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NEW CHARACTERIZATIONS OF TANGENTIAL QUADRILATERALS

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Abstract. We prove 13 new necessary and sufficient conditions for when a convex quadrilateral can have an incircle.

1. Introduction

A *tangential quadrilateral* is a convex quadrilateral that can have an *incircle*, that is, a circle inside of itself that is tangent to all four sides. It has been known since antiquity that all triangles have the capacity of having an incircle, but this is not the case for quadrilaterals.

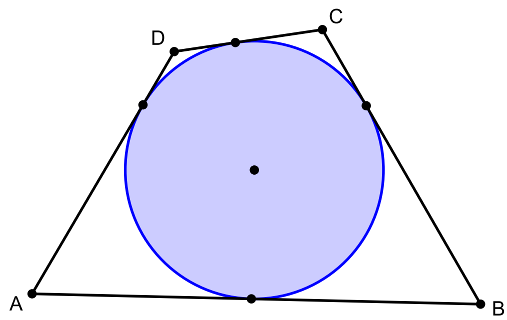

Figure 1. $ABCD$ is tangential $\Leftrightarrow$ $AB + CD = BC + DA$

As late as in 2008, there were only about ten known characterizations of tangential quadrilaterals, but since then, that number has increased rapidly. The papers [12, 5, 6, 7, 8] contain a total of approximately 37 different necessary and sufficient conditions for when a convex quadrilateral can have an incircle, all with proofs or references (where the exact number depends on if very similar corollaries are considered different or not).

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Of all of these previously known characterizations we just remind the reader that two famous and important characterizations (probably the two oldest) are that a convex quadrilateral $ABCD$ can have an incircle if and only if the angle bisectors of all four vertex angles are concurrent; or if and only if the sides satisfy *Pitot's theorem* (see Figure 1):

$$AB + CD = BC + DA. \tag{1}$$

Different proofs of this last theorem as well as some historical background were recently discussed in [10]. Half of the characterizations in this paper will be proved by showing that they are equivalent to (1).

Considering the large number of known characterizations of tangential quadrilaterals, one might think this topic is exhausted. But that is far from the truth. In fact, in the present paper we will prove 13 more that we have not seen published before, making it a total of 50 known necessary and sufficient conditions for when a convex quadrilateral can have an incircle. Almost all of these new characterizations were discovered by the second author using a dynamic geometry computer program, while the first author provided their proofs.

2. Subtriangle incircles

We start with a stronger version of Theorem 2 in [5].

Theorem 2.1. *The incircles in the four overlapping triangles formed by the diagonals in a convex quadrilateral are tangent to the sides in eight points, two per side, making one distance between tangency points for each side. Two adjacent such distances are equal if and only if it is a tangential quadrilateral.*

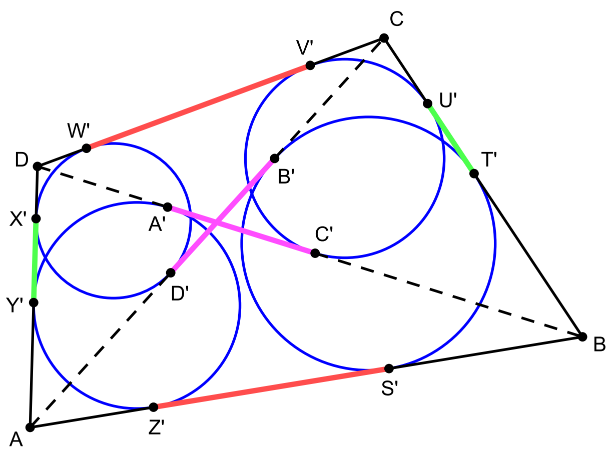

FIGURE 2. $ABCD$ is tangential $\Leftrightarrow$ $V'W' = X'Y'$

Proof. We use notations as in Figure 2. Then, by the two tangent theorem (the two tangents to a circle from an external point have equal lengths), we have

$$\begin{aligned}V'W' &= V'D - DW' \\ &= C'D - DX' \\ &= A'C' + A'D - DX' \\ &= A'C' + Y'D - DX' \\ &= A'C' + X'Y'. \qquad (2)\end{aligned}$$

In the same way

$$Z'S' = B'D' + X'Y'.$$

But

$$A'C' = B'D' = \tfrac{1}{2}|AB - BC + CD - DA| \qquad (3)$$

in all convex quadrilaterals according to the proof of Theorem 1 in [5], which proves that $V'W' = Z'S'$ in all convex quadrilaterals.

From (2), we get $V'W' = X'Y'$ if and only if $A'C' = 0$, which holds if and only if $ABCD$ is tangential according to (3) and (1). □

Theorem 2.2. *In a convex quadrilateral, the incircle in one of the triangles created by a diagonal and the incircles in the two triangles created by the other diagonal are tangent to two of the sides of the first triangle in four points, creating two lines that do not intersect within the quadrilateral. These lines are parallel if and only if it is a tangential quadrilateral.*

