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AN EXAMPLE OF S TRIANGLES ARISING FROM TWO I.M.O. PROBLEMS

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Abstract. In this paper we solve the 6^{th} Problem of I.M.O. 2000 and the 2^{nd} Problem of I.M.O. 1982 by using complex numbers. This approach allows us to prove that two triangles that appear in the geometric configuration in both problems are S triangles.

1. Problem 6, I.M.O. 2000

ABC is an acute-angled triangle. The feet of altitudes from A, B, C are H_1, H_2, H_3 and the incircle touches the sides BC, CA, AB at T_1, T_2, T_3. Let l_1, l_2, l_3 the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2. Show that l_1, l_2, l_3 form a triangle with vertices on the incircle.

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Proof. Let the incircle be the unit circle, and let t_1, t_2, t_3, h_1, h_2, h_3, a, b, c be the complex coordinates of the points T_1, T_2, T_3, H_1, H_2, H_3, A, B, C. Denote by I the incenter. The equation of the line IT_2 is

$$\overline{z}t_2 - z\overline{t_2} = 0.$$

The equation of the tangent line to the unit circle at T_2 is:

$$z - t_2 = \frac{t_2}{-\overline{t_2}}(\overline{z} - \overline{t_2}) \Leftrightarrow$$

$$z + t_2^2\overline{z} = 2t_2. \tag{1}$$

The equation of the tangent line to the unit circle at T_3 is:

$$z + t_3^2\overline{z} = 2t_3. \tag{2}$$

From (1) and (2) we obtain the complex coordinate of A:

$$a = \frac{2t_2t_3}{t_2 + t_3}.$$

Similarly, we have:

$$b = \frac{2t_3t_1}{t_3 + t_1}, \quad c = \frac{2t_1t_2}{t_1 + t_2}.$$

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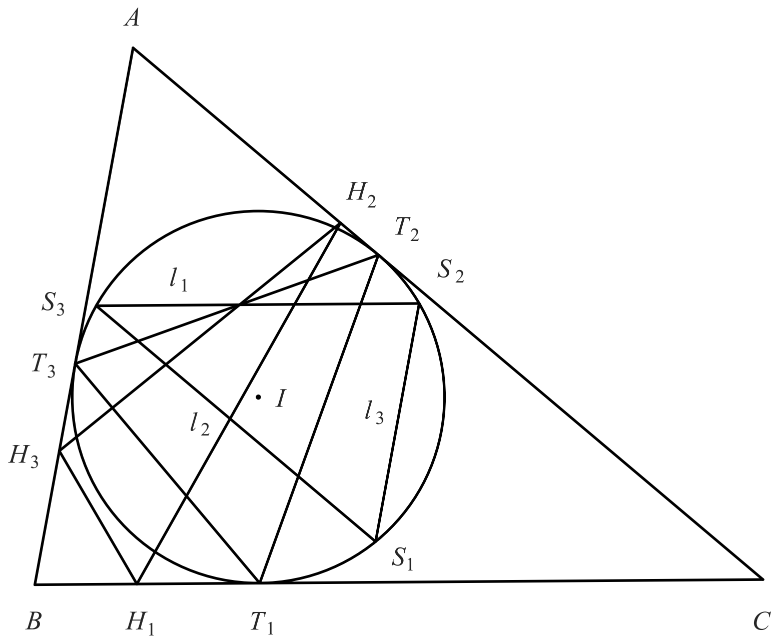

Figure 1

The line BC has the equation:

$$\begin{vmatrix} z & \overline{z} & 1 \\ \dfrac{2t_3t_1}{t_3+t_1} & \dfrac{2\overline{t_3}\,\overline{t_1}}{\overline{t_3}+\overline{t_1}} & 1 \\ \dfrac{2t_1t_2}{t_1+t_2} & \dfrac{2\overline{t_1}\,\overline{t_2}}{\overline{t_1}+\overline{t_2}} & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\begin{vmatrix} z & \overline{z} & 1 \\ \dfrac{2t_3t_1}{t_3+t_1} & \dfrac{2}{t_3+t_1} & 1 \\ \dfrac{2t_1t_2}{t_1+t_2} & \dfrac{2}{t_1+t_2} & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$z + t_1^2\overline{z} - 2t_1 = 0 \Leftrightarrow$$

$$t_1\overline{z} + \overline{t_1}z - 2 = 0.$$

Let $P_0(z_0)$ be a point and let

$$d:\ \overline{\alpha}\ \overline{z} + \alpha z + \beta = 0$$

be a line. The foot of the perpendicular from P_0 to d has the complex coordinate:

$$z = \frac{\alpha z_0 - \overline{\alpha}\ \overline{z_0} - \beta}{2\alpha}, \quad [1] \tag{3}$$

