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ON TWO TRANSFORMATIONS IN ELLIPTIC TRIANGLE GEOMETRY

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Abstract. In this note, we introduce elliptic analogues for the isotomic and isogonal transformations, defined, initially, for Euclidean triangles and we investigate some of their properties and we apply the transformation to some of the remarkable points and lines associated to an elliptic triangle.

1. Introduction

In this article we will adapt the classical isotomic and isogonal transformation from the Euclidean geometry to the case of a non-degenerate triangle from the elliptic plane.

We shall use, throughout the paper, the projective model of the elliptic plane $\mathcal{E}^2$. In this model, the elliptic plane is the entire real projective plane, endowed with a polarity. Due to this polarity, as is the case with the hyperbolic plane, the elliptic plane can be thought of as being the interior of a non-degenerate projective conic, only that, unlike the hyperbolic plane, this time the conic is imaginary, therefore its interior is the entire projective plane.

There are several models of the real projective plane. In our case, it is probably best to see the projective plane as being the 2-dimensional sphere, with the antipodal points identified. Actually, sometimes the spherical geometry itself is called *simple elliptic geometry*, while the projective plane is called *double elliptic geometry.* We want to emphasize that the two geometry are distinct. In fact, the sphere, as a manifold, is orientable, while the projective plane is not.

We shall follow, closely, the classical book of Coxeter [3], in what concerns the notations and the language. The reader can consult, for other approaches to the so-called, *Cayley-Klein geometries*, the books [4] and [8].

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Let us be more specific. A polarity of the projective plane is given by

$$\begin{cases} x_\mu = c_{\mu\nu}\xi_\nu, \\ \xi\mu = C_{\mu\nu}x_\nu, \end{cases} \tag{1}$$

where the indices take all the values from 0 to 2. We recall (see [3]) that a polarity associates points to lines and the other way around. In the formula (1) x_μ are the coordinates of a point (hereafter *point coordinates*), while ξ_μ are the coordinates of a line (hereafter *line coordinates*).

As we are working with the elliptic geometry, both matrices are symmetric and non-degenerate and are inverse to each other. They perform several roles in geometry:

- If they are prescribed, they define the homogeneous coordinate system (both for points and lines).
- They allow the computation of angles and distances;
- They define the limit of the space, which is a conic section, called the *Absolute*.

The equation of the Absolute is

$$c_{\mu\nu}x_\mu x_\nu = 0, \tag{2}$$

in point coordinates or

$$C_{\mu\nu}\xi_\mu\xi_\nu = 0, \tag{3}$$

in line coordinates. In the hyperbolic case, the Absolute is a real non-degenerate conic, meaning that the hyperbolic plane, in the projective model, is a limited part of the projective plane, the interior of the conic. In the elliptic case, the conic, still non-degenerate, is imaginary, which means that the elliptic plane occupies the entire projective plane.

We shall use in this paper two sets of coordinates: trilinear and barycentric (or areal). They were, both, introduced in elliptic geometry by Sommerville, in 1932 (see [10]), by using the standard definition of homogeneous coordinates in the projective plane (by using a reference triangle and a unit point), but they were recast by Coxeter (see [3]) in the language of polarities.

We introduce, now, some notations (see [3] or [1] for details). If (x) and (y) are two points, while $[\xi]$ and $[\eta]$ are two lines (from the elliptic plane), then

(1) $(x, y) = c_{\mu\nu}x_\mu y_\nu$;
(2) $[\xi, \eta] = C_{\mu\nu}\xi_\mu\eta_\nu$;
(3) $\{x, \xi\} = x_\mu\xi_\nu$;
(4) $\{\xi, y\} = \xi_\mu y_\nu$,

where, as usually, we sum after all the possible values of the indices. We mention that, if the lines $[\xi]$ and $[\eta]$ are the polars of the points (x) and (y), with respect to the Absolute, then all the all four brackets defined above are equal[1].

The brackets we introduce are useful for the description of various entities related to elliptic geometry. We mention some of them.

[1]The definition of the polarity (1) is nothing but then the relation between a point and its polar with respect to the Absolute.)

(1) The equation of the Absolute is $(x, x) = 0$ (in point coordinates) or $[\xi, \xi] = 0$ (in line coordinates);
(2) the lines $[\xi]$ and $[\eta]$ are perpendicular iff $[\xi, \eta] = 0$;
(3) if α is the angle between two lines, $[\xi]$ and $[\eta]$, then

$$\cos^2 \alpha = \frac{[\xi, \eta]^2}{[\xi, \xi] \cdot [\eta, \eta]}.$$

(4) If (x) and (y) are two points and d is the distance between them, then

$$\cos d = \frac{|(x, y)|}{\sqrt{(x, x) \cdot (y, y)}}.$$

(5) We can, also, compute the distance d between a point (x) and a line $[\xi]$, by using the formula

$$\sin d = \frac{|\{x, \xi\}|}{\sqrt{(x, x)} \cdot \sqrt{[\xi, \xi]}}.$$

We mention that the elliptic trigonometry is nothing but the spherical trigonometry (see [3]).

The transformations described here were first introduced, in the Euclidean case by Mathieu, in 1865 (the isogonal transformation, see [7]) and Longchamps, in 1866 (the isotomic transformation, see [5] and [6]).

In the rest of the paper, we shall introduce, first, the two sets of coordinates we are going to use, then we introduce and study the two transformations. Finally, we apply the transformation to some remarkable points in the elliptic triangle. We shall not discuss the circumcenters in this work, they will be the subject of another investigation.

2. The polarity associated to trilinear coordinates

In the case of the elliptic geometry, we know that C_{00}, C_{11}, C_{22} are, all of them, strictly positive. As such, we can make the assumption that $C_{00} = C_{11} = C_{22} = 1$. The coordinates corresponding to this gauge are called *trilinear coordinates*. For the elliptic geometry, they were introduced by Sommerville, in 1932, and reformulated, in the language of polarities, by Coxeter, in his classical introduction to non-euclidean geometry. They are the elliptic analogues of the Euclidean trilinear coordinates.