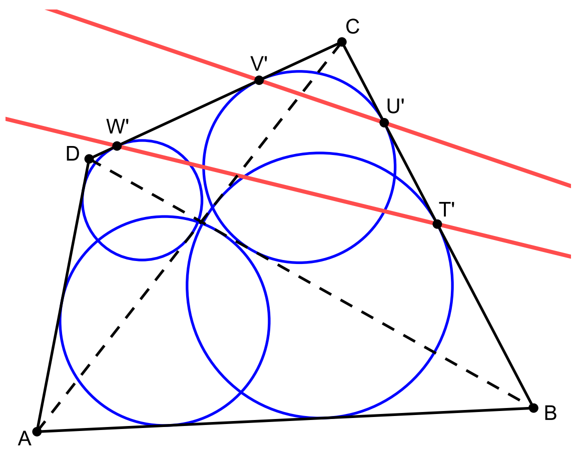

FIGURE 3. $ABCD$ is tangential $\Leftrightarrow$ $U'V' || T'W'$

Proof. With notations as in Figure 3, we have by the intercept theorem and its converse that

$$U'V' || T'W' \quad \Leftrightarrow \quad \frac{CU'}{U'T'} = \frac{CV'}{V'W'} \quad \Leftrightarrow \quad U'T' = V'W'$$

which by Theorem 2.1 is equivalent to that the quadrilateral is tangential. We used that $CU' = CV'$ according to the two tangent theorem. □

To prove the next characterization we need a property of a right kite, that is, a kite with two opposite right angles.

Lemma 2.1. *In a right kite, the length of the diagonal that divides it into two isosceles triangles is equal to the length of the other diagonal times sine of any one of the two angles between the sides of equal length.*

Proof. Let a right kite with consecutive sides a, a, b, b and diagonals p, q have the angle α between the sides of length a. Its area can be calculates in two ways as

$$\tfrac{1}{2}pq = \tfrac{1}{2}a^2 \sin\alpha + \tfrac{1}{2}b^2 \sin(\pi - \alpha)$$

which is simplified into

$$pq = (a^2 + b^2)\sin\alpha$$

and further as

$$pq = p^2 \sin\alpha$$

according to the Pythagorean theorem. We get $q = p\sin\alpha$, as claimed. □

Theorem 2.3. *In a convex quadrilateral $ABCD$ where the diagonals intersect at P, the incircles in triangles ABP, BCP, CDP, DAP are tangent to each of the diagonals, making one chord per circle between such tangency points. The product of the lengths of opposite chords are equal if and only if $ABCD$ is a tangential quadrilateral.*

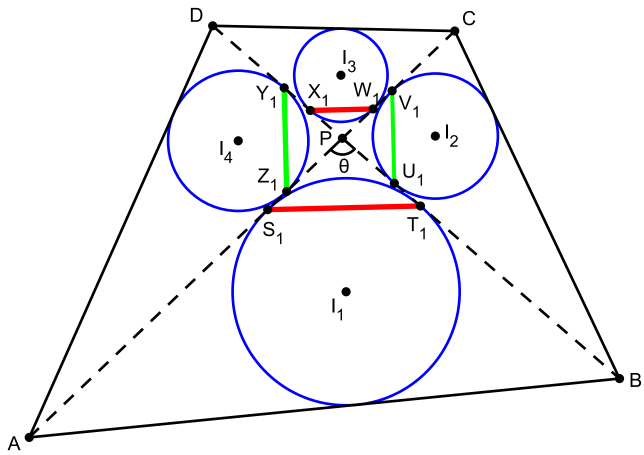

FIGURE 4. $ABCD$ is tangential $\Leftrightarrow$ $S_1T_1 \cdot W_1X_1 = U_1V_1 \cdot Y_1Z_1$

Proof. Suppose the points of tangency are S_1, T_1, U_1, V_1, W_1, X_1, Y_1, Z_1, and let I_1, I_2, I_3, I_4 be the incenters of these circles (see Figure 4). In [14] it was proved that $ABCD$ is tangential if and only if $I_1I_2I_3I_4$ is a cyclic quadrilateral. According to the intersecting chords theorem and its converse, $I_1I_2I_3I_4$ is cyclic if and only if $PI_1 \cdot PI_3 = PI_2 \cdot PI_4$. By applying Lemma 2.1 in the right kites $PS_1I_1T_1$, $PW_1I_3X_1$, $PU_1I_2V_1$ and $PY_1I_4Z_1$, this is equivalent to

$$\frac{S_1T_1}{\sin\theta} \cdot \frac{W_1X_1}{\sin\theta} = \frac{U_1V_1}{\sin(\pi-\theta)} \cdot \frac{Y_1Z_1}{\sin(\pi-\theta)}$$

which is simplified as

$$S_1T_1 \cdot W_1X_1 = U_1V_1 \cdot Y_1Z_1.$$

The last equality is the stated product of the lengths of opposite chords. Since there are equivalence in all the steps, this concludes the proof. □

3. Subtriangle excircles

An *excircle* to a triangle is a circle that is tangent to one of the sides and the extensions of the other two sides. We will need a formula for the distance between a vertex of the triangle and one of the tangency points of the excircle.

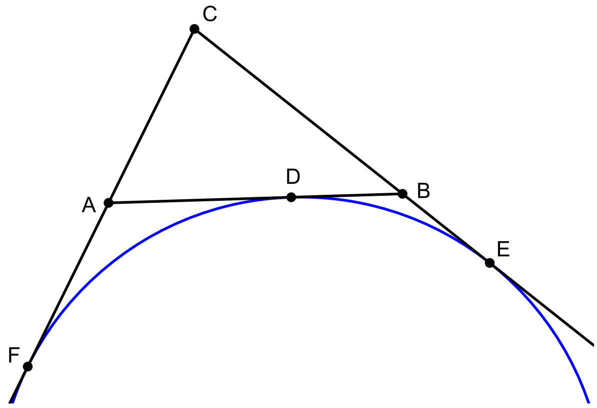

Figure 5. An excircle to a triangle

Suppose the excircle to side AB in a triangle ABC is tangent to the extensions of BC, CA at E, F respectively and to AB at D. We seek a formula for $AF = AD$. Applying the two tangent theorem, we get

$$CA + AF = BC + BE = BC + BD = BC + AB - AD = BC + AB - AF$$

so

$$AF = \tfrac{1}{2}(AB + BC - CA) = AD. \tag{4}$$

For the other part of side AB, we get

$$BD = AB - AD = \tfrac{1}{2}(AB - BC + CA) = BE.$$

These formulas are well-known, but we included them since they are crucial tools in the proofs of the next three characterizations as well as one later on.

