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APOLLONIUS PROBLEM:
IN PURSUIT OF A NATURAL SOLUTION

Liliana Gabriela Gheorghe

Abstract. We give a geometric solution to Apollonius problem, that consist in (geometrically) construct all the circles that tangent three given circles. This method works either if the circles are mutually disjoint, or not. The centres of all the tangent circles are obtained by (geometrically) intercepting conics, which restores the whole merit of Romanus ideas.

The key ingredient is the method of polar reciprocals, crafted by V. Poncelet in order to solve his Porisma.

Our approach is natural, robust and have purely geometric roots.

1. Introduction

Apollonius classic problem consists in finding (and drawing) all the circles that are tangent to three given circles whose interiors are disjoint. Variations of Apollonius problem allows the circles to have non void intersection, and degenerate cases deals with the same problem in which lines and points substitute some (or all) of the circles.

Anyone - even those who think never hear of it- must have solved at least a first instance of a degenerate Apollonius problem: to construct all the circles that pass through three distinct points. There are either one (the circumcircle) or no solution (when the three points are collinear). Another (degenerate) Apollonius problem ask to construct all the circles tangent to three lines. There will be four circles (the inscripted and the ex-inscripted circles of a triangle), if the lines are in general position, two circles, if two of the lines are parallel and no solution if the three lines are either parallel, or concurrent.

In these elementary cases, the circles are easy to construct, and the number of solutions is obvious. But in the general setting, solving this problem is not a trivial task. The proper Apolonius problem, in which we have to construct all the tangent circles to three given circles that have disjoint interiors, admits eight solutions, corresponding to each tangency type: (eee), (iii), (eei), (iie), (eie), (iei), (iee) and (eii).

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Here, the label (*eie*) denotes a circle that tangent externally the first and the third circle and internally, the second one.

The Apollonius problem requires to draw them: to construct them with a line and a compass only.

Nowadays, there is a lot of accessible literature on Apollonius problem; see e.g. [1],[2] and the references therein; as a matter of fact, any curios reader can easily find an abundance of approaches and references by simply making a "search."

The approach we propose here is new and combines the straightforwardness of Romanus solutions, with inversive methods and polar reciprocity.

Romanus (a latinized name of the flemish mathematician Adriaan van Roomen) used the fact that the loci of the centres of circles that tangent a pair of external circles, are branches of hyperbolas; hence, by intercepting these branches of hyperbolas he claimed to obtain the centres of the circles that tangents three circles. The problem was that he did not showed how to intercept these hyperbolas with line and compass only.

Unlike Romanus, we use ellipses, instead of hyperbolas; ellipses will do a better job, since any intersection of any pair of such ellipses will lead to a proper centre of a solution circle of the packed problem. And we also show how we may intercept these conics, with line and compass only (we perform the whole geometric construction).

The approach we give here is natural, fluid and enables a purely geometric construction; it works essentially in the same manner whether the circles are external, tangent, secant or internal, or are reduced to mere points; the same technique easily adapts to solve degenerate cases of Apollonius problem, in which the three elements are chosen arbitrarily between circles, lines or points, and their relative position is arbitrary.

The main ingredients of our proof are the method of polar reciprocals of Poncelet and common sense.

2. The virtues of the packed problem

The (classic) Apollonius problem is the following: given three mutually disjoint circles, to draw all the circles that tangent them all.

One inconvenient of drawing all these circles, is that some of them may be very small, other very big, and will look like lines; except for the rare case, in which the given circle have comparable radius and their centres determine a triangle that is "almost" equilateral, that will be the case.

An inversion w.r. to any of the given circles, transforms this problem into a simpler one and its drawing, into a nicer one.

2.1. **The packed Apolonius problem.** The packed Apollonius problem is the following: given three circles, two of which are mutually disjoints and contained into a third circle, to draw all the circles that tangent (simultaneously) all.

There is a lot of gain by performing this inversion, w.r to one of the circles.