In our case:

$$\overline{\alpha} = t_1, \quad \alpha = \overline{t_1}, \quad \beta = -2, \quad z_0 = a = \frac{2t_2t_3}{t_2+t_3}.$$

It follows that the complex coordinate of H_1 is

$$h_1 = t_1 + \frac{t_2t_3 - t_1^2}{t_2+t_3}.$$

Similarly,

$$h_2 = t_2 + \frac{t_3 t_1 - t_2^2}{t_3 + t_1}.$$

The equation of the line T_1T_2 is:

$$\begin{vmatrix} z & \overline{z} & 1 \\ t_1 & \overline{t_1} & 1 \\ t_2 & \overline{t_2} & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$z + t_1 t_2 \overline{z} - (t_1 + t_2) = 0.$$

Let p_1 be the complex coordinate of the foot of the perpendicular from H_1 to T_1T_2. We have:

$$p_1 = \frac{\dfrac{1}{t_1+t_2}\ h_1 - \dfrac{t_1 t_2}{t_1+t_2}\ \overline{h_1} + 1}{2 \cdot \dfrac{1}{t_1+t_2}} \Leftrightarrow$$

$$p_1 = \frac{h_1 - t_1 t_2 \overline{h_1} + t_1 + t_2}{2}.$$

Let h_1' be the complex coordinate of the symmetric point of H_1 with respect to T_1T_2. We have:

$$h_1' = 2p_1 - h_1 \Leftrightarrow$$

$$h_1' = -t_1 t_2 \overline{h_1} + t_1 + t_2.$$

It follows:

$$h_1' = \frac{t_3(t_1^2 + t_2^2)}{t_1(t_2 + t_3)}.$$

Similarly, the complex coordinate of the symmetric point of H_2 with respect to T_1T_2 is:

$$h_2' = \frac{t_3(t_1^2 + t_2^2)}{t_2(t_1 + t_3)}.$$

The line l_3 has the equation:

$$\begin{vmatrix} z & \overline{z} & 1 \\ h_1' & \overline{h_1'} & 1 \\ h_2' & \overline{h_2'} & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\frac{z - h_2'}{\overline{z} - \overline{h_2'}} = \frac{h_1' - h_2'}{\overline{h_1'} - \overline{h_2'}}.$$

But,

$$\begin{aligned} h_1' - h_2' &= \frac{t_3(t_1^2 + t_2^2)}{t_1(t_2 + t_3)} - \frac{t_3(t_1^2 + t_2^2)}{t_2(t_1 + t_3)} \\ &= \frac{t_3^2(t_1^2 + t_2^2)(t_2 - t_1)}{t_1 t_2 (t_1 + t_3)(t_2 + t_3)}. \end{aligned}$$

Hence,

$$\overline{h_1'} - \overline{h_2'} = \frac{1}{t_1 t_2} \cdot \frac{(t_1^2 + t_2^2)(t_1 - t_2)}{(t_1 + t_3)(t_2 + t_3)}.$$

It follows that:

$$\frac{h'_1 - h'_2}{\overline{h'_1} - \overline{h'_2}} = -t_3^2,$$

hence,

$$\frac{z - h'_2}{\overline{z} - \overline{h'_2}} = -t_3^2.$$

The complex coordinates of the intersections S_1 and S_2 of the line l_3 with the unit circle are the solutions of the system:

$$\begin{cases} z - h'_1 = -t_3^2(\overline{z} - \overline{h'_2}) \\ z\overline{z} = 1. \end{cases}$$