To find the rest of the components of the matrix of the polarity, determined by the triangle ABC, written in line coordinates, we use the coordinates of the of the sides of the fundamental triangle:

$$BC \to [\xi] = [1, 0, 0], \; CA \to [\eta] = [0, 1, 0], \; AB \to [\zeta] = [0, 0, 1].$$

According to the definition. $[\eta, \zeta] = C_{12}$. But, on the other hand, we know that

$$\cos A = -\frac{C_{12}}{C_1 C_2} = -C_{12},$$

hence $C_{12} = C_{21} = -\cos A$. In exactly the same manner, we get $C20 = C_{02} = -\cos B$ and $C_{10} = C_{01} = -\cos C$.

As such, if ξ and η are vectors associated to two arbitrary lines in the elliptic plane, we have

$$[\xi,\eta]=\xi_0\eta_0+\xi_1\eta_1+\xi_2\eta_2-\cos A\,(\xi_1\eta_2+\xi_2\eta_1)-\cos B(\xi_0\eta_2+\xi_2\eta_0)- \\ -\cos C(\xi_0\eta_1+\xi_1\eta_0), \tag{4}$$

which means that the matrix of the polarity in line coordinates is

$$[C_{\mu\nu}]=\begin{pmatrix}1 & -\cos C & -\cos B\\ -\cos C & 1 & -\cos A\\ -\cos B & -\cos A & 1\end{pmatrix}. \tag{5}$$

The determinant of this matrix is

$$\Gamma=1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C. \tag{6}$$

In particular, the equation of the Absolute, in line coordinates, is

$$[\xi,\xi]=\xi_0^2+\xi_1^2+\xi_2^2-2\xi_1\xi_2\cos A-2\xi_0\xi_2\cos B-2\xi_0\xi_1\cos C=0. \tag{7}$$

The matrix of the polarity in *point* coordinates, is, simply, the inverse of the matrix in line coordinates. We get, after some computations,

$$[c_{\mu\nu}]=\frac{1}{\Gamma}\begin{pmatrix}\sin^2 A & \cos A\cos B+\cos C & \cos A\cos C+\cos B\\ \cos A\cos B+\cos C & \sin^2 B & \cos B\cos C+\cos A\\ \cos A\cos C+\cos B & \cos B\cos C+\cos A & \sin^2 C\end{pmatrix}. \tag{8}$$

The determinant of the matrix $[c_{\mu\nu}]$ is, obviously, $\gamma=\dfrac{1}{\Gamma}$.

The formula (8) can be simplified if we notice, using spherical trigonometry, that

$$\begin{cases}\cos A\cos B+\cos C=\sin A\sin B\cos c,\\ \cos B\cos C+\cos A=\sin B\sin C\cos a,\\ \cos C\cos A+\cos B=\sin C\sin A\cos b.\end{cases}$$

Therefore, the matrix of the polarity in point coordinates finally becomes

$$[c_{\mu\nu}]=\frac{1}{\Gamma}\begin{pmatrix}\sin^2 A & \sin A\sin B\cos c & \sin A\sin C\cos b\\ \sin A\sin B\cos c & \sin^2 B & \sin B\sin C\cos a\\ \sin A\sin C\cos b & \sin B\sin C\cos a & \sin^2 C\end{pmatrix}. \tag{9}$$

This means, in particular, that the equation of the Absolute, in trilinear point coordinates is

$$(x,x)=\frac{1}{\Gamma}\left(\sin^2 A\cdot x_0^2+\sin^2 B\cdot x_1^2+\sin^2 C\cdot x_2^2+2\sin A\sin B\cos c\cdot x_0x_1+ \right. \\ \left. +2\sin A\sin C\cos b\cdot x_0x_1+2\sin B\sin C\cos a\cdot x_1x_2\right)=0. \tag{10}$$

To justify the name of trilinear coordinates, let us denote by d_0, d_1, d_2 the distances from an arbitrary point $M(x_0,x_1,x_2)$ in the elliptic plane, given through its trilinear coordinates, to the sides BC, CA and AB, respectively, of the fundamental triangle.

We recall that the distance between d the point of coordinates (x) and the line of coordinates $[\eta]$ is given by the formula

$$\sin d=\frac{\{x,\eta\}}{\sqrt{(x,x)}\cdot\sqrt{[\eta,\eta]}}. \tag{11}$$

Let's apply this formula for an arbitrary point (x) and the line BC, given by the coordinates $[\eta_1] = (1, 0, 0)$. Then

$$\{x, \eta_1\} = x_0$$

3. The polarity associated to barycentric (areal) coordinates

We start, this time, with the matrix of the polarity in point coordinates. Exactly as we did for the hyperbolic plane, we get, immediately,

$$[c_{\mu\nu}] = \begin{pmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{pmatrix}. \tag{12}$$

As such, the equation of the Absolute in barycentric point coordinate is

$$(X, X) = X_0^2 + X_1^2 + X_2^2 + 2X_0X_1 \cos c + 2X_0X_2 \cos b + 2X_1X_2 \cos C = 0.$$

The determinant of the polarity matrix in barycentric point coordinates is

$$\gamma = 1 + 2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c.$$

The matrix of the polarity written in line coordinates is the inverse of the matric corresponding to point coordinates and we get

(13)

$$[C_{\mu\nu}] = \frac{1}{\gamma} \begin{pmatrix} \sin^2 A & -\sin A \sin B \cos C & -\sin A \sin C \cos B \\ -\sin A \sin B \cos C & \sin^2 B & -\sin B \sin C \cos A \\ -\sin A \sin C \cos B & -\sin B \sin C \cos A & \sin^2 C \end{pmatrix}.$$

