scalar curvature $R < 1$ and nonnegative sectional curvatures is totally umbilical. Later, Cheng and Ishikawa [7] showed that Zheng's result in [16] is also true without additional assumptions on the sectional curvatures of the hypersurface. Later on, answering a question proposed by Li in Section 4 of [13], Camargo, Chaves and Sousa Jr. [4] proved that a complete spacelike hypersurface in $\mathbb{S}^{n+1}_1$, $n \geq 3$, with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leq R \leq 1$ and having bounded mean curvature, must be totally umbilical. They also showed that a complete spacelike hypersurface in $\mathbb{S}^{n+1}_1$, with constant normalized scalar curvature $R \leq 1$ and whose square norm of the second fundamental form S satisfies $\sup S < 2\sqrt{n-1}$, must be totally umbilical.

Extending the ideas of these previous works, an interesting Goddard type problem is to obtain characterizations of the so-called *linear Weingarten* spacelike hypersurfaces (that is, spacelike hypersurfaces whose mean and scalar curvatures are linearly related) immersed in $\mathbb{S}^{n+1}_1$. Many authors have approached investigations into this branch (see, for instance, [5, 6, 10, 12, 13]). More recently, Alías jointly with the first and second authors [2] obtained new characterizations concerning complete linear Weingarten spacelike hypersurfaces immersed in $\mathbb{S}^{n+1}_1$. Under appropriated constrains on the scalar curvature function, they applied a suitable extension of the generalized maximum principle of Omori-Yau giving a sharp estimate of the total umbilicity tensor of the hypersurface, which allowed them to characterize hyperbolic cylinders $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ of $\mathbb{S}^{n+1}_1$ when $n \geq 3$ and totally umbilic 2-spheres in $\mathbb{S}^3_1$ when $n = 2$.

Proceeding with this picture, here we deal with closed linear Weingarten spacelike hypersurfaces immersed in the de Sitter space $\mathbb{S}^{n+1}_1$. Using the technique developed by Alías and Meléndez in [3], we obtain a sharp integral inequality involving the total umbilicity tensor Φ of these spacelike hypersurfaces and we apply it to characterize totally umbilical round spheres of $\mathbb{S}^{n+1}_1$. More precisely, we prove the following result.

Theorem 1.1. *Let M^n be a closed linear Weingarten spacelike hypersurface isometrically immersed in the de Sitter space $\mathbb{S}_1^{n+1}$, $n \geq 2$, such that $R = aH + b$ with $b \leq 1$. In the case where $b = 1$, suppose that $a > 0$. Then*

$$\int_M |\Phi|^{q+2}\varphi_{a,b}(|\Phi|)dM \leq 0, \tag{1}$$

for every real number $q > 2$, where the real function $\varphi_{a,b}(x)$ is given by

$$\begin{aligned}(2)\ \varphi_{a,b}(x) = {} & \frac{n-2}{n-1}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x\right)\sqrt{\frac{x^2}{n(n-1)} + \frac{a^2}{4} - b + 1} \\ & + \frac{n(n-2)}{\sqrt{n(n-1)}}\frac{a}{2}x - n\left(\frac{a^2}{2} - b\right).\end{aligned}$$

Moreover, assuming in addition that $n \geq 3$ and $0 < b \leq R < \frac{n-2}{n}$, the equality holds in (1) if, and only if, M^n is a totally umbilical round sphere $\mathbb{S}^n(r) \hookrightarrow \mathbb{S}_1^{n+1}$, with $r = \frac{1}{R} > 1$.

Taking into account that when $n = 2$, $a \geq 0$ and $0 < b \leq 1$ we have that $\varphi_{a,b}(x) > 0$ for all $x \geq 0$, and noting that $R = K$ is the Gaussian curvature of M^2, it is not difficult to verify that from the integral inequality (1) we get the following consequence.

Corollary 1.1. *Let M^2 be a closed linear Weingarten spacelike surface isometrically immersed in the de Sitter space $\mathbb{S}_1^3$, such that $K = aH + b$.*

- (i) *If $0 < b < 1$ and $a \geq 0$, then M^2 is a totally umbilical round sphere $\mathbb{S}^2(r) \hookrightarrow \mathbb{S}_1^3$, with $r = \frac{1}{K} > 1$.*
- (ii) *If $b = 1$ and $a > 0$, then M^2 is a totally geodesic unit round sphere $\mathbb{S}^2 \hookrightarrow \mathbb{S}_1^3$.*

The proof of Theorem 1.1 is given in Section 4.

2. Preliminaries

As before, let M^n be a spacelike hypersurface immersed in the de Sitter space $\mathbb{S}_1^{n+1}$. We choose a local field of semi-Riemannian orthonormal frame $\{e_A\}_{1\leq A\leq n+1}$ in $\mathbb{S}_1^{n+1}$, with dual coframe $\{\omega_A\}_{1\leq A\leq n+1}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+1,\ 1 \leq i, j, k, \ldots \leq n.$$

In this setting, denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{S}_1^{n+1}$, we have that the structure equations of $\mathbb{S}_1^{n+1}$ are given by:

$$d\omega_A = \sum_i \omega_{Ai} \wedge \omega_i - \omega_{An+1} \wedge \omega_{n+1}, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \varepsilon_C\varepsilon_D K_{ABCD}\omega_C \wedge \omega_D,$$

$$K_{ABCD} = \varepsilon_A\varepsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}),$$

where $\varepsilon_i = 1$ and $\varepsilon_{n+1} = -1$.