The following is an excircle version of Theorem 1 in [5].

Theorem 3.1. *In a convex quadrilateral, the excircles to the two triangles formed by a diagonal are tangent to each other on that diagonal if and only if it is a tangential quadrilateral.*

Proof. Suppose the excircles to triangles ABD and BCD are tangent to diagonal BD at Q_1 and Q_2 respectively (see Figure 6). Then

$$BQ_1 = \tfrac{1}{2}(BD + DA - AB), \qquad BQ_2 = \tfrac{1}{2}(BD + CD - BC),$$

so

$$2(BQ_2 - BQ_1) = AB - BC + CD - DA.$$

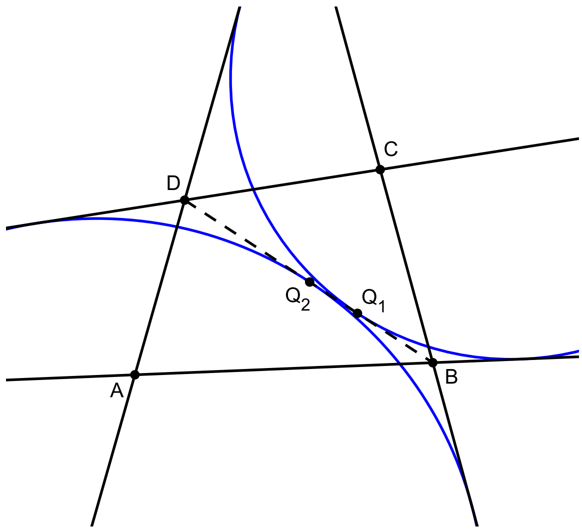

FIGURE 6. $ABCD$ is tangential $\Leftrightarrow$ $Q_1Q_2 = 0$

We have that

$$Q_1 \equiv Q_2 \quad \Leftrightarrow \quad BQ_1 = BQ_2 \quad \Leftrightarrow \quad AB + CD = BC + DA$$

which proves the theorem for triangles ABD and BCD according to Pitot's theorem. In the same way it is proved true for triangles ACD and ABC. $\square$

In the next two characterizations we have different excircle versions of Theorem 2.1 in this paper.

Theorem 3.2. *In a convex quadrilateral there are four excircles to the four overlapping triangles created by the diagonals that are tangent to the diagonals. The distance between tangent points on the extensions of two adjacent sides are equal if and only if it is a tangential quadrilateral.*

Proof. In a convex quadrilateral, we have (see Figure 7)

$$\begin{aligned} S''Z'' &= S''A + AB + BZ'' \\ &= \tfrac{1}{2}(-AB + BC + CA) + AB + \tfrac{1}{2}(-AB + BD + DA) \\ &= \tfrac{1}{2}(AC + CB + BD + DA) \end{aligned}$$

and in the same way

$$V''W'' = \tfrac{1}{2}(AC + CB + BD + DA)$$

so $S''Z'' = V''W''$. Similarly, it holds that

$$X''Y'' = \tfrac{1}{2}(BD + DC + CA + AB) = U''T''.$$

Hence

$$S''Z'' - X''Y'' = \tfrac{1}{2}(CB + DA - DC - AB)$$

so the distance between tangent points on the extensions of two adjacent sides are equal if and only if it is a tangential quadrilateral according to Pitot's theorem. $\square$

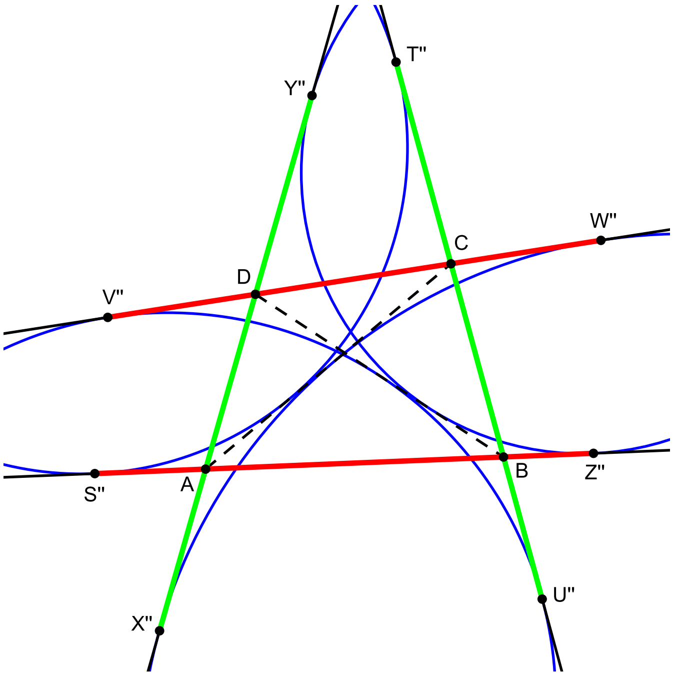

FIGURE 7. $ABCD$ is tangential $\Leftrightarrow$ $S''Z'' = X''Y''$

Theorem 3.3. *In a convex quadrilateral there are eight excircles to the four overlapping triangles formed by the diagonals that are tangent to the sides of the quadrilateral. Two adjacent distances between tangency points on each side are equal if and only if it is a tangential quadrilateral.*