- The first is simply logistic (or aesthetic): all the tangent circles are easy to visualise, they are all contained into the inversion circle.

- The second is that such an inversion eliminates the external tangency type, w.r. to one of the three given circles: all the solutions will automatically tangent the inversion circle internally! And this facilitates the drawing itself.
- The third gain is that this new configuration led to intersection of pairs of ellipses, instead of branches of hyperbolas, as Romanus solution; and ellipses are more accountable, since their intersection point never led to false positive results, as the branches of hyperbolas do.

When we perform back the inversion w.r. to the external circle, we obtain the full solution to the (original) Apollonius problem.

2.2. **The locus of the centres of tangents circles.** Let $\mathbf{O_0}, \mathbf{O_1}, \mathbf{O_2}$ be three circles centred in O_0, O_1, O_2, such that the circles $\mathbf{O_1}, \mathbf{O_2}$ are mutually disjoint and contained into $\mathbf{O_0}$.

By convention, we say that two tangent circles tangent internally, if their interior have non void intersection, and externally, if their interiors are disjoint.

With this convention, all the tangent circles of the packed problem will tangent internally the bigger (external) circle, and may tangent either externally, or internally the internal one.

Let us look closer to the location of the centres of all the circles that tangent a pair of circles $\mathbf{O_0}$ and $\mathbf{O_1}$, where the circle $\mathbf{O_1}$ is contained into $\mathbf{O_0}$.

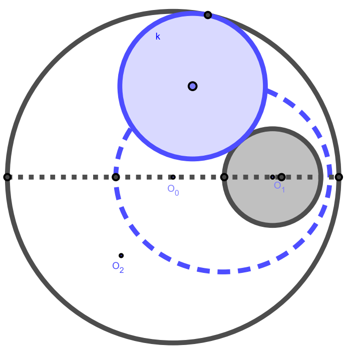

FIGURE 1. the locus of the centres of all tangent circles (ie)

Lemma 2.1. *i) The locus of the centres of all circles that tangent internally the circle* $\mathbf{O_0}$ *and externally the circle* $\mathbf{O_1}$ *is an ellipse, whose foci are in* O_0 *and* O_1*, and whose main axis is* $2a = R_0 + R_1$.

ii) The locus of the centres of all circles that tangent internally both the circle $\mathbf{O_0}$ *and the circle* $\mathbf{O_1}$ *is an ellipse, whose foci are in* O_0 *and* O_1*, and whose main axis is* $2a = R_0 - R_1$.

The proofs are straightforward and we omit it.

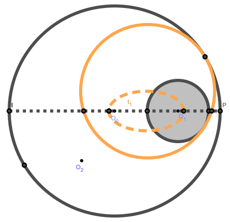

FIGURE 2. the locus of the centres of all tangent circles (ii)

2.3. **Drawing the solution.** At this point, any geometric software, such as Geogebra, that we employed here, is able to "intercept" any two such ellipses and thus "to solve" the packed Apollonius problem; thus, we are able to draw all the circles that tangent internally the circle $\mathbf{O_0}$, and that tangents (either internally or externally) the other two circles, $\mathbf{O_1}$ and $\mathbf{O_2}$.

The centres of these tangent circles are precisely the intersection points of four ellipses:

- two ellipses with foci in O_0, O_1 and main axis $R_0 + R_1$ and $R_0 - R_1$;
- two ellipses with foci in O_0, O_2 and main axis $R_0 + R_2$ and $R_0 - R_2$.

There will be eight interception points, say $o_1, o_2, \ldots, o_8$; these are the centres of the tangent circles of the packed problem. All these circles will tangent internally the circle $\mathbf{O_0}$; therefore, their tangency points with $\mathbf{O_0}$, say $T_1, T_2, \ldots, T_8$ are the interception of the half-line $O_0 o_k$, with the circle $\mathbf{O_0}$ itself. The solution of the packed problem will therefore be the circles $\mathbf{o_1}$, $\mathbf{o_2}$, $\ldots \mathbf{o_8}$, centred at o_k, and of radius $r_k = o_k T_k$, $k = 1, 2, \ldots 8$.

