INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 9 (2020), No. 2, 81-92

# Two new refinements of a linear geometric inequality

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**Abstract.** For a consequence of the Erdös-Mordell inequality, we establish two new refinements with the help of Maple software. Some applications are also given by new results. Several interesting conjectures checked by computer are presented as well.

## 1. Introduction

The famous Erdös-Mordell inequality is a beautiful linear geometric inequality, which can be stated as follows:

Let $P$ be an interior point of the triangle $ABC$. Denote by $R_1, R_2, R_3$ the distances from $P$ to the vertices $A, B, C$ respectively, and denote by $r_1, r_2, r_3$ the distances from $P$ to the sides $BC, CA, AB$ ,respectively. Then

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3), \tag{1}$$

with equality if and only if $\triangle ABC$ is equilateral and $P$ is its center.

There exist various proofs, generalizations, sharpness and refinements of the Erdös-Mordell inequality in the literature. Some recent results on the Erdös-Mordell inequality can be found in [2],[6]-[10].

By the Erdös-Mordell inequality, it is easy to obtain the following inequality:

$$R_1 + R_2 + R_3 \geq \frac{2}{3}(h_a + h_b + h_c), \tag{2}$$

where $h_a, h_b, h_c$ are the altitudes of the triangle $ABC$. In fact, by the triangle inequality we have

$$R_1 + r_1 \geq h_a \tag{3}$$

and two similar relations. Adding $2(R_1 + R_2 + R_3)$ to both sides of (1) and then using inequality (3) etc., we obtain one inequality equivalent with (2) immediately.

Clearly, inequality (2) can be regarded as a consequence of the Erdös-Mordell inequality. We wish to point out that it is valid for any point $P$ in the plane of the triangle $ABC$, because we have the following conclusion:

**Keywords and phrases:** the Erdös-Mordell inequality, Triangle, Point, Altitude, Median
**(2010)Mathematics Subject Classification:** 51M16
Received: 29.04.2020. In revised form: 10.07.2020. Accepted: 07.06.2020.

For any point $Q$ outside triangle $ABC$, there exist a point $P$ inside triangle $ABC$ such that $QA \geq PA, QB \geq PB, QC \geq PC$ at the same time. This conclusion is not difficult to be proved, we omit here.

From [12], we know that Ji Chen conjectured that the following inequality stronger than (2) holds:

$$R_1 + R_2 + R_3 \geq \frac{2}{3}(w_a + w_b + w_c), \tag{4}$$

where $w_a, w_b, w_c$ are the lengths of angle bisectors of triangle $ABC$.

In [16], Xiao-Guang Chu proved inequality (4). As $w_a \geq h_a$ etc., thus we have the following refinement of inequality (2):

$$R_1 + R_2 + R_3 \geq \frac{2}{3}(w_a + w_b + w_c) \geq \frac{2}{3}(h_a + h_b + h_c). \tag{5}$$

In [4], the author and Xiao-Guang Chu established the following inequality:

$$R_1 + R_2 + R_3 \geq \frac{1}{2}(m_a + m_b + m_c + 3r), \tag{6}$$

where $m_a, m_b, m_c$ are the lengths of medians of $\triangle ABC$ and $r$ is the inradius of $ABC$.

By inequality (6), we can further obtain the following refinement of inequality (2):

$$\begin{aligned} &R_1 + R_2 + R_3 \geq \frac{1}{2}(m_a + m_b + m_c + 3r) \\ &\geq \frac{1}{3}(m_a + m_b + m_c + h_a + h_b + h_c) \geq \frac{2}{3}(h_a + h_b + h_c). \end{aligned} \tag{7}$$

where the second inequality has been proved in [4] and the last inequality is obvious by $m_a \geq h_a$ etc.

Our aim of this paper is to give two new refinements of the linear geometric inequality (2).

In what follows, we shall continue to use the previous symbols. Also, denote by $a, b, c$ the side lengths of the triangle $ABC$ and denote by $s, R, r$ the sem-perimeter, the circumradius and the inradius of the triangle $ABC$ respectively.