Next, we restrict all the tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n, so $\sum_i \omega_{n+1i} \wedge \omega_i = d\omega_{n+1} = 0$ and by Cartan's Lemma we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \;\; h_{ij} = h_{ji}. \tag{3}$$

This gives the second fundamental form of M^n, $h = \sum_{ij} h_{ij}\omega_i \otimes \omega_j e_{n+1}$. Furthermore, the mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

The structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l.$$

Using the structure equations we obtain the Gauss equation

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - h_{ik}h_{jl} + h_{il}h_{jk}, \tag{4}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

The Ricci curvature and the normalized scalar curvature of M^n are given, respectively, by

$$R_{ij} = (n-1)\delta_{ij} - nHh_{ij} + \sum_k h_{ik}h_{kj} \tag{5}$$

and

$$R = \frac{1}{n(n-1)}\sum_i R_{ii}. \tag{6}$$

From (5) and (6) we obtain

$$S = n^2H^2 + n(n-1)(R-1), \tag{7}$$

where $S = \sum_{i,j} h_{ij}^2$ is the square norm of the second fundamental form h of M^n.

3. Set up and auxiliary results

In what follows, we will consider M^n as being a *linear Weingarten* spacelike hypersurface immersed in $\mathbb{S}_1^{n+1}$, that is, a spacelike hypersurface of $\mathbb{S}_1^{n+1}$ whose mean curvature H and normalized scalar curvature R satisfy

$$R = aH + b,$$

for some $a, b \in \mathbb{R}$.

Let $\phi = \sum_{i,j} \phi_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$\phi_{ij} = nH\delta_{ij} - h_{ij}.$$

According to Cheng and Yau [8], the $\Box$ operator associated to ϕ and acting on a C^2 function f is given by

$$\Box f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{8}$$

Here, we will consider the following Cheng-Yau's modified operator

$$L = \Box + \frac{n-1}{2}a\Delta. \tag{9}$$

In other words, for any $u \in \mathcal{C}^2(M)$,

$$L(u) = \operatorname{tr}(P \circ \nabla^2 u), \tag{10}$$

with

$$P = \left(nH + \frac{n-1}{2}a\right) I - h, \tag{11}$$

where I is the identity in the algebra of smooth vector fields on M^n and $\nabla^2 u$ stands for the self-adjoint linear operator metrically equivalent to the Hessian of u.

Remark 3.1. From equation (7), since $R = aH + b$, we have that

$$n^2H^2 = S - n(n-1)(aH + b - 1). \tag{12}$$

When $b < 1$, it follows from (12) that $H(p) \neq 0$ for every $p \in M^n$. In this case, we can choose the orientation of M^n such that $H > 0$. On the other hand, when $b = 1$, we will assume that H does not change sign on M^n and we will choose the orientation of M^n such that $H \geq 0$.

The next lemma gives a sufficient criteria for the ellipticity of the operator L. For its proof, see Lemma 3 of [2].

Lemma 3.1. *Let M^n be a linear Weingarten spacelike hypersurface in $\mathbb{S}_1^{n+1}$ such that $R = aH + b$. Let μ_- and μ_+ be, respectively, the minimum and the maximum of the eigenvalues of the operator P at every point $p \in M^n$. If $b < 1$, then the operator L is elliptic, with*

$$\mu_- > 0 \quad \textit{and} \quad \mu_+ < 2nH + (n-1)a.$$

In the case where $b = 1$, assume further that the mean curvature function H does not change sign and $R \geq 1$. Then the operator L is semi-elliptic, with

$$\mu_- \geq 0 \quad \textit{and} \quad \mu_+ \leq 2nH + (n-1)a,$$

unless M^n is totally geodesic.

Remark 3.2. Also related to the ellipticity of operator L, when M^n is totally geodesic, we observe that the operator L reduces to $L = \frac{n-1}{2}a\Delta$, which is elliptic if, and only if, $a > 0$. For this reason, in order to keep the validity of Lemma 3.1 when $b = 1$, even in the totally geodesic case, we will choose a to be a positive constant.

We will also consider the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i \otimes \omega_j$$

associated to the second fundamental form of a hypersurface M^n in $\mathbb{S}_1^{n+1}$, whose components are given by $\Phi_{ij} = h_{ij} - H\delta_{ij}$. Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the square of the length of Φ. It is not difficult to check that Φ is traceless and, from (6), we get

$$|\Phi|^2 = S - nH^2 = n(n-1)H^2 + n(n-1)\left(R - 1\right). \tag{13}$$

In particular, $|\Phi|^2 = 0$ at the umbilic points of M^n. For that reason Φ is usually called the total umbilicity tensor of M^n.