As such, the equation of the Absolute in line coordinates is

$$\begin{aligned} [\Sigma, \Sigma] = {} & \sin^2 A \cdot \Sigma_0^2 + \sin^2 B \cdot \Sigma_1^2 + \sin^2 C \cdot \Sigma_2^2 - \\ & - 2\sin A \cdot \sin B \cdot \cos C \cdot \Sigma_0 \cdot \Sigma_1 - \\ & - 2\sin A \cdot \sin C \cdot \cos B \cdot \Sigma_0 \cdot \Sigma_2 - \\ & - 2\sin B \cdot \sin C \cdot \cos A \cdot \Sigma_1 \cdot \Sigma_2. \end{aligned} \tag{14}$$

4. The connection between the two kind of coordinates

There is a very simple way to pass from trilinear coordinates to barycentric coordinates:

$$\begin{cases} X_0 = x_0 \cdot \sin a, \\ X_1 = x_1 \cdot \sin b, \\ X_2 = x_2 \cdot \sin c. \end{cases} \tag{15}$$

5. The isotomic transformation

As we mentionead, already, the isotomic transformation for Euclidean triangles has been introduced by G. de Longchamps in 1866 (see [5] and [6]). We shall give here a similar definition, using the hyperbolic barycentric coordinates.

Definition 1. We define, by analogy to the Euclidean case, the *isotomic transformation* as being a map

$$\mathrm{Isot} : \mathbb{P}^2(\mathbb{R}) \setminus \mathcal{T} \to \mathbb{P}^2(\mathbb{R}),$$

defined by

$$\text{Isot}\,(X_0, X_1, X_2) = \left(\frac{1}{X_0}, \frac{1}{X_1}, \frac{1}{X_2}\right). \tag{16}$$

Here $\mathcal{T}$ is the union of the three sides of the triangle ABC (thought of as *projective* lines). We shall say that the points M and M' form an *isotomic pair*. We shall also say that M' is the *isotomic conjugate* or the *isotomic inverse* of M. We may, as well, say, again inspired from the classical case, that the two points are *reciprocal* to each other (with respect to the triangle ABC).

As Isot is defined on $\mathbb{P}^2(\mathbb{R})\setminus\mathcal{T}$, none of the coordinates X_i vanishes, hence Isot is well defined.

Remark. By looking at the formula (16), the reader may think that the definition of the isotomic transformation is identical to the definition from the Euclidean/projective case. This is not the case, however, because the barycentric coordinates from the elliptic case are not the same with the classical barycenter coordinates.

Definition 2. We shall say that two points on the side BC of the elliptic triangle ABC are *isotomically symmetric* with respect to the midpoint A' of the side BC if they coordinates are $A_1(0, \alpha_1, \alpha_2)$ and $A'(0, 1/\alpha_1, 1/\alpha_2)$ or $A_1'(0, \alpha_2, \alpha_1)$.

Remark. As A_1 and A_1' are not vertices of the triangle, the numbers α_1 and α_2 are both different from zero, we can write $A_1(0, \alpha_1\alpha_2, a)$ or $A_1(0, \alpha, 1)$. Then its isotomically symmetric point can be written as $A_1'(0, 1, \alpha)$.

The following theorem justifies the name of "isotomic transformation".

Theorem 1. *If A_1 is a point on BC, then its isotomic symmetric A_1' verifies the equality and $A'A_1 = A'A_1'$ (as elliptic lengths).*

Proof.

We know, already, that the barycentric coordinates of A' are $(0, 1, 1)$. We compute, first, the length of the segment $A'A_1$. We have

$$\cos A'A_1 = \frac{|(A', A_1)|}{\sqrt{(A', A') \cdot (A_1, A_1)}}.$$

On the other hand,

$$\begin{aligned}
(A', A') &= 2(1 + \cos a) = 4\cos^2\frac{a}{2},\\
(A_1, A_1) &= \alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2\cos a,\\
(A', A_1) &= 2(\alpha_1 + \alpha_2)\cos^2\frac{a}{2}.
\end{aligned}$$

We have, therefore

$$\cos A'A_1 = \frac{\left|2(\alpha_1 + \alpha_2)\cos^2\frac{a}{2}\right|}{2\cos\frac{a}{2}\sqrt{\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2\cos a}} = \frac{|\alpha_1 + \alpha_2|\cos\frac{a}{2}}{\sqrt{\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2\cos a}}.$$

Now, it is easy to check that $(A_1', A_1') = (A_1, A_1)$ and $(A', A_1') = (A', A_1)$, therefore $\cos A'A_1' = \cos A'A_1$, hence $A'A_1' = A'A_1$.

The previous theorem justifies the following definition:

Definition 3. Two cevians of an elliptic triangle ABC, starting from the same vertex, are called *isotomic* if they cut the opposite side at isotomically symmetric points. We shall, also, say that the cevians are *isotomically conjugated* (or *reciprocal*).

Theorem 2. *If three cevians (starting from different vertices) are concurrent at a point, then their isotomic conjugates are also concurrent and the intersection points are, as well, isotomically conjugated.*

Proof. Let $M\left(X_0^0, X_1^0, X_2^0\right)$ be the intersection point of the three given cevians. It is easy to see that the equations of these cevians are

$$AM : X_2^0 X_1 - X_1^0 X_2 = 0,$$
$$BM : X_2^0 X_0 - X_0^0 X_2 = 0,$$
$$CM : X_1^0 X_0 - X_0^0 X_1 = 0.$$

As such, their intersection points with the sides BC, CA and AB, respectively, will be $A_1\left(0, X_1^0, X_2^0\right)$, $B_1\left(X_0^0, 0, X_2^0\right)$ and $C_1\left(X_0^0, X_1^0, 0\right)$, respectively. Then, according to the previous lemma, their symmetrics with respect to the midpoints of the respective sides will be $A_1'\left(0, X_2^0, X_1^0\right)$, $B_1'\left(X_2^0, 0, X_0^0\right)$ and $C_1'\left(X_1^0, X_0^0, 0\right)$, respectively.