The main results are as follows:

**Theorem 1.1.** *For any point $P$ in the plane of the triangle $ABC$, it holds:*

$$R_1 + R_2 + R_3 \geq \frac{\sqrt{3}}{2}a + h_a \geq \frac{2}{3}(h_a + h_b + h_c), \tag{8}$$

*where the first equality in (8) holds if and only if $b = c$, $P$ lies on the altitude from the vertex $A$ to the side $BC$ and $\angle BPC = \frac{2}{3}\pi$, the second inequality holds if and only if the triangle $ABC$ is equilateral.*

**Theorem 1.2.** *For any point $P$ in the plane of the triangle $ABC$, it holds:*

$$\begin{aligned} &R_1 + R_2 + R_3 \geq 2r + \frac{2}{3}\sqrt{a^2 + b^2 + c^2} \geq \frac{2}{3}\left(1 + \frac{r}{R}\right)\sqrt{a^2 + b^2 + c^2} \\ &\geq \frac{2}{3}(h_a + h_b + h_c), \end{aligned} \tag{9}$$

*where the first equality in (9) holds if and only if the triangle $ABC$ is equilateral and $P$ is its center, the others hold if and only if the triangle $ABC$ is equilateral.*

## 2. Proof of Theorem 1.1

Before proving Theorem 1.1, we first give the following lemma.

**Lemma 2.1.** *In any triangle $ABC$, holds:*

$$b + c \geq \frac{\sqrt{3}}{2}a + h_a, \tag{10}$$

*with equality if and only if $b = c$ and $A = \frac{2}{3}\pi$.*

**Proof.** By the known formula

$$h_a = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{a} \tag{11}$$

(where $s = (a + b + c)/2$) and the fact that $\sqrt{(s-b)(s-c)} \leq 2a$, we have

$$h_a \leq \sqrt{s(s-a)}. \tag{12}$$

Squaring both sides and using $s = (a + b + c)/2$, we easily get

$$b + c \geq \sqrt{a^2 + 4h_a^2}. \tag{13}$$

But

$$a^2 + 4h_a^2 = \left(\frac{\sqrt{3}}{2}a + h_a\right)^2 + \left(\frac{1}{2}a - \sqrt{3}h_a\right)^2, \tag{14}$$

thus from (13) and (14) we see that (10) holds. Observe that the equality in (13) holds if and only if $b = c$, thus by (13) and (14) one sees that the equality in (10) holds if and only if $b = c$ and $a/2 - \sqrt{3}h_a = 0$, from which and $ah_a = 2S$ ($S$ being the area of $ABC$) it follows that $a^4 = 12S^2$. Further, by the equivalent form of Heron's formula:

$$2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 = 16S^2, \tag{15}$$

we get

$$2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 = \frac{4}{3}a^4.$$

Substituting $b = c$ into the above identity, we easily get $a = \sqrt{3}c$ and then conclude that $B = C = \pi/3$ and $A = 2\pi/3$. Hence the equality condition of (10) is just as mentioned as in Lemma 2.1. This completes the proof of Lemma 2.1.

Next, we prove Theorem 1.1.

**Proof.** Firstly, we prove the first inequality of (9), i.e.,

$$R_1 + R_2 + R_3 \geq \frac{\sqrt{3}}{2}a + h_a. \tag{16}$$

Applying inequality (10) to triangle $BPC$, we have

$$R_2 + R_3 \geq \frac{\sqrt{3}}{2}a + r_1. \tag{17}$$

Adding $R_1$ to both sides of this inequality and then using inequality (3), one obtains inequality (16) immediately. Note that the equality in (3) holds if and only if $P$ lies on the altitude from the vertex $A$ to the side $BC$. Thus by the equality condition of (10), we further know that the equality condition of (16) is just as mentioned in Theorem 1.1.

Next, we prove the second inequality of (8), i.e.,

$$\frac{\sqrt{3}}{2}a + h_a \geq \frac{2}{3}(h_a + h_b + h_c). \tag{18}$$

This inequality is equivalent to

$$\frac{3\sqrt{3}}{2}a + h_a \geq 2(h_b + h_c). \tag{19}$$

Taking squares on both sides and then using $ah_a = 2S = 2sr$, it becomes

$$\frac{27}{4}a^2 + h_a^2 + 6\sqrt{3}rs \geq 4(h_b + h_c)^2. \tag{20}$$

Multiplying both sides by 4 and noticing the following known inequality

$$s \geq 3\sqrt{3}r, \tag{21}$$

we only need to prove that

$$27a^2 + 4h_a^2 + 216r^2 \geq 16(h_b + h_c)^2, \tag{22}$$

i.e.,

$$27a^2 + \frac{16S^2}{a^2} + \frac{216S^2}{s^2} \geq 64S^2\left(\frac{1}{b} + \frac{1}{c}\right)^2.$$