It follows:

$$z^2 - (h'_2 + t_3^2\overline{h'_2})z - t_3^2 = 0$$

and we obtain after a few simple computations:

$$t_1t_2z^2 - t_3(t_1^2 + t_2^2)z + t_3^2t_1t_2 = 0.$$

The solutions are $\dfrac{t_2t_3}{t_1}$ and $\dfrac{t_3t_1}{t_2}$, the complex coordinates of S_1 and S_2. Similarly the point of intersections of the line l_1 with the unit circle are S_2 and S_3 with the complex coordinates $\dfrac{t_3t_1}{t_2}$ and $\dfrac{t_1t_2}{t_3}$ and the points of intersections of the line l_2 with the unit circle are S_3 and S_1 with the complex coordinates $\dfrac{t_1t_2}{t_3}$ and $\dfrac{t_2t_3}{t_1}$, which finishes the proof.

2. Problem 2, I.M.O. 1982

A non-isosceles triangle ABC is given. M_1, M_2, M_3 are the midpoints of the sides BC, CA, AB and T_1, T_2, T_3 are the points where the incircle touches sides BC, CA, AB. Denote by S_1, S_2, S_3 the reflections of T_1, T_2, T_3 in the interior bisectors of angles A, B, C. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

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Proof. Let the incircle be the unit circle. If the complex coordinates of T_1, T_2, T_3 are t_1, t_2, t_3, then the complex coordinates of A, B, C are $\dfrac{2t_2t_3}{t_2 + t_3}$, $\dfrac{2t_3t_1}{t_3 + t_1}$, $\dfrac{2t_1t_2}{t_1 + t_2}$, and the complex coordinate of the incenter I is 0.

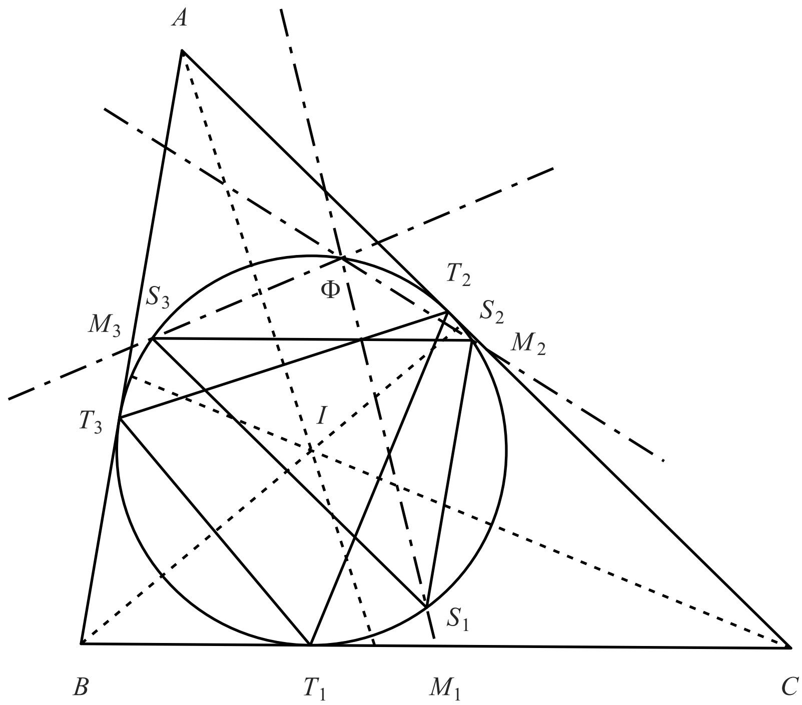

Figure 2

The equation of the interior bisector AI is:

$$\begin{vmatrix} z & \overline{z} & 1 \\ \dfrac{2t_2t_3}{t_2+t_3} & \dfrac{2\overline{t_2}\,\overline{t_3}}{\overline{t_2}+\overline{t_3}} & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$z\ \frac{\overline{t_2}\,\overline{t_3}}{\overline{t_2}+\overline{t_3}} - \overline{z}\ \frac{t_2t_3}{t_2+t_3} = 0 \Leftrightarrow$$

$$z\ \overline{i}\ \frac{\overline{t_2}\,\overline{t_3}}{\overline{t_2}+\overline{t_3}} + \overline{z}\ i\ \frac{t_2t_3}{t_2+t_3} = 0.$$