The solution of the Apollonius (original) problem are the inverse w.r. to the (external) circle $\mathbf{O_0}$ of all these circles.

The next pictures shows either how to obtain the packed solution, as well as the original one.

Here, we adopt the folllowing convention. The inversion circle $\mathbf{O_0}$ is common black; the circles $\mathbf{O_1}, \mathbf{O_2}$, as well as their inverse w.r. to $\mathbf{O_0}$, are solid coloured in grey scale; the solid coloured circles are the solution of the packed problem, while the coloured circles, their inverses w.r. to $\mathbf{O_0}$, are the solution of the original problem.

For convenience, we shall proceed in pairs (see Fig 3 to Fig 8).

The reader must be aware that this does not provide yet a geometric solution, since we did not show how to intercept these ellipses with a line and compass only.

3. The main tool: polar reciprocity

The solution of the Apolonius packed problem, described above, only became legitimate if we are able to provide a geometric construction for the intersection point of those remarkable ellipses, with a line and a compass only!

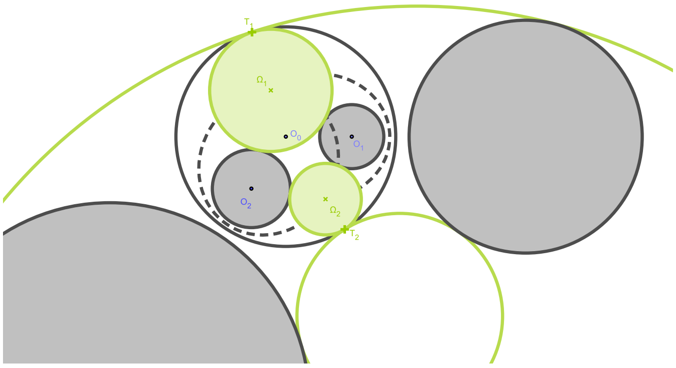

FIGURE 3. the first packed and unpacked pair of tangent circles

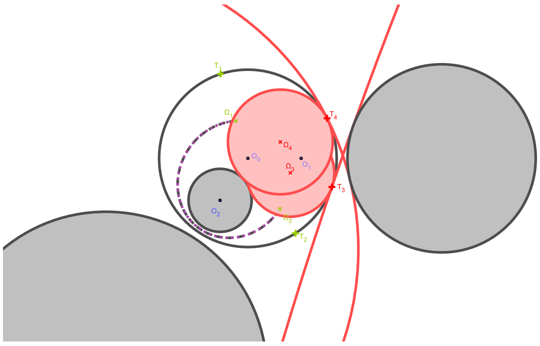

FIGURE 4. the second packed and unpacked tangent circles

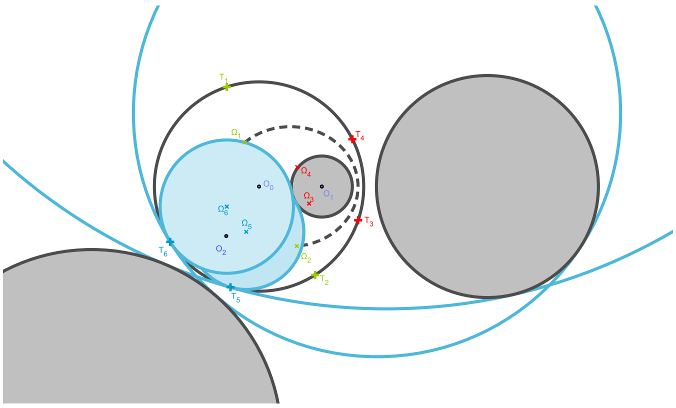

FIGURE 5. the third pair of packed and unpacked circles

We show how this is possible, by using polar reciprocity, a tool crafted by Poncelet, in order to solve his Porisma (see [3] or [4]), but that was never use in order to intercept conics.