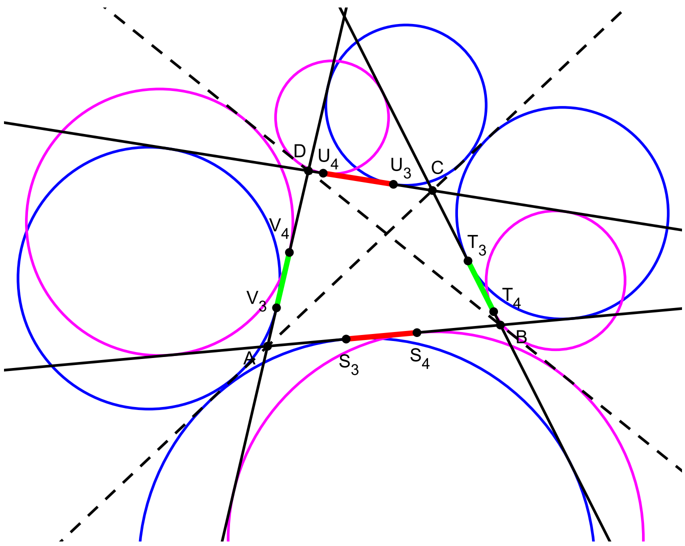

FIGURE 8. $ABCD$ is tangential $\Leftrightarrow$ $S_3S_4 = V_3V_4$

Proof. We use notations as in Figure 8. Then

$$AS_4 = \tfrac{1}{2}(AB + BD - DA), \qquad AS_3 = \tfrac{1}{2}(AB + BC - AC)$$

so

$$S_3S_4 = |AS_4 - AS_3| = \tfrac{1}{2}|BD + AC - DA - BC|.$$

In the same way

$$U_3U_4 = \tfrac{1}{2}|BD + AC - DA - BC| = S_3S_4$$

and

$$T_3T_4 = \tfrac{1}{2}|BD + AC - AB - CD| = V_3V_4.$$

Here we see that opposite distances between tangency points on each side are always equal, but for adjacent distances we have that

$$S_3S_4 = V_3V_4 \quad \Leftrightarrow \quad AB + CD = BC + DA$$

which according to Pitot's theorem is true if and only if $ABCD$ can have an incircle. □

Next we have an excircle version of Theorem 2.2.

Theorem 3.4. *In a convex quadrilateral, consider the excircle to one of the triangles created by a diagonal that is tangent to that diagonal, and the excircles to the two triangles created by the other diagonal that are tangent to this second diagonal. These three excircles are tangent to the extensions of two adjacent sides of the quadrilateral in four points, creating two lines of which one connects two tangent points on the first mentioned excircle. These lines are parallel if and only if it is a tangential quadrilateral.*

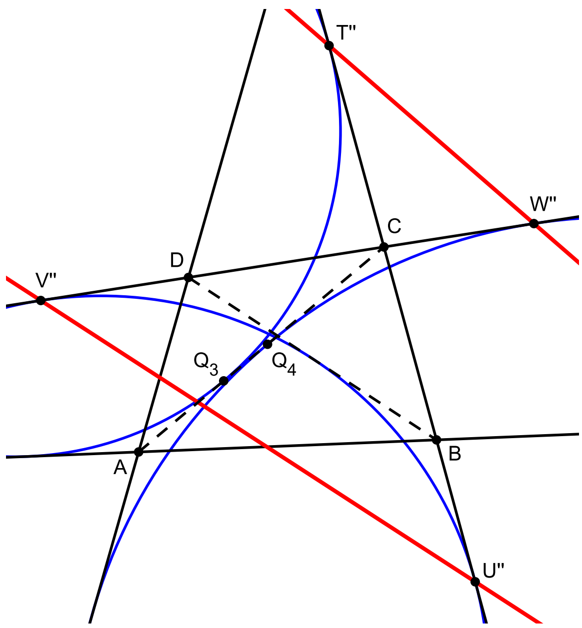

FIGURE 9. $ABCD$ is tangential $\quad\Leftrightarrow\quad U''V''||W''T''$

Proof. Using notations as in Figure 9, where T'', U'', V'', W'' are tangent points on the extensions of sides BC, CD and Q_3, Q_4 are tangent points on diagonal AC, then the lines $U''V''$ and $W''T''$ are parallel if and only if the angles $U''V''W''$ and $T''W''V''$ are equal. Since triangle $CU''V''$ is isosceles, this is equivalent to that triangle $CT''W''$ is also isosceles with $CT'' = CW''$. This in turn is (according to the two tangent theorem) equivalent to that $CQ_4 = CQ_3$, which is equivalent to that $ABCD$ is tangential according to Theorem 3.1. □

The following is an excircle version of Theorem 2.3.

Theorem 3.5. *In a convex quadrilateral $ABCD$ where the diagonals intersect at P, the excircles to triangles ABP, BCP, CDP, DAP that are tangent to the quadrilateral sides, make one chord per circle between their tangency points on the extensions of the diagonals. The product of the lengths of opposite chords are equal if and only if $ABCD$ is a tangential quadrilateral.*