We are, thus, led to the equations of the isotomically conjugated of the cevians AM, BM and CM:

$$AA_1' : X_1^0 X_1 - X_2^0 X_2 = 0,$$
$$BB_1' : X_0^0 X_0 - X_2^0 X_2 = 0,$$
$$CC_1' : X_0^0 X_0 - X_1^0 X_1 = 0.$$

It turns out that the three cevians *do* intersect, at the point

$$M'\left(1/X_0^0, 1/X_1^0, 1/X_2^0\right),$$

as we expected.

6. The isogonal transformation

We consider now, on the elliptic plane, the trilinear polarity (and trilinear coordinates).

Definition 4. We define, by analogy to the Euclidean case, the *isogonal transformation* as being a map

$$\mathrm{Isog} : \mathbb{P}^2(\mathbb{R}) \setminus \mathcal{T} \to \mathbb{P}^2(\mathbb{R}),$$

defined by

$$\mathrm{Isog}\left(x_0, x_1, x_2\right) = \left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}\right). \tag{17}$$

Here $\mathcal{T}$ is the union of the three sides of the triangle ABC (thought of as *projective* lines). We shall say that the points M and M' form an *isogonal pair*. We shall also say that M' is the *isogonal conjugate* or the *isogonal inverse* of M. We may, as well, say, again inspired from the classical case, that the two points are, simply, *inverse* (with respect to the triangle ABC).

Definition 5. Two cevians of an elliptic triangle ABC, passing through the same vertex of the triangle, are called *isogonal* if they are symmetric with respect to the internal bisector of the corresponding angle (i.e. they form the same angle with the bisector, whence the name). We shall also, say that the two cevians are *isogonally conjugate* with respect to the given triangle, or *inverse* with respect to the triangle.

We would like to recast the definition of the isogonal cevians, by the model of the isotomic cevians. To this end, we begin by giving the following definition:

Definition 6. Let ABC be a triangle in the elliptic plane. We shall say the two points A_1 and A_1' from the side BC are *isotomically symmetric* if, written in trilinear coordinates, the can be described by $A_1(0,1,\alpha)$ and $A_1'(0,\alpha,1)$.

The next theorem clarifies the meaning of the definition.

Theorem 3. *Let ABC be a triangle in the elliptic plane. Then two distinct*[2] *cevians AA_1 and AA_1' are isogonal iff their intersections with the side BC, A_1 and A_1', respectively, are isogonally symmetric.*

Proof. Let AA_1 a cevian passing through the vertex A of the elliptic triangle ABC, with $A_1 \in BC$. As the cevian is passing through A, its equation (in trilinear coordinates) can be written as

$$\alpha x_1 + x_2 = 0, \tag{18}$$

where $\alpha \in \mathbb{R}^*$. As such, its foot will be $A_1(0,1,-\alpha)$.

Consider now another cevian through A, AA_1', with $A_1' \in BC$. Its equation will be

$$\beta x_1 + x_2 = 0, \tag{19}$$

where $\beta \in \mathbb{R}^*$, and the foot – $A_1'(0,1,-\beta)$.

We want the relation between α and β so that AA_1' is the isogonal of AA_1. We know that the two cevians are isogonal iff they are symmetric with respect to the internal bisector AA' of the angle A of the triangle ABC or, which is the same, iff

$$\angle\left(AA_1, AA'\right) = \angle\left(AA_1', AA'\right) \tag{20}$$

Let $\xi = [0,\alpha,1]$ be the vector associated to the first cevian and $\zeta = [0,\beta,1]$ the vector associated to the second cevian. As we shall see, the equation of the internal bisector of the angle A, in trilinear coordinates, is

$$x_1 - x_2 = 0, \tag{21}$$

which means that the vector associated to it is $\eta = [0,1,-1]$. Thus, the equality (20), where we pass to cosines, is equivalent to

$$\frac{[\xi,\eta]}{\sqrt{[\xi,\xi]\cdot[\eta,\eta]}} = \frac{[\zeta,\eta]}{\sqrt{[\zeta,\zeta]\cdot[\eta,\eta]}} \tag{22}$$

or, after we simplify with the factor $1/\sqrt{[\eta,\eta]}$,

$$[\xi,\eta]\cdot\sqrt{[\zeta,\zeta]} = [\zeta,\eta]\cdot\sqrt{[\xi,\xi]}. \tag{23}$$

[2]Only the internal bisectors coincide with their isogonals.

By using the formula (4, we get

$$[\xi,\eta]=[[0,\alpha,1],[0,1,-1]]=(\alpha-1)\,(1+\cos A)\,, \tag{24}$$

$$[\zeta,\eta]=[[0,\beta,1],[0,1,-1]]=(\beta-1)\,(1+\cos A)\,, \tag{25}$$

$$[\xi,\xi]=[[0,\alpha,1],[0,\alpha,1]]=\alpha^2-2\alpha\cos A+1, \tag{26}$$

$$[\zeta,\zeta]=[[0,\beta,1],[0,\beta,1]]=\beta^2-2\beta\cos A+1, \tag{27}$$

therefore, assuming the triangle is non-degenerate ($A\neq\pi$), the equation (23) becomes

$$(\alpha-1)\sqrt{\beta^2-2\beta\cos A+1}=(\beta-1)\sqrt{\alpha^2-2\alpha\cos A+1}. \tag{28}$$

After squaring both sides of the equation and performing the computations, we get

$$(1-\cos A)(\alpha-\beta)(\alpha\beta-1)=0. \tag{29}$$

Again, because the triangle is non-degenerate, we have $A\neq 0$, therefore the equation (29) gives us either $\alpha=\beta$ (i.e. the two lines should coincide, which we assumes is not the case), either $\alpha\beta=1$, or $\beta=1/\alpha$, i.e. we have $A_1'\left(0,1,-\dfrac{1}{\beta}\right)$, which means that the foots of the two cevians are isogonally symmetric (with respect to the foot of the bisector).