From now on, let $\mathbf{\Gamma}$ be a circle centred in Ω and of ray R, which we shall call the inversion circle.

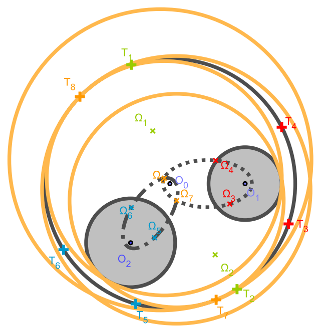

Figure 6. the fourth pair of packed and unpacked circles

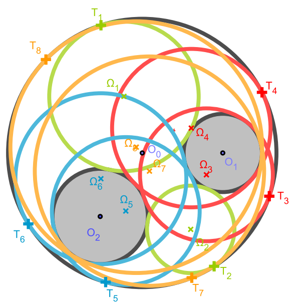

Figure 7. The packed solution

Definition 3.1. *The reciprocal of a given curve, w.r. to an inversion circle, is the curve whose points are the poles of its tangents.*

Due to the fundamental theorem on poles and polars, there is also a dual (equivalent) definition.

Definition 3.2. *The reciprocal of a given curve, w.r. to an inversion circle, is the curve whose tangents are the polars of its points.*

Polar reciprocity extends the definition of polarity: the "reciprocal" of a line, w.r. to a circle, is a point: its pole! Also, the reciprocal of a reciprocal curve is the original.

One of the most remarkable facts, that we shall heavily use, is the following.

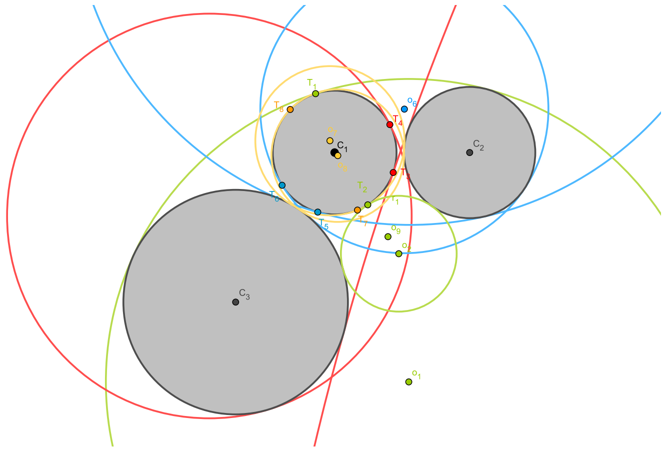

FIGURE 8. The solution of the Apolonius problem: eight tangent circles

Theorem 3.1. *(see [4]) The reciprocal of a circle $\gamma = C(O, r)$, w.r. to an inversion circle $\mathbf{\Gamma_0} = \mathbf{C}(\mathbf{\Omega}, \mathbf{R})$ is a conic, $\mathbf{\Gamma}$. To be specific:*

- *$\mathbf{\Gamma}$ is an ellipses, if $r < \Omega O$;*
- *$\mathbf{\Gamma}$ is a parabola, if $r = \Omega O$;*
- *$\mathbf{\Gamma}$ is a hyperbola, if $r > \Omega O$.*

Moreover,

- *one of the foci of $\mathbf{\Gamma}$ is precisely Ω, the centre of the inversion circle;*
- *the polar of O, is the directrix of $\mathbf{\Gamma}$;*
- *the eccentricity of the reciprocal conic $\mathbf{\Gamma}$ is $e = \frac{r}{O\Omega}$.*

This result is not new, but since the references are somehow scarce and since the result lack the popularity if fully deserves, we give here its full (elementary) proof.