And, by Heron formula:

$$S = \sqrt{s(s-a)(s-b)(s-c)}, \tag{23}$$

we easily know again the inequality is equivalent to

$$\begin{aligned} Q_0 \equiv\ & 4(s-a)(s-b)(s-c)\left[16b^2c^2s^2 + 216(abc)^2 - 64s^2a^2(b+c)^2\right] \\ & + 108sa^4b^2c^2 \geq 0. \end{aligned} \tag{24}$$

Substituting $s = (a+b+c)/2$ into (24) and then arranging with the help of Maple software, we know again that (24) is equivalent to

$$\begin{aligned} Q_0 \equiv\ & 4(b+c)^2a^7 + 4(b+c)^3a^6 - (8b^4 + 16b^3c + 44b^2c^2 + 16bc^3 + 8c^4)a^5 \\ & - 8(b+c)(b^4 + 2b^3c - 8b^2c^2 + 2bc^3 + c^4)a^4 + (4b^6 + 8b^5c + 52b^4c^2 \\ & - 124b^3c^3 + 52b^2c^4 + 8bc^5 + 4c^6)a^3 + 4(b+c)(b^6 + 2b^5c - 14b^4c^2 \\ & + 23b^3c^3 - 14b^2c^4 + 2bc^5 + c^6)a^2 - (b-c)^2(b+c)^2b^2c^2a \\ & - (b-c)^2(b+c)^3b^2c^2 \geq 0, \end{aligned} \tag{25}$$

which is required to prove.

Now, we let $b+c-a = 2x, c+a-b = 2y, a+b-c = 2z$, then $a = y+z, b = z+x, c = x+y$. Substituting them into $Q_0$ and using Maple software, it is easy to get the following identity:

$$Q_0 = 2Q_1, \tag{26}$$

where

$$Q_1 \equiv 16yzx^7 + 64(y+z)yzx^6 + (27y^4 + 164y^3z + 306y^2z^2 + 164yz^3 + 27z^4)x^5 + (y+z)(81y^4 + 52y^3z + 38y^2z^2 + 52yz^3 + 81z^4)x^4 + (81y^6 - 60y^5z - 441y^4z^2 - 584y^3z^3 - 441y^2z^4 - 60yz^5 + 81z^6)x^3 + (y+z)(27y^6 - 114y^5z - 235y^4z^2 - 156y^3z^3 - 235y^2z^4 - 114yz^5 + 27z^6)x^2 - yz(10y^4 + 13y^3z - 226y^2z^2 + 13yz^3 + 10z^4)(y+z)^2x + 27(y+z)^5y^2z^2.$$

Thus, it remains to prove that $Q_1 \geq 0$ holds for positive real numbers $x, y, z$. Since $Q_1$ is symmetric in $y$ and $z$, we may assume without loss of generality that $y \geq z$ and let $y = z + m (m \geq 0)$. Substituting $y = z + m$ into $Q_1$ and using Maple software, we immediately obtain the following identity:

$$\begin{aligned} Q_1 = {} & (27z^2 - 10zx + 27x^2)m^7 + (324z^3 - 103z^2x + 102zx^2 + 81x^3)m^6 \\ & + (1647z^4 - 218z^3x - 304z^2x^2 + 426zx^3 + 81x^4)m^5 \\ & + (4590z^5 + 531z^4x - 2496z^3x^2 + 474z^2x^3 + 538zx^4 + 27x^5)m^4 \\ & + 8z(945z^5 + 348z^4x - 780z^3x^2 - 166z^2x^3 + 179zx^4 + 34x^5)m^3 \\ & + 16z(459z^6 + 268z^5x - 506z^4x^2 - 264z^3x^3 + 123z^2x^4 \\ & + 60zx^5 + 4x^6)m^2 + 16z(z-x)(z+x)(243z^5 + 180z^4x \\ & - 107z^3x^2 - 87z^2x^3 - 12zx^4 - x^5)m + 16(54z^3 + 45z^2x \\ & + 8zx^2 + x^3)(z-x)^2(z+x)^2z^2. \end{aligned} \tag{27}$$

To show that inequality $Q_1 \geq 0$ is valid for $y > 0, z > 0$ and $m \geq 0$, we shall consider two cases to finish the proof.