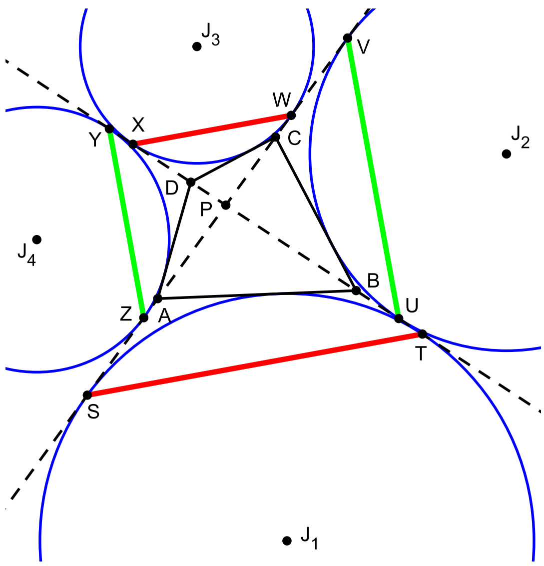

FIGURE 10. $ABCD$ is tangential $\Leftrightarrow$ $ST \cdot WX = UV \cdot YZ$

Proof. Suppose the points of tangency are S, T, U, V, W, X, Y, Z, and let J_1, J_2, J_3, J_4 be the excenters of the excircles (see Figure 10). Theorem 5 in [5, p. 73] states that $ABCD$ is tangential if and only if $J_1J_2J_3J_4$ is a cyclic quadrilateral. According to the intersecting chords theorem and its converse, $J_1J_2J_3J_4$ is cyclic if and only if $PJ_1 \cdot PJ_3 = PJ_2 \cdot PJ_4$. By applying Lemma 2.1 in the right kites PSJ_1T, PWJ_3X, PUJ_2V and PYJ_4Z, this is equivalent to

$$\frac{ST}{\sin\theta} \cdot \frac{WX}{\sin\theta} = \frac{UV}{\sin(\pi-\theta)} \cdot \frac{YZ}{\sin(\pi-\theta)}$$

which is simplified as

$$ST \cdot WX = UV \cdot YZ.$$

The last equality is the stated product of the lengths of opposite chords. Since there are equivalence in all the steps, this concludes the proof. □

4. Areas and radii

The three points where an excircle is tangent to a side of a triangle and the extensions of the adjacent two sides determine a new triangle that is called the *contact triangle*. If the original triangle has area S, circumradius R and the considered excircle has radius r_a, then this contact triangle has area

$$S_a = \frac{r_a}{2R} S. \tag{5}$$

Several different proofs of this formula can be found at [3].

Theorem 4.1. *In a convex quadrilateral, the product of the areas of opposite triangles determined by the points of tangency of the excircles to the two triangles created by a diagonal which are tangent to a side of the quadrilateral are equal if and only if it is a tangential quadrilateral.*

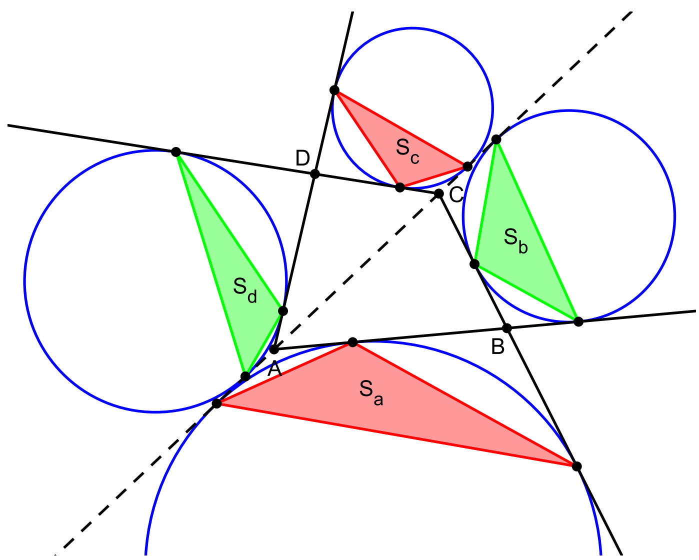

Figure 11. $ABCD$ is tangential $\Leftrightarrow$ $S_aS_c = S_bS_d$

Proof. In a convex quadrilateral $ABCD$, let triangles ABC, CDA have areas S_1, S_2 and circumradii R_1, R_2 respectively. Let the excircles tangent to sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ have radii r_a, r_b, r_c, r_d and their corresponding contact triangles have areas S_a, S_b, S_c, S_d respectively (see Figure 11). Applying (5) yields

$$S_aS_c - S_bS_d = \frac{r_aS_1}{2R_1} \cdot \frac{r_cS_2}{2R_2} - \frac{r_bS_1}{2R_1} \cdot \frac{r_dS_2}{2R_2} = \frac{S_1S_2}{4R_1R_2}\left(r_ar_c - r_br_d\right).$$

Now we use the well-known formula for the radius of an excircle to a triangle ($r_a = \frac{S}{s-a}$ where S is the triangle area and s is the semiperimeter) to get

$$r_ar_c - r_br_d = \frac{2S_1}{-a+b+p} \cdot \frac{2S_2}{-c+d+p} - \frac{2S_1}{-b+p+a} \cdot \frac{2S_2}{-d+p+c}$$

where $p = AC$ is the length of one of the diagonals. The right hand side is factorized as

$$\frac{8S_1S_2p(a-b+c-d)}{(-a+b+p)(-c+d+p)(-b+p+a)(-d+p+c)}.$$

Thus we have derived that

$$S_aS_c - S_bS_d = \frac{2S_1^2S_2^2p(a-b+c-d)}{R_1R_2(-a+b+p)(-c+d+p)(-b+p+a)(-d+p+c)}$$

where the denominator is never zero according to the triangle inequality. Hence

$$S_aS_c = S_bS_d \qquad \Leftrightarrow \qquad a+c = b+d$$

which proves this characterization according to Pitot's theorem (1). □

There is a similar version for the excircles related to the triangles created by the other diagonal. We also note that the theorem could be stated in terms of the product of the considered exradii instead (see Figure 12):