For the complex coordinate q_1 of the foot of perpendicular from T_1 to AI we will use again formula (3) with

$$\alpha = \overline{i}\ \frac{\overline{t_2}\,\overline{t_3}}{\overline{t_2}+\overline{t_3}}, \quad \overline{\alpha} = i\ \frac{t_2t_3}{t_2+t_3}, \quad \beta = 0, \quad z_0 = t_1.$$

Hence,

$$q_1 = \frac{-i\ \dfrac{\overline{t_2}\,\overline{t_3}}{\overline{t_2}+\overline{t_3}}\ t_1 - i\ \dfrac{t_2t_3}{t_2+t_3}\ \overline{t_1}}{2(-i)\ \dfrac{\overline{t_2}\,\overline{t_3}}{\overline{t_2}+\overline{t_3}}} \Leftrightarrow$$

$$q_1 = \frac{t_1^2 + t_2t_3}{2t_1}.$$

Let z_1 be the complex coordinate of S_1, where S_1 is the reflection of T_1 in AI. We have:

$$z_1 = 2q_1 - t_1 \Leftrightarrow$$

$$z_1 = \frac{t_2t_3}{t_1}.$$

Hence, S_1 from this problem is the same point S_1 from the previous problem. Similarly, the complex coordinates of S_2 and S_3 are $\frac{t_3t_1}{t_2}$ and $\frac{t_1t_2}{t_3}$ just like in Problem 6.

Let m_1 be the complex coordinate of the midpoint of the side BC. We have

$$\begin{aligned} m_1 &= \frac{t_3t_1}{t_3+t_1} + \frac{t_1t_2}{t_1+t_2} = t_1 \cdot \frac{t_3(t_1+t_2)+t_2(t_3+t_1)}{(t_3+t_1)(t_1+t_2)} \\ &= \frac{(t_1t_2+t_2t_3+t_3t_1+t_2t_3)(t_1t_2+t_1t_3)}{(t_1+t_2)(t_2+t_3)(t_3+t_1)} \\ &= \frac{(\sigma_2+t_2t_3)(\sigma_2-t_2t_3)}{\sigma_1\sigma_2-\sigma_3}. \end{aligned}$$

Hence,

$$m_1 = \frac{\sigma_2^2-(t_2t_3)^2}{\sigma_1\sigma_2-\sigma_3},$$

where

$$\begin{aligned} \sigma_1 &= t_1+t_2+t_3 \\ \sigma_2 &= t_1t_2+t_2t_3+t_3t_1 \\ \sigma_3 &= t_1t_2t_3. \end{aligned}$$

Similarly,

$$m_2 = \frac{\sigma_2^2-(t_3t_1)^2}{\sigma_1\sigma_2-\sigma_3}, \quad m_3 = \frac{\sigma_2^2-(t_1t_2)^2}{\sigma_1\sigma_2-\sigma_3}.$$

The line M_1S_1 has the equation:

$$\begin{vmatrix} z & \overline{z} & 1 \\ \dfrac{\sigma_2^2-(t_2t_3)^2}{\sigma_1\sigma_2-\sigma_3} & \dfrac{\overline{\sigma_2^2}-(\overline{t_2}\ \overline{t_3})^2}{\overline{\sigma_1}\ \overline{\sigma_2}-\overline{\sigma_3}} & 1 \\ \dfrac{t_2t_3}{t_1} & \dfrac{\overline{t_2}\ \overline{t_3}}{\overline{t_1}} & 1 \end{vmatrix} = 0.$$

Using the relations:

$$\overline{\sigma_1} = \frac{\sigma_2}{\sigma_3}, \quad \overline{\sigma_2} = \frac{\sigma_1}{\sigma_3}, \quad \overline{\sigma_3} = \frac{1}{\sigma_3},$$

we obtain after a few simple computations the equivalent equation:

$$\sigma_1t_1^2z + \sigma_2\sigma_3\overline{z} = \sigma_1\sigma_3 + \sigma_2t_1^2.$$

Similarly, the equations of M_2S_2 and M_3S_3 are:

$$\sigma_1t_2^2z + \sigma_2\sigma_3\overline{z} = \sigma_1\sigma_3 + \sigma_2t_2^2,$$

$$\sigma_1t_3^2z + \sigma_2\sigma_3\overline{z} = \sigma_1\sigma_3 + \sigma_2t_3^2.$$

Subtracting the first two equations we obtain:

$$\sigma_1(t_1^2-t_2^2)z = \sigma_2(t_1^2-t_2^2) \Leftrightarrow$$

$$z = \frac{\sigma_2}{\sigma_1}.$$

This complex number also verifies the equation of M_3S_3, hence the lines M_1S_1, M_2S_2, M_3S_3 are concurrent, and the problem is solved.