**Case 1.** $z \geq x$.

In this case, we set $z = x + n (n \geq 0)$ and substitute it into (27), then the inequality $Q_1 \geq 0$ becomes

$$\begin{aligned} Q_2 \equiv {} & (2304m^2 + 6912mn + 6912n^2)x^7 + 128(m+2n)(35m^2 + 146mn \\ & + 146n^2)x^6 + (3664m^4 + 33472m^3n + 119104m^2n^2 + 171264mn^3 \\ & + 85632n^4)x^5 + 32(m+2n)(51m^4 + 494m^3n + 2176m^2n^2 \\ & + 3364mn^3 + 1682n^4)x^4 + (404m^6 + 5752m^5n + 42072m^4n^2 \\ & + 152752m^3n^3 + 276656m^2n^4 + 240336mn^5 + 80112n^6)x^3 \\ & + 4(m+2n)(11m^6 + 195m^5n + 1841m^4n^2 + 7700m^3n^3 \\ & + 14870m^2n^4 + 13224mn^5 + 4408n^6)x^2 + n(m+n)(44m^4 \\ & + 649m^3n + 2773m^2n^2 + 4248mn^3 + 2124n^4)(m+2n)^2x \\ & + 27(m+n)^2(m+2n)^5n^2 \geq 0. \end{aligned} \tag{28}$$

Since $m \geq 0, n \geq 0$ and $x > 0$, we see that $Q_2 \geq 0$ holds.

**Case 2.** $x > z$.

In this case, we set $x = z + n (n \geq 0)$ and substitute it into (27), then the inequality $Q_1 \geq 0$ becomes

$$\begin{aligned} Q_3 \equiv\ & (2304m^2 - 6912mn + 6912n^2)z^7 + (4480m^3 - 11520m^2n \\ & + 7680mn^2 + 11008n^3)z^6 + (3664m^4 - 6592m^3n + 1600m^2n^2 \\ & + 19968mn^3 + 6528n^4)z^5 + 16(3m + 4n)(34m^4 - 61m^3n + 104m^2n^2 \\ & + 164mn^3 + 28n^4)z^4 + (404m^6 + 776m^5n + 2424m^4n^2 + 7120m^3n^3 \\ & + 7728m^2n^4 + 2864mn^5 + 240n^6)z^3 + 4(m + n)(11m^6 + 75m^5n \\ & + 290m^4n^2 + 434m^3n^3 + 264m^2n^4 + 72mn^5 + 4n^6)z^2 \\ & + mn(44m^4 + 257m^3n + 192m^2n^2 + 32mn^3 + 16n^4)(m + n)^2z \\ & + 27(m + n)^3m^4n^2 \geq 0. \end{aligned} \tag{29}$$

Since

$$2304m^2 - 6912mn + 6912n^2 > 0,$$
$$4480m^3 - 11520m^2n + 7680mn^2 = 640m(7m^2 - 18mn + 12n^2) > 0,$$
$$3664m^4 - 6592m^3n + 1600m^2n^2 = 16m^2(229m^2 - 412mn + 100n^2) > 0,$$
$$34m^4 - 61m^3n + 104m^2n^2 = m^2(34m^2 - 61mn + 104n^2) > 0,$$

we conclude that $Q_3 \geq 0$ holds for $m \geq 0, n \geq 0$, and $z > 0$ from (29).

Combining the arguments of the above two cases, inequality $Q_1 \geq 0$ is proved for all positive numbers $x, z$ and non-positive real number $m$. Hence, inequality $Q_0 \geq 0$ and inequality (18) are proved too. Also, it is easy to conclude that the equality in (18) holds if and only if $a = b = c$, i.e., $\triangle ABC$ is equilateral. This completes the proof of Theorem 1.1.