Theorem 4.2. *In a convex quadrilateral, the product of the radii in opposite excircles to those triangles created by a diagonal which are tangent to a side of the quadrilateral are equal if and only if it is a tangential quadrilateral.*

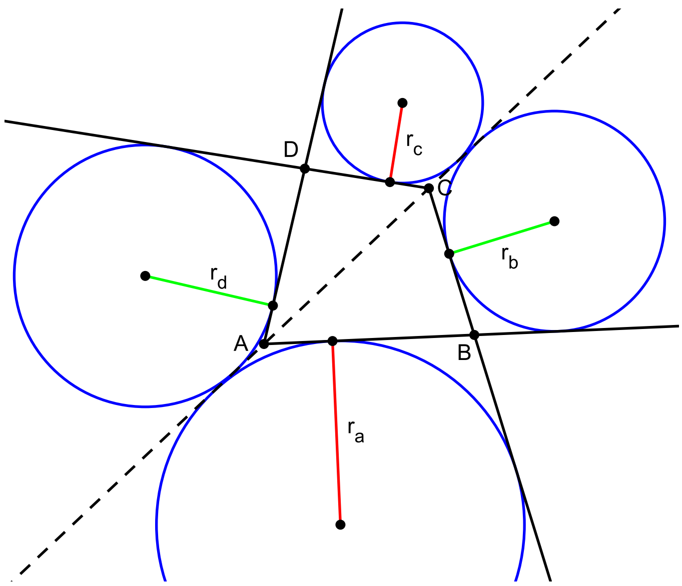

FIGURE 12. $ABCD$ is tangential $\quad\Leftrightarrow\quad r_ar_c = r_br_d$

Proof. In the previous proof, we showed that

$$r_ar_c - r_br_d = \frac{8S_1S_2p(a-b+c-d)}{(-a+b+p)(-c+d+p)(-b+p+a)(-d+p+c)}.$$

Hence

$$r_ar_c = r_br_d \qquad \Leftrightarrow \qquad a+c = b+d$$

completing the proof of this exradii characterization. □

Note that the excircles we just considered are all tangent to one side in the quadrilateral, the extension of an adjacent side and the extension of one diagonal. There is an identical looking characterization for the excircles that are tangent to one side in the quadrilateral and both of the adjacent sides, which was proved as Theorem 5 in [6].

5. Cyclic quadrilaterals

A cyclic quadrilateral is a quadrilateral whose vertices all lie on a circle. Detailed studies of their characterizations were conducted in [9, 11, 2]. The following theorem is an excircle version of Theorem 8 in [7], which in turn is a generalization of a theorem proved in [15, pp. 197–198].

Theorem 5.1. *In a convex quadrilateral $ABCD$, the excircles to the two triangles formed by a diagonal that are tangent to that diagonal are also tangent to the extensions of all four sides. These last four points are the vertices of a cyclic quadrilateral if and only if $ABCD$ is a tangential quadrilateral.*

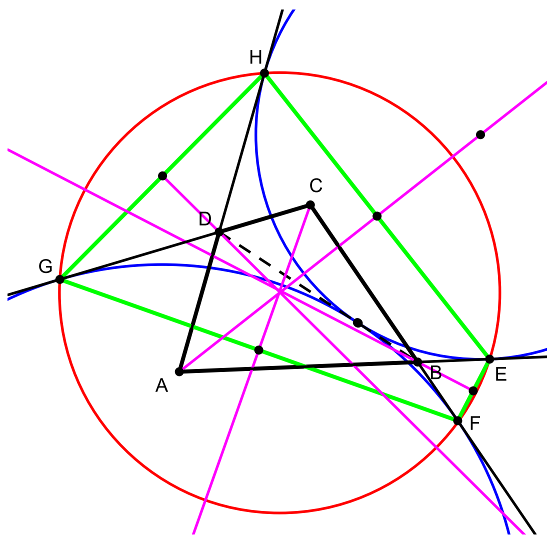

Figure 13. If $ABCD$ is tangential, then $EFGH$ is cyclic

Proof. We consider the excircles to triangles ABD and BCD. Let E, F, G, H be their tangency points on AB, BC, CD, DA respectively (see Figure 13). Then $AE = AH$ and $CF = CG$ according to the two tangent theorem.

($\Rightarrow$) If $ABCD$ is tangential, then $AB+CD = BC+DA$. Thus $DA-AB = CD - BC$ and we have that $BE = BF$ and $DH = DG$ since

$$BE = \tfrac{1}{2}(BD + DA - AB), \qquad BF = \tfrac{1}{2}(BD + CD - BC)$$

and

$$DH = \tfrac{1}{2}(BD + AB - DA), \qquad DG = \tfrac{1}{2}(BD + BC - CD).$$

This implies that the triangles HAE, FCG, EBF, GDH are all isosceles, so their angle bisectors to the angles HAE, FCG, EBF, GDH are also perpendicular bisectors to the bases EH, GF, FE, HG. Since the angle bisectors in the tangential quadrilateral $ABCD$ are concurrent, then so are the perpendicular bisectors in quadrilateral $EFGH$, confirming that it is cyclic.