Our proof is a variation of the one in art. 309, [4], and is based on the following elementary fact.

Lemma 3.1. *Let O, P two points and let o, p, be their polars w.r. to an inversion circle centred in Ω. Then $\frac{\Omega O}{\Omega P} = \frac{d(O,p)}{d(P,o)}$.*

Proof: With the notations in Fig. 9, we have to prove that $\frac{\Omega O}{\Omega P} = \frac{OT}{PP_1}$. Let $PB' \perp O\Omega$ and B be the intersection between $O\Omega$ and p, the polar of P. First note that $\triangle \Omega PB' \sim \triangle \Omega BP'$; hence $\frac{\Omega P}{\Omega B} = \frac{\Omega B'}{\Omega P'}$; this implies that $\Omega P \cdot \Omega P' = \Omega B \cdot \Omega B'$, so B and B' are symmetric w.r. to the inversion circle.

Since, by construction, OT and ΩP are perpendicular to p, they are parallel hence, by Thales theorem, $\triangle \Omega P'B \sim \triangle OTB$; this guarantees that $\frac{OT}{\Omega P'} = \frac{OB}{\Omega B}$.

Since $(PP_1O'B')$ is a rectangle,

$$PP_1 = B'O' = B'\Omega + \Omega O' = \frac{R^2}{\Omega B} + \frac{R^2}{\Omega O} = R^2 \frac{B\Omega + \Omega O}{\Omega O \cdot \Omega B} = R^2 \frac{OB}{\Omega O \cdot \Omega B}.$$

Hence $PP_1 \cdot \Omega O = R^2 \cdot \frac{OB}{\Omega B}$. On the other hand, $OT \cdot \Omega P = OT \cdot \frac{R^2}{\Omega P'} = R^2 \cdot \frac{OB}{\Omega B}$, hence $OT \cdot \Omega P = PP_1 \cdot \Omega O$, and this ends the proof of Lemma.

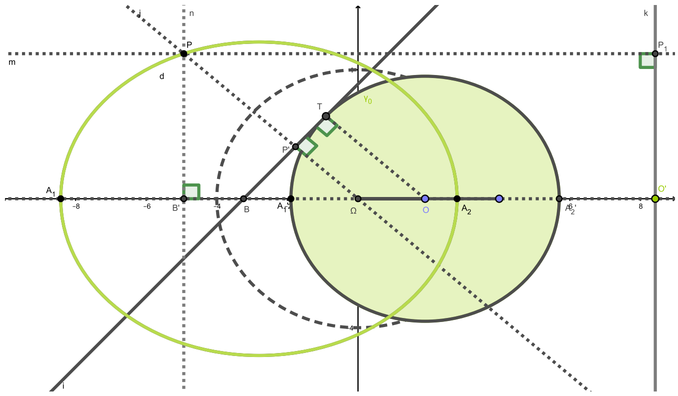

FIGURE 9. the reciprocal of a circle: the key lemma

Finally, in order to proof the theorem, consider a circle γ centred in O and radius r, (the reciprocated circle, the solid green in Fig. 9) let p be a current tangent to that circle and P its pole w.r. to the inversion circle. Then $OT = r$, and by the lemma above,

$$\frac{\Omega P}{PP_1} = \frac{\Omega O}{r} = constant.$$

This mean that the point P is contained into a conic that have a focus in Ω and whose directrix is the line o, the polar of O.

The reverse inclusion is similar and we omit it.

This theorem has a very useful consequence, that we shall use systematically.

Corrolary 3.1. *The reciprocal of a conic* $\mathbf{\Gamma}$, *w.r. to an inversion circle centred into its focus,* F, *is a circle,* γ.

The symmetric of the vertices of the conic $\mathbf{\Gamma}$, *are a pair of diametrically opposite points of the reciprocal circle,* γ.

The pole of the directrix of $\mathbf{\Gamma}$, *is the centre of the circle* γ.