**Remark 2.1.** *By inequality (17), we have*

$$\frac{R_2 + R_3 - r_1}{a} + \frac{R_3 + R_1 - r_2}{b} + \frac{R_1 + R_2 - r_3}{c} \geq \frac{3\sqrt{3}}{2} \tag{30}$$

*which is a special case of a conjecture posed by the author in [5].*

**Remark 2.2.** *By inequality (17) and Gerber's inequality (see [3]):*

$$r_2r_3 + r_3r_1 + r_1r_2 \leq \frac{S}{\sqrt{3}}, \tag{31}$$

*we easily obtain the following known inequality (see [11]):*

$$\begin{aligned} & R_1(r_2 + r_3) + R_2(r_3 + r_1) + R_3(r_1 + r_2) \\ & \geq (r_3 + r_1)(r_1 + r_2) + (r_1 + r_2)(r_2 + r_3) + (r_2 + r_3)(r_3 + r_1). \end{aligned} \tag{32}$$

## 3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we shall use the following known result (see [1, inequality 12.55]):

**Lemma 3.1.** *Let $P$ be a point in the plane of the triangle $ABC$.*

(i) *If $A \geq \frac{2}{3}\pi$, then*

$$R_1 + R_2 + R_3 \geq b + c, \tag{33}$$

*with equality if and only if* $P = A$.

(ii) *If* $A < \frac{2}{3}\pi$, *then*

$$R_1 + R_2 + R_3 \geq \sqrt{\frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}S}, \tag{34}$$

*with equality if and only if* $\angle BPC = \angle CPA = \angle APB$.

**Lemma 3.2.** *In triangle* $ABC$, *if* $max\{A, B, C\} \leq \frac{2}{3}\pi$ *then*

$$\sqrt{3}s \geq 2R + 5r, \tag{35}$$

*with equality if and only if* $\triangle ABC$ *is equilateral.*

Inequality (35) can be obtained by Xue-Zhi Yang's result in [15], which was used to prove inequality (4) by Xiao Chu in [16].

We are now read to prove Theorem 1.2.

**Proof.** We first prove the first inequality of (9), i.e.,

$$R_1 + R_2 + R_3 \geq 2r + \frac{2}{3}\sqrt{a^2 + b^2 + c^2}. \tag{36}$$

Based on Lemma 1, we divide the following two cases to complete the proof.

**Case 1.** $A > \frac{2}{3}\pi$.

In this case, by Lemma 1 we only need to prove that

$$b + c > 2r + \frac{2}{3}\sqrt{a^2 + b^2 + c^2}. \tag{37}$$

Note that $b + c > 2h_a > 4r$, thus we only need to show that

$$9(b + c - 2r)^2 - 4(a^2 + b^2 + c^2) > 0,$$

i.e.,

$$9(b + c)^2 + 36r^2 - 4a^2 - 4b^2 - 4c^2 > 36r(b + c). \tag{38}$$

Multiplying both sides by $s$ and using $S = rs$ and the known identity

$$(s - a)(s - b)(s - c) = sr^2, \tag{39}$$

we know that inequality (38) is equivalent to

$$\begin{aligned} &s\left(5b^2 + 5c^2 + 18bc - 4a^2\right) + 36(s - a)(s - b)(s - c) \\ &> 36(b + c)\sqrt{s(s - a)(s - b)(s - c)}. \end{aligned} \tag{40}$$

For proving the above inequality, we set $s - a = x, s - b = y, s - c = z$, then $a = y + z, b = z + x, c = x + y, s = x + y + z (x, y, z > 0)$ and we see that inequality (40) is equivalent to

$$\begin{aligned} &(x + y + z)\left[5(z + x)^2 + 5(x + y)^2 + 18(z + x)(x + y) - 4(y + z)^2\right] \\ &+ 36xyz > 36(2x + y + z)\sqrt{xyz(x + y + z)}, \end{aligned}$$

i.e.,

$$\begin{aligned} &28x^3 + 56(y + z)x^2 + (29y^2 + 102yz + 29z^2)x + (y + z)(y^2 + 10yz + z^2) \\ &> 36(2x + y + z)\sqrt{xyz(x + y + z)}. \end{aligned}$$

Thus, we need to prove

$$\left[28x^3+56(y+z)x^2+(29y^2+102yz+29z^2)x+(y+z)(y^2+10yz+z^2)\right]^2$$
$$-36^2xyz(x+y+z)(2x+y+z)^2>0.$$