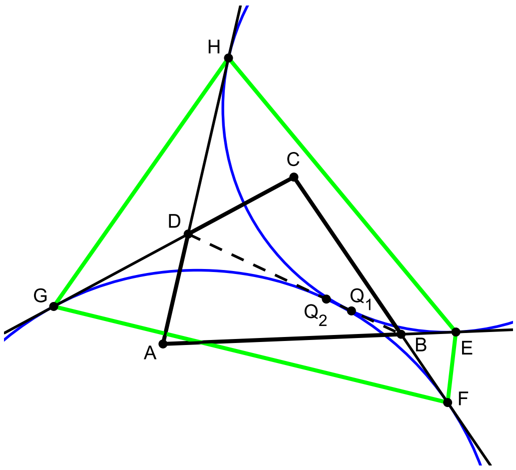

FIGURE 14. If $ABCD$ is not tangential, then $EFGH$ is not cyclic

($\Leftarrow$) We do an indirect proof of the converse. If $ABCD$ is not tangential, we shall prove that $EFGH$ cannot be cyclic. We have that $\angle CGF = \frac{1}{2}(\pi - \angle C)$ and $\angle AEH = \frac{1}{2}(\pi - \angle A)$. Assume without loss of generality that the point Q_2 where the excircle to triangle ABD is tangent to BD is closer to D than the point Q_1 where the excircle to triangle BCD is tangent to that diagonal (see Figure 14). Then

$$DG = DQ_2 < DQ_1 = DH \quad \Rightarrow \quad \angle CGH = \angle DGH > \tfrac{1}{2}(\pi - \angle D)$$

since a longer side in a triangle is opposite a larger angle. In the same way

$$BE = BQ_1 < BQ_2 = BF \quad \Rightarrow \quad \angle AEF = \angle BEF > \tfrac{1}{2}(\pi - \angle B).$$

Hence for two opposite angles in $EFGH$, we have

$$\angle FGH + \angle FEH > \frac{4\pi - (\angle A + \angle B + \angle C + \angle D)}{2} = \frac{2\pi}{2} = \pi$$

which proves that $EFGH$ is not cyclic.

The same is true for the four tangency points on the extended sides related to the two excircles tangent to AC of the triangles formed by that diagonal.

□

From the first part of the proof we get that when $ABCD$ is tangential, then its incircle is concentric with the circumcircle of $EFGH$.

Next we return to the configuration of Theorem 2.1 in this paper.

Theorem 5.2. *In a convex quadrilateral $ABCD$, let the incircles in triangles ABC, BCD, CDA, DAB be tangent to the sides AB, BC, CD, DA, AB at points S', T', U', V', W', X', Y', Z' respectively. The lines $Z'U'$, $T'W'$, $V'Y'$, $X'S'$ create a quadrilateral, which is cyclic if and only if $ABCD$ is a tangential quadrilateral.*

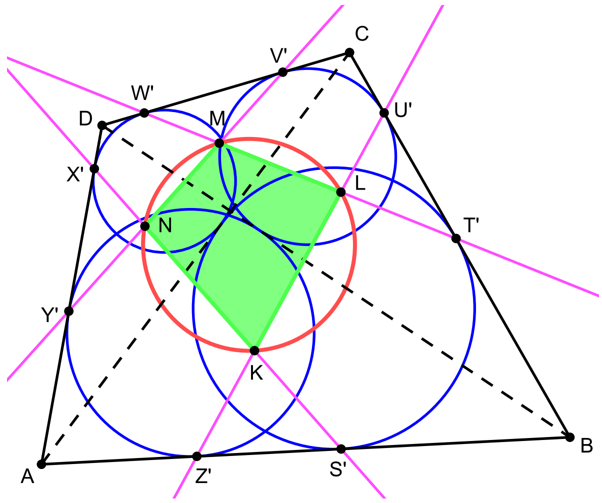

FIGURE 15. $ABCD$ is tangential $\quad\Leftrightarrow\quad$ $KLMN$ is cyclic

Proof. ($\Rightarrow$) Let $KLMN$ be the quadrilateral created by the lines $Z'U'$, $T'W'$, $V'Y'$, $X'S'$. When $ABCD$ has an incircle (see Figure 15), we have

$$\angle K = \pi - \left(\frac{\pi - \angle B}{2} + \frac{\pi - \angle A}{2}\right) = \frac{\angle A + \angle B}{2}$$

since $AS' = AX'$ and $BZ' = BU'$ according to the two tangent theorem and Theorem 2.1. In the same way

$$\angle M = \frac{\angle C + \angle D}{2}.$$

Thus

$$\angle K + \angle M = \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

confirming that $KLMN$ is cyclic.

($\Leftarrow$) We do an indirect proof of the converse. When $ABCD$ is not tangential, suppose without loss of generality that $Z'S' > X'Y'$. This implies $AS' > AX'$, so

$$\angle KS'Z' < \frac{\pi - \angle A}{2}, \qquad \angle KZ'S' < \frac{\pi - \angle B}{2},$$
$$\angle MV'W' < \frac{\pi - \angle D}{2}, \qquad \angle MW'V' < \frac{\pi - \angle C}{2}$$

since a longer side in a triangle is opposite a larger angle. Hence

$$\begin{aligned}\angle K + \angle M &= 2\pi - (\angle KS'Z' + \angle KZ'S' + \angle MV'W' + \angle MW'V') \\ &> 2\pi - \frac{\pi - \angle A + \pi - \angle B + \pi - \angle D + \pi - \angle C}{2} \\ &= \frac{\angle A + \angle B + \angle C + \angle D}{2} \\ &= \pi\end{aligned}$$

proving that the quadrilateral created by the four lines $Z'U'$, $T'W'$, $V'Y'$, $X'S'$ is not cyclic. $\square$

We conclude by studying an excircle version of the previous theorem.