The next theorem justifies the name of the point transformation, connecting the isogonal points and isogonal cevians.

Theorem 4. *Let ABC be an elliptic triangle. If three cevians of the triangle, passing through different vertices, intersect at the same point, then their isogonal conjugates also intersect, at the isogonal conjugate of the first intersection point.*

Proof. The proof is similar to the analogue theorem for the isotomic transformation, only that now we work in trilinear coordinates and we use the isogonal symmetry of point on the sides instead of isotomic symmetry. Thus, let $M(x_0^0,x_1^0,x_2^0)$ be the intersection point of the given cevians of the elliptic triangle ABC. It is easy to check that the equations of the cevians can be written, in terms of the coordinates of the intersection point, as

$$\begin{aligned} AM &: x_2^0x_1-x_1^0x_2=0,\\ BM &: x_2^0x_0-x_0^0x_2=0,\\ CM &: x_1^0x_0-x_0^0x_1=0, \end{aligned} \tag{30}$$

while the intersections of these cevians with the corresponding opposite sides of the triangle are $A_1(0,x_1^0,x_2^0), B_1(x_0^0,0,x_2^0)$ and $C_1(x_0^0,x_1^0,0)$, respectively.

Based on the previous theorem, we can write down the feet of the cevians isogonal to the given ones, these being the isogonal symmetric of the feet of the initial cevians: $A_1'(0,x_2^0,x_1^0), B_1'(x_2^0,0,x_0^0)$ and $C_1'(x_1^0,x_0^0,0)$. Thus, the equations of the isogonal of the cevians (30) are

$$\begin{aligned} AA_1' &: x_1^0 x_1 - x_2^0 x_2 = 0, \\ BB_1' &: x_0^0 x_0 - x_2^0 x_2 = 0, \\ CC_1' &: x_0^0 x_0 - x_1^0 x_1 = 0. \end{aligned} \tag{31}$$

It is trivial to check that the cevians (31) pass through the point $M'\left(1/x_0^0, 1/x_1^0, 1/x_2^0\right)$, which is the isogonal of the point M.

7. The transformations in other coordinate systems

Having in mind the connection between the two coordinate systems, we have the following result:

Proposition 7.1. *(1) In trilinear coordinates, the isotomic transformation is given by*

$$\operatorname{Isot}(x_0, x_1, x_2) = \left(\frac{1}{x_0 \cdot \sin^2 A}, \frac{1}{x_1 \cdot \sin^2 B}, \frac{1}{x_2 \cdot \sin^2 C}\right). \tag{32}$$

(2) In barycentric coordinates, the isogonal transformation is given by

$$\operatorname{Isog}(X_0, X_1, X_2) = \left(\frac{\sin^2 A}{X_0}, \frac{\sin^2 B}{X_1}, \frac{\sin^2 C}{X_2}\right). \tag{33}$$

This proposition, very easy to prove, is useful in many situations, because barycentric and trilinear coordinates have the peculiarity that in one of them the polarity is simpler in point coordinates and more complex in line coordinates and the other way around. Thus, for instance, when working with lines, it is easier to work with trilinear coordinates, while when working with points it is easier to work with barycentric coordinates. It is handy, therefore, to have the transformations expressed in both coordinate systems.

Remark. Due to the sine theorem in elliptic trigonometry, in the formulae (32) and (33) we can use the sines of the sides of the triangle, instead of the sines of the angles.

8. Remarkable points and lines in an elliptic triangle

8.1. **Centroids and medians.** In projective geometry, it is considered that a segment has two midpoints: the "ordinary" one and its harmonic conjugate. Put it another way, they are the points that divide the segment internal and external in the ratio 1/2 (see [11] for a recent discussion). This means that, in particular, we have not three but six medians, three internal and three external, the external ones being, in fact, the harmonical conjugates of the internal ones in the pencils of line defined by the vertices of the triangle. Exactly as two external bisectors and one internal intersect at the same point, the same happens in the case of the medians, therefore we have four centroids of a given triangle, one internal and three external.

The difference from the Euclidean case is only apparent, because these supplementary lines are present in the case of Euclidean triangles, as well. For instance, the harmonic conjugate of the median through the vertex A of the Euclidean triangle ABC is the line through A, parallel to BC.

The external centroids are the vertices of the so-called *anticomplementary triangle* of ABC (see [9]).

When studying the medians and centroids, it is convenient to work in barycentric coordinates, because these were, originally, constructed by having the centroid as unit point. It is well known (see [3]) that, in barycentric coordinates, the four centroids are $G(1,1,1), G_a(-1,1,1), G_b(1,-1,1)$ and $G_c(1,1,-1)$.

It is easy, now to write down the equations of the medians, as lines passing through a vertex and a centroid. For instance, the medians from A are:

$$AG: \ X_1 - X_2 = 0 \tag{34}$$

and

$$AG_a: \ X_1 + X_2 = 0. \tag{35}$$

Once we have the centroids and the medians in barycentric coordinates, it is easy to find them in trilinear coordinates, by using the coordinate transformations (15). Thus, the centroids, in trilinear coordinates are

$$\begin{gathered} G(\csc A, \csc B, \csc C), \\ G_a(-\csc A, \csc B, \csc C), \\ G_b(\csc A, -\csc B, \csc C) \end{gathered}$$

and

$$G_c(\csc A, \csc B, -\csc C),$$

where we used the sine theorem to pass from the sines of sides to the sines of angles.