Expanding out and collecting like terms gives

$$\begin{aligned}&784x^6+3136(y+z)x^5+(4760y^2+6800yz+4760z^2)x^4\\&+8(y+z)(413z^2+202yz+413y^2)x^3+(953y^4+780y^3z\\&+1590y^2z^2+780yz^3+953z^4)x^2+2(y+z)(29y^4-256yz^3\\&-218y^2z^2-256y^3z+29z^4)x+(y+z)^2(y^2+10yz+z^2)^2>0.\end{aligned}\tag{41}$$

Obviously, to prove this inequality it is enough to prove that

$$\begin{aligned}&(953y^4+780y^3z+1590y^2z^2+780yz^3+953z^4)x^2\\&+2(y+z)(29y^4-256yz^3-218y^2z^2-256y^3z+29z^4)x\\&+(y+z)^2(y^2+10yz+z^2)^2>0.\end{aligned}\tag{42}$$

The left is a quadratic function on $x$ and its discriminant $F_x$ can easily be calculated as follows:

$$F_x=-64(7y^6+2154y^5z-471y^4z^2+940y^3z^3-471y^2z^4+2154yz^5+7z^6)(y+z)^4.$$

Since

$$\begin{aligned}&2154y^5z-471y^4z^2-471y^2z^4+2154yz^5\\&=3yz\left[(561(y^4+z^4)+157(y^2+yz+z^2)(y-z)^2\right]>0,\end{aligned}$$

so that $F_x<0$, thus we conclude that inequality (42) holds and inequality (37) is proved.

**Case 2** $A\le\frac{2}{3}\pi$.

In this case, by Lemma 2, to prove inequality (36) we only need to prove that

$$\sqrt{\frac{1}{2}(a^2+b^2+c^2)+2\sqrt{3}S}\ge 2r+\frac{2}{3}\sqrt{a^2+b^2+c^2}.\tag{43}$$

Squaring both sides, it becomes

$$\frac{1}{2}(a^2+b^2+c^2)+2\sqrt{3}S\ge 4r^2+\frac{4}{9}(a^2+b^2+c^2)+\frac{8}{3}r\sqrt{a^2+b^2+c^2},$$

that is

$$(a^2+b^2+c^2)+36\sqrt{3}rs\ge 72r^2+48r\sqrt{a^2+b^2+c^2}.$$

Squaring both sides again, we have to prove that

$$\begin{aligned}&(a^2+b^2+c^2)^2+3888r^2s^2+72\sqrt{3}rs(a^2+b^2+c^2)\\&\ge 5184r^4+2304r^2(a^2+b^2+c^2)+6912r^3\sqrt{a^2+b^2+c^2}.\end{aligned}\tag{44}$$

According to Lemma 3 and the following known identity:

$$a^2+b^2+c^2=2(s^2-4Rr-r^2),\tag{45}$$

we only need to prove the following inequality:

$$\begin{aligned}&4(s^2-4Rr-r^2)^2+3888r^2s^2+144(2R+5r)r(s^2-4Rr-r^2)\\&\ge 5184r^4+4608r^2(s^2-4Rr-r^2)+6912r^3\sqrt{2(s^2-4Rr-r^2)},\end{aligned}$$

i.e.,

$$\begin{aligned} &4s^4 + (256Rr - 8r^2)s^2 - 1088R^2r^2 + 15296Rr^3 - 1292r^4 \\ &\geq 6912r^3\sqrt{2(s^2 - 4Rr - r^2)}. \end{aligned} \tag{46}$$

According to the Gerretsen inequality (see [1, inequality 5.8])

$$s^2 \geq 16Rr - 5r^2 \tag{47}$$

and Euler inequality $R \geq 2r$, we have

$$\begin{aligned} &4s^4 + (256Rr - 8r^2)s^2 - 1088R^2r^2 + 15296Rr^3 - 1292r^4 \\ &> 4(16Rr - 5r^2)^2 + (256Rr - 8r^2)(16Rr - 5r^2) - 1088R^2r^2 \\ &\quad + 15296Rr^3 - 1292r^4 \\ &= 576r^2(7R^2 + 23Rr - 2r^2) > 0. \end{aligned}$$