At this point, the reader may convince himself that is able to draw, with a line and compass only, the reciprocal of a conic, w.r. to a circle centred into one of its foci: it will be a circle, whose diameter are the symmetric of the vertices of that conic (or a circle centred at the pole of the directrix and passing through the symmetric of one vertex).

The second basic result, that is also a consequence of the fundamental propriety poles and polars is the following (see [S])

Proposition 3.1. *The intersection of two conics that shares a common focus, are the poles, w.r. to an arbitrary inversion circle centred at their (common) focus, of the common tangents their reciprocal circles have.*

The figure 10 illustrates how to (geometrically) intercept, two ellipses that have a common focus; but the procedure is the same for any pair of conics:

- chose an inversion circle centred into the common focus of the conics and perform the reciprocal of each conic;
- draw the common tangents to these reciprocal circles;

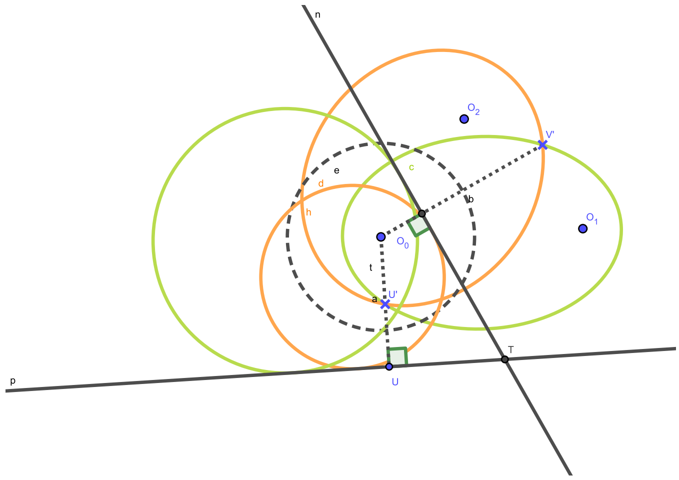

FIGURE 10. the interception of two ellipses are the poles of their common tangents to their reciprocal circles

- the interception points of the given conics, are the poles of the common tangents to their reciprocal circles.

4. APOLONIUS PROBLEM FOR SECANT CIRCLES

Now we study the Apolonius problem when some of the three circles may intercept. As in the classic setting, we first need to find the loci of the centres of all circles that tangents a pair of secant circles.

Lemma 4.1. *Let* $(\mathbf{O_0}, \mathbf{O_1})$ *be a pair of secant circles.*

i) The locus of the centres of the circles that have distinct tangency type with respect to a given pair of secant circles, is an ellipses with focus in the centres of the circles and that pass through their intersection point.

ii) The locus of the centres of the circles that have the same tangency type, w.r. to a pair of secant circles is a hyperbola, with focus in the centres of the circles and that pass through their intersection point.

To be specific, the branch closer to the smaller circle, is the locus of the centres of circles that either tangents both circles externally, or are contained into their common set and tangent both internally.

The branch closer to the bigger circle, is the locus of the centres of circles that tangents both circles internally.

When the two given circles have same radius, this hyperbola reduces to the mediatrix of the segment O_1O_2.

There exists two exceptional points on these branches of these hyperbolas or on ellipses, that led to circles of null radius: the intersection point of the two given circles.

A final detail. As we already point out earlier, the loci of the centres of the tangent circles to a pair of secant circles that have the same radius is the mediatrix of the segment determined by their centres. A scrupulous reader, may object, at this point, that it is not clear how to draw, with a line and a compass only, the intersection between a conic and a line. Proceed as follows:

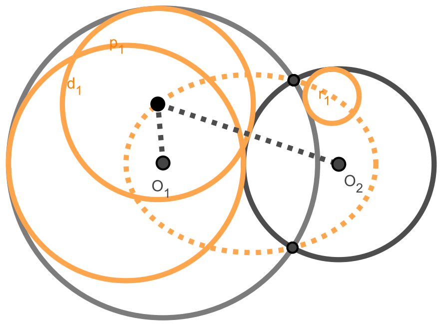

FIGURE 11. the locus of the centres of the circles that have opposite tangency type, with respect to a pair of secant circles is an ellipse