In order to prove Theorem 1.1, it will be essential the following lower boundedness for the operator L acting on the square norm of the total umbilicity tensor. For its proof, see Proposition 6 of [2].

Lemma 3.2. *Let M^n be a linear Weingarten spacelike hypersurface immersed in $\mathbb{S}_1^{n+1}$, $n \geq 2$, such that $R = aH + b$ with $b \leq 1$. In the case where $b = 1$, assume that the mean curvature function H does not change sign and $R \geq 1$. Then,*

$$L(|\Phi|^2) \geq 2(n-1)|\Phi|^2 \varphi_{a,b}(|\Phi|)\sqrt{\frac{|\Phi|^2}{n(n-1)} + \frac{a^2}{4} + 1 - b},$$

where the real function $\varphi_{a,b}(x)$ is defined in (2).

4. Proof of Theorem 1.1

Making $u = |\Phi|^2$, from Lemma 3.2 we have that

$$L(u) \geq 2(n-1)u\varphi_{a,b}(\sqrt{u})\sqrt{\frac{u}{n(n-1)} + \frac{a^2}{4} + 1 - b}. \tag{14}$$

Taking into account that $u \geq 0$ and $b \leq 1$, with $b = 1$ only for $a > 0$, from (14) we get

$$u^{\frac{q+2}{2}}\varphi_{a,b}(\sqrt{u}) \leq \sqrt{\frac{n}{n-1}}\frac{u^{\frac{q}{2}}}{\sqrt{4u + n(n-1)(a^2 + 4(1-b))}}L(u), \tag{15}$$

for every real number q. By the compactness of M^n, we can integrate both sides of (15) in order to obtain

$$\int_M u^{\frac{q+2}{2}}\varphi_{a,b}(\sqrt{u})dM \leq \sqrt{\frac{n}{n-1}}\int_M \frac{u^{\frac{q}{2}}}{\sqrt{4u + n(n-1)(a^2 + 4(1-b))}}L(u)dM. \tag{16}$$

But, from (10) we have

$$f(u)L(u) = \operatorname{div}(f(u)P(\nabla u)) - f'(u)\langle P(\nabla u), \nabla u\rangle, \tag{17}$$

for every smooth function $f \in \mathcal{C}^1(\mathbb{R})$. Integrating both sides of (17) and using Stokes' theorem, we deduce that

$$\int_M f(u)L(u)dM = -\int_M f'(u)\langle P(\nabla u), \nabla u\rangle dM, \tag{18}$$

for every smooth function f. In our case, we choose

$$f(t) = \frac{t^{q/2}}{\sqrt{4t + n(n-1)(a^2 + 4(1-b))}}, \quad \text{for } t \geq 0. \tag{19}$$

Hence, since we are supposing $b \leq 1$ and that $b = 1$ only for $a > 0$, we get

$$f'(t) = \frac{4(q-1)t^{q/2} + n(n-1)(a^2 + 4(1-b))qt^{\frac{q-2}{2}}}{2\left(4t + n(n-1)(a^2 + 4(1-b))\right)^{3/2}} \geq 0, \tag{20}$$

for every real number $q > 2$. Using (19) and (20) into (16), we can estimate

$$\int_M u^{\frac{q+2}{2}}\varphi_{a,b}(\sqrt{u})dM \leq -\sqrt{\frac{n}{n-1}}\int_M f'(u)\langle P(\nabla u), \nabla u\rangle dM \leq 0, \tag{21}$$

since Lemma 3.1 assures that the operator P is semi-positive definite. Therefore, we conclude

$$\int_M u^{\frac{q+2}{2}} \varphi_{a,b}(\sqrt{u}) dM \leq 0. \tag{22}$$

This proves inequality (1).

Now, let us suppose that $n \geq 3$ and $0 < b \leq R < \frac{n-2}{n}$. If the equality holds in (1), from (21) we get

$$\int_M f'(u) \langle P(\nabla u), \nabla u \rangle dM = 0. \tag{23}$$

But, since $q > 2$ and assuming that $b < 1$, from (20) we have

$$f'(u) = \frac{4(q-1)u^{q/2} + n(n-1)(a^2 + 4(1-b))qu^{\frac{q-2}{2}}}{2\left(4u + n(n-1)(a^2 + 4(1-b))\right)^{3/2}} > 0. \tag{24}$$

Consequently, taking into account Lemma 3.1, (23) and (24) imply

$$\langle P(\nabla u), \nabla u \rangle = 0. \tag{25}$$

Thus, since P is positive definite, from (25) we get that $\nabla u = 0$ on M^n. Hence, the function $u = |\Phi|^2$ must be constant.

In the case that $|\Phi| = 0$, we conclude that M^n must be a totally umbilical round sphere $\mathbb{S}^n(r)$, with $r = \frac{1}{R} > 1$. Otherwise, $|\Phi|$ is a positive constant and, since we are assuming in addition that $n \geq 3$ and $0 < b \leq R < \frac{n-2}{n}$, we can apply Theorem 8 of [2] to conclude that M^n should be isometric to a hyperbolic cylinder $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ of radius $r > 0$. Therefore, since we are supposing that M^n is closed, this last situation which cannot occur.

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