Thus, to prove (46) we only need to prove that

$$\begin{aligned} &\left[4s^4 + (256Rr - 8r^2)s^2 - 1088R^2r^2 + 15296Rr^3 - 1292r^4\right]^2 \\ &- \left[6912r^3\sqrt{2(s^2 - 4Rr - r^2)}\right]^2 \geq 0. \end{aligned}$$

After arranging, it becomes

$$\begin{aligned} Q_0 \equiv\ & 16s^8 + 64r(32R - r)s^6 + 96(592R^2 + 1232Rr - 107r^2)r^2s^4 \\ & - 64(8704R^3 - 122640R^2r + 14160Rr^2 + 1492669r^3)r^3s^2 \\ & + 16(73984R^4 - 2080256R^3r + 14798688R^2r^2 \\ & + 21417568Rr^3 + 6076297r^4)r^4 \geq 0, \end{aligned} \tag{48}$$

which is required to prove.

Putting

$$\begin{aligned} e &= R - 2r, \\ g_1 &= s^2 - 16Rr + 5r^2, \\ g_2 &= 4R^2 + 4Rr + 3r^2 - s^2, \end{aligned}$$

then we have $e \geq 0$ and Gerretsen inequalities $g_1 \geq 0, g_2 \geq 0$(see [1]). Through analysis, we obtain the following identity:

$$Q_0 = m_1g_2 + e(m_2g_1 + m_3) + m_4g_1^2, \tag{49}$$

where

$$\begin{aligned} m_1 &= 35831808r^6, \\ m_2 &= 55296(56R^2 + 289Rr + 539r^2)r^3, \\ m_3 &= 331776(49R^3 + 420R^2r + 909Rr^2 - 434r^3)r^4, \\ m_4 &= 16\left[s^4 + 2r(80R - 7r)s^2 + (8416R^2 + 5504Rr - 527r^2)r^2\right]. \end{aligned}$$

Therefore, by Euler inequality $R \geq 2r$ and Gerretsen inequalities $g_1 \geq 0$, $g_2 \geq 0$, we see that $Q_0 \geq 0$ holds true.

Combining the arguments of the above two cases, we finish the proof of inequality (36).

The last two inequalities of (9) are easily proved.

The second inequality of (9) is actually equivalent to

$$\sqrt{a^2+b^2+c^2} \le 3R, \tag{50}$$

which is also equivalent to the simple known inequality $a^2+b^2+c^2 \le 9R^2$. Since $bc = 2Rh_a$, we see that the third inequality of (9) is equivalent to

$$(R+r)\sqrt{a^2+b^2+c^2} \ge bc+ca+ab. \tag{51}$$

In view of $a^2+b^2+c^2 \ge bc+ca+ab$ and the following known inequality (see [1, inequality 5.17]):

$$bc+ca+ab \le 4(R+r)^2, \tag{52}$$

we conclude that inequality (51) holds and finish the proof of inequality chain (9). In addition, we can easily determine the equality conditions of (9). This completes the proof of Theorem 1.2.

## 4. Corollaries and conjectures

In this section, we give several corollaries of Theorem 1.1 and Theorem 1.2 and present a few relevant conjectures.

Since

$$\left(\frac{\sqrt{3}}{2}a+h_a\right)^2 \ge 2\sqrt{3}ah_a = 4\sqrt{3}S,$$

thus by the first inequality of (8) we get the following known inequality (see [1, inequality 12.18]):

**Corrolary 4.1.** *For any point $P$ in the plane of $\triangle ABC$, it holds:*

$$R_1+R_2+R_3 \ge 2\sqrt{\sqrt{3}S}. \tag{53}$$

Let $P$ be the centroid of $\triangle ABC$, then $R_1 = \frac{2}{3}m_a, R_2 = \frac{2}{3}m_b, R_3 = \frac{2}{3}m_c$. Thus by the first inequality of (8) we have

**Corrolary 4.2.** *In any triangle $ABC$, holds:*

$$m_a+m_b+m_c \ge \frac{3\sqrt{3}}{4}a+\frac{3}{2}h_a. \tag{54}$$

For the acute triangle $ABC$, the author conjectures that the following stronger inequality holds.