As for the medians, the equations of the medians from A written in trilinear coordinates, are

$$AG: \ x_1 \sin B - x_2 \sin C = 0 \tag{36}$$

and

$$AG_a: \ x_1 \sin B + x_2 \sin C = 0. \tag{37}$$

The external medians of a triangle are not concurrent. Actually, as it happens with the bisectors (see below), one external centroid is the intersection of two external medians and one internal median. For instance,

$$\{G_a\} = AG \cap BG_b \cap CG_c.$$

indeed, using the equations of the three lines, we get the system (in barycentric coordinates)

$$\begin{cases} X_1 - X_2 = 0, \\ X_0 + X_2 = 0, \\ X_0 + X_1 = 0. \end{cases}$$

Clearly, the coordinates of G_a form a solution of the system.

8.2. **The angle bisectors, the incenter and the excenters.** The incenter and the excenters of an elliptic triangle (as intersections of the three internal bisectors and two external and one internal bisector, respectively) can be written very easily in trilinear coordinates. They are (see [3]) $I(1,1,1)$, $I_a(-1,1,1)$, $I_b(1,-1,1)$ and $I_c(1,1,-1)$.

Knowing the centers, the equations of the bisectors are easy to write. For instance, the bisectors of the angle A are:

$$AI: \; x_1 - x_2 = 0 \tag{38}$$

and

$$x_1 + x_2 = 0. \tag{39}$$

If we pass, instead, to barycentric coordinates, the incenter and the excenters will be $I(\sin A, \sin B, \sin C)$, $I_a(-\sin A, \sin B, \sin C)$, $I_b(\sin A, -\sin B, \sin C)$ and $I_c(\sin A, \sin B, \sin C)$, while the equations of the bisectors of the angle A become

$$X_1 \sin C - X_2 \sin B = 0 \tag{40}$$

and

$$X_1 \sin C + X_2 \sin B = 0. \tag{41}$$

8.3. **The altitudes and the orthocenter.** For the orthocenter we have an expression in terms of the components of the polarity matrix $[C]$, namely the orthocenter is

$$H = H\left(\frac{1}{C_{12}}, \frac{1}{C_{20}}, \frac{1}{C_{01}}\right) \tag{42}$$

(see [3]). Thus, in trilinear coordinates, the orthocenter is given by

$$H = H(\sec A, \sec B, \sec C). \tag{43}$$

Instead of using the formula (42) for barycentric coordinates, it is more convenient to apply the coordinates change formula (15) to the equation (42) and we get the orthocenter in barycentric coordinates:

$$H = H(\tan A, \tan B, \tan C). \tag{44}$$

Because the polarity matrix in trilinear coordinates is simpler, we shall start working in this coordinate. The altitude from A is of the form

$$\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 = 0,$$

which means that it can be described by a triple $\xi\,[0, \alpha_1, \alpha_2]$, while the side BC can be described by the triple $\eta\,[1,0,0]$. As we saw earlier, the perpendicularity condition can be written as

$$[\xi, \eta] = \xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 - \cos C \cdot (\xi_0\eta_1 + \xi_1\eta_0) - \cos B \cdot (\xi_0\eta_2 + \xi_2\eta_0) - \\ - \cos A \cdot (\xi_1\eta_2 + \xi_2\eta_1) = 0$$

or

$$-\alpha_1 \cos C - \alpha_2 \cos B = 0,$$

whence $\xi = [0, \cos B, -\cos C]$. Thus, the equations of the three altitudes are

$$\begin{cases} x_1 \cos B - x_2 \cos C = 0, \\ x_0 \cos A - x_2 \cos C = 0, \\ x_0 \cos A - x_1 \cos B = 0, \end{cases} \tag{45}$$

while orthocenter is $H(\sec A, \sec B, \sec C)$.

The barycentric equations of the altitudes are

$$\begin{cases} X_1 \cot B - X_2 \cot C = 0, \\ X_0 \cot A - X_2 \cot C = 0, \\ X_0 \cot A - X_1 \cot B = 0, \end{cases} \tag{46}$$

while the orthocenter is $H(\tan A, \tan B, \tan C)$.

8.4. **The Gergonne point and its adjoints.** The Gergonne points and its adjoints are an important part of Euclidean geometry of the triangle. We showed, in another paper ([2]) that they can be adapted to hyperbolic triangles and the happens to the Nagel point and its associate points that will be treated in the next subsection. The computation we are going to make are, merely, adaptation to those made in [2] to the elliptic settings. As we shall be dealing with perpendicular lines, it is easier to work in trilinear coordinates, since the matrix of the polarity is simpler. We have, first of all, the following result:

Theorem 5. *Let ABC be a non-degenerate triangle in the elliptic plane. We connect the contact points of the incircle of the triangle with each side to the opposite vertex. Then, the three cevians intersect each other at a point that we will call the* Gergonne point *of the triangle.*

Proof. To begin with, we find the contact points of the incircle with the sides. These are nothing but the feet of the perpendicular lines from the incenter to the sides. We shall denote them by K_a, K_b, K_c. Let's start with the side BC. The equation of this side is $X_0 = 0$, hence the associated line vector is $\eta = [1, 0, 0]$. As we saw already, the incenter I has the trilinear coordinates $(1, 1, 1)$. Let assume that the equation of the line IK_a is

$$\xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0,$$

which means the the line vector of this line is $\xi = [\xi_0, \xi_1, \xi_2]$. Since the lines BC and IK_1 are perpendicular, we need to have $[\xi, \eta] = 0$, i.e.

$$\xi_0 - \xi_1 \cos C - \xi_2 \cos B = 0. \tag{47}$$

On the other hand, IK_a has to pass through I, therefore

$$\xi_1 + \xi_2 + \xi_3 = 0. \tag{48}$$

We get the equation:

$$IK_a: \ (\cos C - \cos B)x_0 + (1 + \cos B)x_1 - (1 + \cos C)x_2 = 0 \tag{49}$$