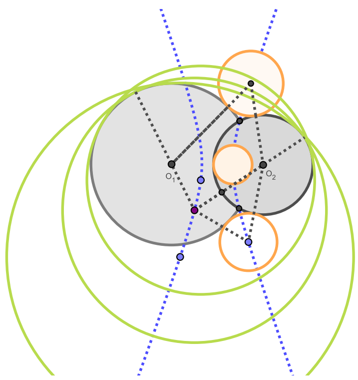

FIGURE 12. the locus of centres of circles that have the same tangency type, with respect to a pair of secant circles, is a hyperbola

- choose an inversion circle centred at the focus of the conic;
- reciprocate the conic into a circle;
- find the pole of the line;
- draw the tangents from the pole to the circle;
- draw the poles of these tangents.

The poles of these common tangents are the interception points between the conic and the line.

4.1. **Apollonius problem for one pair of secant circles.** Assume that the circles $(\mathbf{O_1}, \mathbf{O_2})$ are secant, while $(\mathbf{O_0}, \mathbf{O_1})$, $(\mathbf{O_0}, \mathbf{O_2})$, are disjoint. By an inversion w.r.t. to $\mathbf{O_0}$, we may always assume that both $\mathbf{O_1}$ and $\mathbf{O_2}$ are contained into $\mathbf{O_0}$. We therefore repeat the procedure employed earlier, in the disjoint case: we find the loci of the centres of the tangent circles to the disjoint pairs $(\mathbf{O_0}, \mathbf{O_1})$ and $(\mathbf{O_0}, \mathbf{O_2})$, (that are ellipses) and intercept them. The solution of the packed problem is illustrated in the figure 13 below.

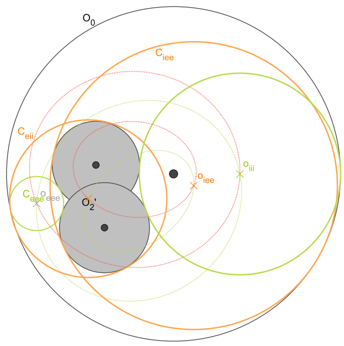

FIGURE 13. Apollonius problem for a pair of secant circles; four circles; the packed solution

4.2. **Apollonius problem for two pairs of secant circles.** If the pairs $(\mathbf{O_0}, \mathbf{O_1})$ and $(\mathbf{O_1}, \mathbf{O_2})$ are secant and $(\mathbf{O_0}, \mathbf{O_2})$ are disjoint, perform an inversion w.r.to $\mathbf{O_0}$.

This reduce the problem to one in which one circle is contained into another, (say $\mathbf{O_2}$ contained into $\mathbf{O_0}$), while the circle $\mathbf{O_1}$ intercept both $\mathbf{O_0}$ and $\mathbf{O_2}$, and it is not contained into O_0.

Any admissible solution will tangent $\mathbf{O_0}$ internally, and the other two circles, externally; this observation facilitates the drawing; in any case, the centres of the tangent circles are interception points of the ellipses, that correspond to the pair $(\mathbf{O_0}, \mathbf{O_2})$, with ellipse and hyperbola, that correspond to the pair $(\mathbf{O_0}, \mathbf{O_1})$. The solution of the packed problem, is in figure 14.