**Conjecture 4.1.** *In the acute triangle $ABC$, holds:*

$$m_a+m_b+m_c \ge \frac{3\sqrt{3}}{4}a+\frac{3}{2}w_a. \tag{55}$$

For the first inequality of (8),i.e.,(10), we also have two similar inequalities. Adding these three inequalities gives

**Corrolary 4.3.** *For any point $P$ in the plane of $\triangle ABC$, it holds:*

$$R_1+R_2+R_3 \ge \frac{1}{3}(\sqrt{3}s+h_a+h_b+h_c). \tag{56}$$

Let $P$ be the centroid of $\triangle ABC$, then by (56) we can obtain

**Corrolary 4.4.** *In any triangle $ABC$, holds:*

$$2(m_a + m_b + m_c) \geq \sqrt{3}s + h_a + h_b + h_c. \tag{57}$$

It follows from inequalities (19) and (17) that

$$3(R_2 + R_3) \geq 2(h_b + h_c) - h_a + 3r_1. \tag{58}$$

Adding $R_1$ and then using the previous inequality (3), the following linear inequality is obtained:

**Corrolary 4.5.** *For any point $P$ in the plane of $\triangle ABC$, it holds:*

$$R_1 + 3(R_2 + R_3) \geq 2(r_1 + h_b + h_c), \tag{59}$$

**Remark 4.1.** *The previous inequalities (3) and (16) are both actually valid for any point $P$ in the plane, thus inequality (59) holds in the same situation too.*

For inequality (59), we present the following stronger inequality.

**Conjecture 4.2.** *For any point $P$ in the plane of $\triangle ABC$, it holds:*

$$R_1 + 3(R_2 + R_3) \geq 2(r_1 + w_b + w_c). \tag{60}$$

Let $P$ be the centroid of $\triangle ABC$, then by the first inequality of (9) we get

**Corrolary 4.6.** *In any triangle $ABC$, holds:*

$$m_a + m_b + m_c \geq 3r + \sqrt{a^2 + b^2 + c^2}. \tag{61}$$

Inspired by this inequality, we pose the following conjecture:

**Conjecture 4.3.** *In any triangle $ABC$, holds:*

$$3r + \sqrt{a^2 + b^2 + c^2} \geq k_a + k_b + k_c, \tag{62}$$

*where $k_a, k_b, k_c$ are the symmedians of $\triangle ABC$.*

**Remark 4.2.** *If (62) is true then by Theorem 2 we conclude that the following linear inequality similar to (2) and (4) holds:*

$$R_1 + R_2 + R_3 \geq \frac{2}{3}(k_a + k_b + k_c), \tag{63}$$

*which was proved by Xiao-Guang Chu in [16].*

**Remark 4.3.** *For the acute triangle $ABC$, the author has already proved the following inequality:*

$$3r + \sqrt{a^2 + b^2 + c^2} \geq w_a + w_b + w_c, \tag{64}$$

*which shows that the first inequality of (9) is stronger than inequality (4) for the acute triangle $ABC$.*

Adding inequality (17) and its two analogues gives

$$2(R_1 + R_2 + R_3) - (r_1 + r_2 + r_3) \geq \sqrt{3}s, \tag{65}$$

which is valid for any interior point $P$ of triangle $ABC$. Here, we put forward the following similar conjecture:

**Conjecture 4.4.** *For any interior point $P$ of $\triangle ABC$, it holds:*

$$2(R_1 + R_2 + R_3) - (r_1 + r_2 + r_3) \geq m_a + m_b + m_c. \tag{66}$$

**Remark 4.4.** *If the above inequality holds, then by inequality (3) one easily obtains*

$$R_1 + R_2 + R_3 \geq \frac{1}{3}(m_a + m_b + m_c + h_a + h_b + h_c), \tag{67}$$

*which follows from the previous inequality chain (7).*

Finally, we present an inequality which is stronger than the Erdös-Mordell inequality (1).

**Conjecture 4.5.** *For any interior point $P$ of $\triangle ABC$, it holds:*

$$R_1 + R_2 + R_3 \geq 2\left(\sqrt{\frac{m_a}{h_a}}r_1 + \sqrt{\frac{m_b}{h_b}}r_2 + \sqrt{\frac{m_c}{h_c}}r_3\right). \tag{68}$$

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