By intersecting this line with the side BC ($x_0 = 0$), we get the intersection point

$$K_a\left(0, \cos^2 \frac{C}{2}, \cos^2 \frac{B}{2}\right).$$

In the same manner, we get

$$K_b\left(\cos^2\frac{C}{2}, 0, \cos^2\frac{A}{2}\right)$$

and

$$K_c\left(\cos^2\frac{B}{2}, \cos^2\frac{A}{2}, 0\right).$$

It is straightforward to check that the Gergonne cevians are

$$AK_a:\ x_1\cos^2\frac{B}{2} - x_2\cos^2\frac{C}{2} = 0,$$

$$BK_b:\ x_0\cos^2\frac{A}{2} - x_2\cos^2\frac{C}{2} = 0$$

and

$$CK_a:\ x_0\cos^2\frac{A}{2} - x_1\cos^2\frac{B}{2} = 0.$$

The three line intersect at the Gergonne's point,

$$Ge\left(\frac{1}{\cos^2\frac{A}{2}}, \frac{1}{\cos^2\frac{B}{2}}, \frac{1}{\cos^2\frac{C}{2}}\right). \tag{50}$$

Remark. The formula (50) gives us the coordinates of the Gergonne point in trilinear coordinates. As an easy computation shows, in barycentric coordinates it is given by

$$Ge\left(\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}\right). \tag{51}$$

The adjoint Gergonne points are constructed in exactly the same manner as the Gergonne point, but we use one of the excenters instead of the incenter of the triangle. Take, for instance, the excenter $I_a(-1,1,1)$ (we still work in trilinear coordinates). We denote by K_{aa}, K_{ab} and K_{ac} the projections of the excenter on the sides of the triangle.

We start by finding the projection K_{aa}. To this end, we consider, again, the equation of I_aK_{aa} in the form

$$I_aK_{aa}:\ \xi_0x_0 + \xi_1x_1 + \xi_2x_2 = 0,$$

i.e. it has the line vector $\xi = [\xi_0, \xi_1, \xi_2]$. This line has to be perpendicular to the side BC, of line vector $\eta = [1, 0, 0]$, hence

$$0 = [\xi, \eta] = \xi_0 - \xi_1\cos C - \xi_2\cos B.$$

On the other hand, the line passes through I_a, hence

$$-\xi_0 + \xi_1 + \xi_2 = 0.$$

By solving the system, we get the equation of the line:

$$I_aK_{aa}:\ (\cos B - \cos C)\,x_0 - (1-\cos B)x_1 + (1-\cos C)x_2 = 0.$$

If we intersect this line with the side BC, of equation $x_0 = 0$, we obtain the point

$$K_{aa}\left(0, \sin^2\frac{C}{2}, \sin^2\frac{B}{2}\right).$$

By an analogue procedure, we get

$$K_{ab}\left(\sin^2\frac{C}{2}, 0, -\cos^2\frac{A}{2}\right)$$

and

$$K_{ac}\left(\sin^2\frac{B}{2}, -\cos^2\frac{A}{2}, 0\right).$$

It is easy know to write the equations of the cevians associated to the contact points of the excircle of center I_a with the sides of the triangle:

(52) $$AK_{aa}:\ \sin^2\frac{B}{2}x_1 - \sin^2\frac{C}{2}x_2 = 0,$$

$$BK_{ab}:\ \cos^2\frac{A}{2}x_0 + \sin^2\frac{C}{2}x_2 = 0$$

and

$$CK_{ac}:\ \cos^2\frac{A}{2}x_0 + \sin^2\frac{B}{2}x_1 = 0.$$

These three lines do intersect at the *adjoint Gergonne point* J_a:

(53) $$J_a\left(-\frac{1}{\cos^2\frac{A}{2}}, \frac{1}{\sin^2\frac{B}{2}}, \frac{1}{\sin^2\frac{C}{2}}\right).$$

In the same manner, we get the other two adjoint Gergonne points:

(54) $$J_b\left(\frac{1}{\sin^2\frac{A}{2}}, -\frac{1}{\cos^2\frac{B}{2}}, \frac{1}{\sin^2\frac{C}{2}}\right)$$

and

(55) $$J_c\left(\frac{1}{\sin^2\frac{A}{2}}, \frac{1}{\sin^2\frac{B}{2}}, -\frac{1}{\cos^2\frac{C}{2}}\right).$$

Remark. In barycentric coordinates, the adjoint Gergonne points are:

(56) $$J_a\left(-\tan\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}\right),$$

(57) $$J_b\left(\cot\frac{A}{2}, -\tan\frac{B}{2}, \cot\frac{C}{2}\right)$$

and

(58) $$J_c\left(\cot\frac{A}{2}, \cot\frac{B}{2}, -\tan\frac{C}{2}\right).$$

8.5. **The Nagel point and their adjoint points.** In the Euclidean triangle geometry, the Nagel point is obtained by intersecting the cevians that connect a vertex to the contact point of the opposite excircle with the opposite side. We shall prove below that these cevians also intersect in the case of an elliptic triangle. Using the same notation as in the previous subsection, we need to prove that the lines AK_{aa}, BK_{bb} and CK_{cc} intersect. We already have the equation of the cevian AK_{aa} (see the equation (52)). The other ones are obtained in the same way:

$$BK_{bb}: \ \sin^2\frac{A}{2}x_0 - \sin^2\frac{C}{2}x_2 = 0 \tag{59}$$

and

$$CK_{cc}: \ \sin^2\frac{A}{2}x_0 - \sin^2\frac{B}{2}x_2 = 0. \tag{60}$$

Their intersection is the Nagel point,

$$N\left(\frac{1}{\sin^2\frac{A}{2}}, \frac{1}{\sin^2\frac{B}{2}}, \frac{1}{\sin^2\frac{C}{2}}\right). \tag{61}$$