Theorem 5.3. *In a convex quadrilateral $ABCD$, let the excircles to triangles ABC, BCD, CDA, DAB that are tangent to the diagonals, be tangent to the sides AB, BC, CD, DA, AB at points S'', T'', U'', V'', W'', X'', Y'', Z'' respectively. The lines $U''Z''$, $W''T''$, $Y''V''$, $S''X''$ create a quadrilateral, which is cyclic if and only if $ABCD$ is a tangential quadrilateral.*

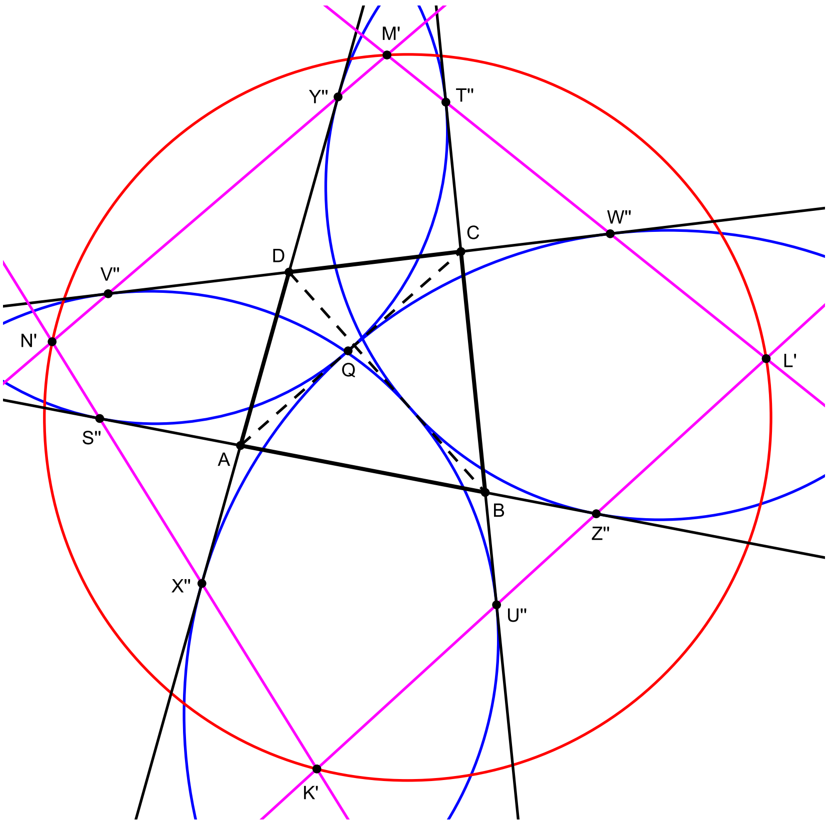

FIGURE 16. $ABCD$ is tangential $\Leftrightarrow$ $K'L'M'N'$ is cyclic

Proof. Let $K'L'M'N'$ be the quadrilateral created by the lines $U''Z''$, $W''T''$, $Y''V''$, $S''X''$. When $ABCD$ is tangential, the excircles to triangles ABC and ADC are tangent to diagonal AC at the same point, say Q (see Figure 16), according to Theorem 3.1. Then $AS'' = AQ = AX''$, so $\angle AX''S'' = \frac{1}{2}(\pi - \angle A)$. Similarly, we have $\angle BZ''U'' = \frac{1}{2}(\pi - \angle B)$, so in triangle $K'S''Z''$, we get

$$\angle K' = \pi - \left(\frac{\pi - \angle B}{2} + \frac{\pi - \angle A}{2}\right) = \frac{\angle A + \angle B}{2}.$$

In the same way

$$\angle M' = \frac{\angle C + \angle D}{2};$$

thus

$$\angle K' + \angle M' = \frac{\angle A + \angle B + \angle C + \angle D}{2} = \pi$$

confirming that $K'L'M'N'$ is cyclic.

The converse result can be proved with an indirect proof in a similar way as in the proof of the previous theorem. We leave the details as an exercise for the reader. □

6. Corrections

For those readers who want to study the history of known characterizations of tangential quadrilaterals (see the reference list), we note a few minor corrections to previous papers.

In [7, p. 1] Simionescu's theorem was misquoted. The correct formulation is according to [13, p. 133]: *Let $ABCD$ be a convex quadrilateral which has sides of lengths a, b, c, d and diagonals of lengths e and f, and let φ be the measure of the angle between the diagonals which is opposite to the side of length a. Then quadrilateral $ABCD$ is tangential if and only if*

$$ac - bd = ef \cos\varphi.$$

The first author of the present paper erroneously used an absolute value on the left hand side when quoting this theorem, but that would also include the possibility of quadrilaterals with an excircle (extangential quadrilaterals), as has been noted by Jean-Pierre Ehrmann.

Theorem 1 in [7] needs a minor reformulation to be true: *Let $ABCD$ be a convex quadrilateral where the interior bisectors of angles A and C intersect at a point I that is inside the quadrilateral. Then*

$$\angle AIB + \angle CID = \pi = \angle AID + \angle BIC$$

if and only if it is a tangential quadrilateral. The original formulation neglected to point out the necessity of I being an internal point of the quadrilateral. This slip was found by Rudolf Fritsch at Munich university.

As stated, Theorem 3 in [5] is not a valid characterization of tangential quadrilaterals, since the property of the subtriangle incircles it addresses is also true in parallelograms. However, the equation $T_1'T_3' = T_2'T_4'$, which was derived first in the same proof, is a correct characterization of tangential quadrilaterals.

Finally, regarding Theorem 5 in [5]: It was claimed that this theorem is due to N. Dergiades in 2004, but since the publication of that paper, we have found out that the theorem stating that the excenters are the vertices of a cyclic quadrilateral if and only if the original quadrilateral is tangential was in fact the first problem on the Iranian Team Selection Test in 2002 according to [1].

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