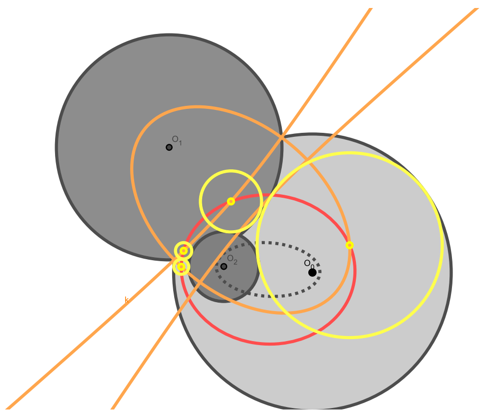

FIGURE 14. Apollonius problem for two pairs of secant circles: there are four circles; the packed solution

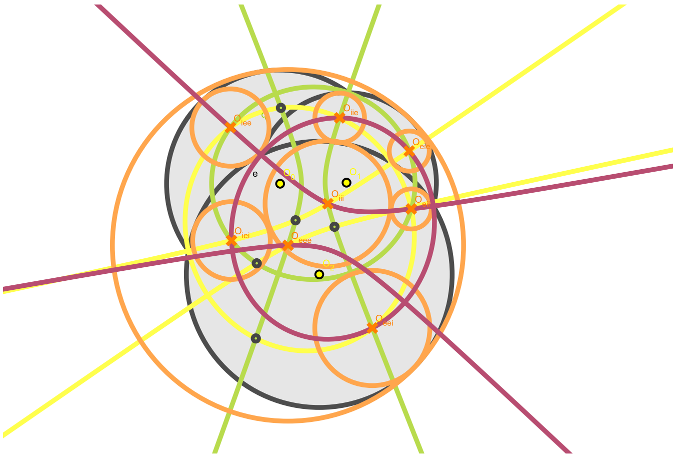

Figure 15. Apollonius problem for three mutually tangent circles: the centres are the interception of three conics

4.3. **Three pairs of secant circles.** There exists only one instance of Apollonius that does not benefit itself by an inversion with respect to a circle: it is when the given circles are mutually secant and still do no have one common point.

Any inversion with respect to any of the circles, will led to a same configuration.

In this case, proceed straightforwardly: intercept all possible conics that carry the centres of the tangent circles to each pair of circles. The centres of the tangent circles will be precisely the intersection points of three such conics. Be aware that the intersection points of two conics only, led to "false positive" points. See figure 15.

Of course, there are also some other cases left, that can be handle easily by inversion.

- If the three circles have a common point and are pairwise secant, then an inversion w.r. to a circle centred into their common point, transform the problem into a familiar problem: to draw the circles that tangents three given lines. There will be four solutions: the i-circle and the three ex-icircles.
- If the circles have one commune tangency point, there are an infinity of solutions.
- If two of the circles are tangent, and the third is secant, then an inversion with respect to a circle centred into the tangency point, transform the problem into that of constructing circles that tangents two parallel lines and a secant circle. There will be four solutions.
- If all the three circles have two common points, there will be no common tangent circle; to see that, perform an inversion w.r. to a circle centred at one of the common points, transform the three circles into three concurrent lines.

5. Final Remarks

We provide a geometric solution to the Apollonius problem for three circles. With few exceptions, this method works no matter what the relative position of the three given circles of the Apollonius problem is (either external, tangent, secant or internal). By sticking to it, one can easily solve degenerate instances of Apollonius problem, those that involve lines and points, as well, instead of circles, only. What will change in those cases will be the number of solutions; the conics that are the loci of the corresponding tangent circles, will always have a common focus, so that our strategy, that uses the method of polar reciprocity in order to intercept them, is applicable.

The whole algorithm that allows to perform the solution, consists in drawing some circles, their common tangents and some poles. No conic need to be drawn in the process: only their vertices and foci really matter. The computer-aided solution, that enable us to visualise all these centres of the circles solutions of Apollonius problem, as intersection points of these conics that carries the centres of tangent circles, turn this geometric approach, into a natural solution and accessible to any reader.

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DEPARTMENT OF MATHEMATICS
FEDERAL UNIVERSITY OF PERNAMBUCO,
BRAZIL
E-mail address: liliana@dmat.ufpe.br