Remark. In barycentric coordinates, the Nagel point is

$$N\left(\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}\right). \tag{62}$$

In Euclidean geometry, the Nagel point also have three adjoint points. It is a little bit more difficult to describe the way they are constructed. Suppose we want to construct the adjoint Nagel point N_a associated to the side BC. To this end, we draw the cevian which connects B to the contact point of the excircle of center I_c (the contact point is the point K_{cb} with the notations introduced earlier)with the side AC, the cevian connecting C to the contact point of the excircle of center I_b with the side AB (the point K_{bc} and the cevian connecting the vertex A to the contact point of the incircle with side BC (i.e. the point K_a). It turns out that three cevians intersect at a point that was called the adjoint Nagel point N_a. We shall prove that the result is,also, true for eelptic triangles.

We find the contact points K_{bc} and K_{cb}, by the same procedure we found already contact points, for the excircle with the center at I_a. We get

$$K_{bc} = \left(-\cos^2\frac{B}{2}, \sin^2\frac{A}{2}, 0\right)$$

and

$$K_{cb} = \left(-\cos^2\frac{C}{2}, 0, \sin^2\frac{A}{2}\right).$$

Thus, the cevians we are looking for are

$$AK_a: \ x_1\cos^2\frac{B}{2} - x_2\cos^2\frac{C}{2} = 0,$$

$$BK_{cb}: \ x_0\sin^2\frac{A}{2} + x_2\cos^2\frac{C}{2} = 0$$

and

$$CK_{bc}: \ x_0\sin^2\frac{A}{2} + x_2\cos^2\frac{B}{2} = 0.$$

These lines intersect and their intersection point is the point

$$N_a\left(-\frac{1}{\sin^2\frac{A}{2}}, \frac{1}{\cos^2\frac{B}{2}}, \frac{1}{\cos^2\frac{C}{2}}\right). \tag{63}$$

In the same manner, we obtain the remaining adjoint Nagel points,

$$N_b\left(\frac{1}{\cos^2\frac{A}{2}}, -\frac{1}{\sin^2\frac{B}{2}}, \frac{1}{\cos^2\frac{C}{2}}\right) \tag{64}$$

and

$$N_c\left(\frac{1}{\cos^2\frac{A}{2}}, \frac{1}{\cos^2\frac{B}{2}}, -\frac{1}{\sin^2\frac{C}{2}}\right). \tag{65}$$

Remark. In barycentric coordinates, the adjoint Nagel points given by:

$$N_a\left(-\cot\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}\right), \tag{66}$$

$$N_b\left(\tan\frac{A}{2}, -\cot\frac{B}{2}, \tan\frac{C}{2}\right) \tag{67}$$

and

$$N_c\left(\tan\frac{A}{2}, \tan\frac{B}{2}, -\cot\frac{C}{2}\right). \tag{68}$$

9. Application of the transformations to remarkable points

We shall investigate now the images of the points discussed above through the two transformations.

9.1. **Isotomic transformation.**

We have the following result, whose proof means only checking the computation. All the point are given by they barycentric coordinates.

Theorem 6. *(i) The isotomic conjugates of the centroids are themselves.*

(ii) The isotomic conjugate of the incenter is the point

$$I'\left(\csc A, \csc B, \csc C\right).$$

(iii) The isotomic of excenters are the points

$$I'_a\left(-\csc A, \csc B, \csc C\right),\ I'_b\left(\csc A, -\csc B, \csc C\right),\ I'_c\left(\csc A, \csc B, -\csc C\right).$$

(iv) The isotomic of the orthocenter is the point

$$H'(\cot A, \cot B, \cot C).$$

(v) The isotomic of the Gergonne point is the Nagel point and the other way around.

(vi) The isotomic of the adjoint Gergonne points are the Nagel adjoint points and the other way around.

9.2. **Isogonal transformation.** We have the following result, whose proof means only checking the computation. All the point are given by they trilinear coordinates.

Theorem 7. *(i) The isogonal of the centroid is the point*

$$K(\sin A, \sin B, \sin C).$$

(ii) The isogonal of the external centrois are the points

$$K_a(-\sin A, \sin B, \sin C),\ K_b(\sin A, -\sin B, \sin C),\ K_c(\sin A, \sin B, -\sin C).$$

(iii) The isogonals of the incenter and the excenters are themselves.

(iv) The isogonal of the orhocenter is the point

$$H''\left(\cos A, \cos B, \cos C\right).$$

(v) The isogonal of the Gergonne point is the point

$$Ge'(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}).$$

(vi) The isogonals of the adjoint Gergonne points are the points

$$J_a'(-\cos^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}),$$
$$J_b'(\sin^2\frac{A}{2}, -\cos^2\frac{B}{2}, \sin^2\frac{C}{2}),$$
$$J_c'(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, -\cos^2\frac{C}{2}).$$

(vii) The isogonal of the Nagel point is the point

$$N'\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{B}{2}\right).$$

(viii) The isogonal of the adjoint Nagel points are the points

$$N_a'\left(\sin^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right),$$

$$N_b'\left(\cos^2\frac{A}{2}, -\sin^2\frac{B}{2}, \cos^2\frac{C}{2}\right)$$

and

$$N_c'\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, -\sin^2\frac{C}{2}\right).$$

Remarks. a) The point K is called the *Lemoine point* and it plays an important role in triangle geometry. The associated cevians (the isogonal of medians) are called *symmedians.* The points K_a, K_b and K_c are called *external Lemoine points* and the associated cevians are called *external symmedians.* They are the isogonal of the external medians.

b) The point H' is not the circumcenter,as it happens in Euclidean geometry. Both topics will be treated in another paper.

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