Remark. There is more to say about the point of concurrency of the line M_1S_1, M_2S_2, M_3S_3.

Theorem. *The lines M_1S_1, M_2S_2, M_3S_3 are concurrent in the Feuerbach point.*

Proof. The Feuerbach point is the point at which the incircle and the nine point circle are tangent.
Let Ω be the point which has the complex coordinate

$$\omega = \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3}.$$

Obvious,

$$\Omega M_1 = \Omega M_2 = \Omega M_3 = \frac{1}{|\sigma_1\sigma_2 - \sigma_3|},$$

hence Ω is the circumcenter of the medial triangle $M_1M_2M_3$, namely the center of the nine point circle of the triangle ABC, and the radius of the nine point circle is $\dfrac{1}{|\sigma_1\sigma_2 - \sigma_3|}$.

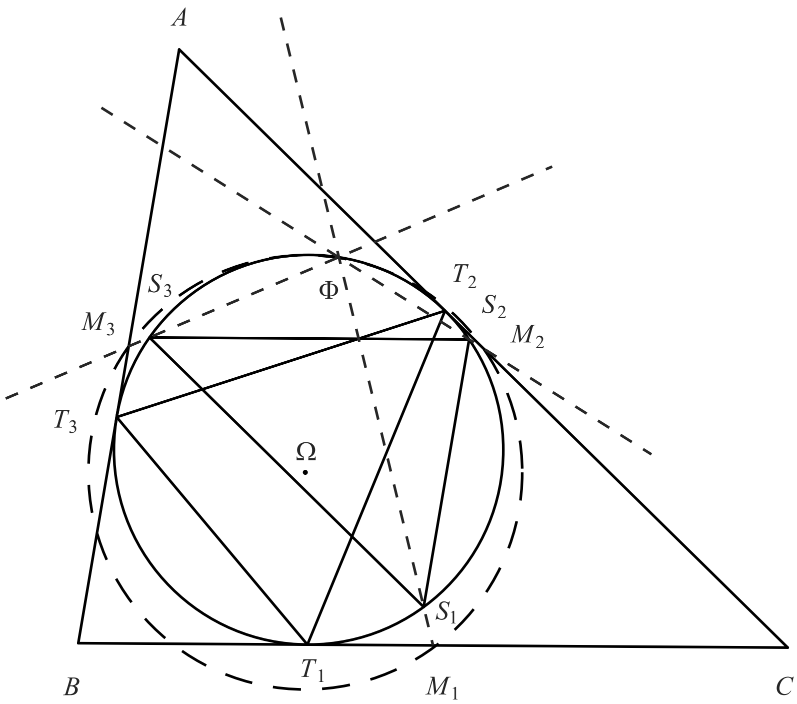

Figure 3

It follows that the equation of the nine point circle is:

$$\left|z - \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3}\right| = \frac{1}{|\sigma_1\sigma_2 - \sigma_3|} \Leftrightarrow$$

$$\left(z - \frac{\sigma_2^2}{\sigma_1\sigma_2 - \sigma_3}\right)\left(\overline{z} - \frac{\sigma_1^2}{\sigma_1\sigma_2 - \sigma_3}\right) = \left(\frac{\sigma_3^2}{\sigma_1\sigma_2 - \sigma_3}\right) \Leftrightarrow$$

$$z\overline{z}-\left(\frac{\sigma_1^2 z}{\sigma_1\sigma_2-\sigma_3}+\frac{\sigma_2^2\overline{z}}{\sigma_1\sigma_2-\sigma_3}\right)+\frac{\sigma_1\sigma_2+\sigma_3}{\sigma_1\sigma_2-\sigma_3}=0.$$

The incircle has the equation

$$z\overline{z}=1.$$

The complex coordinates of the intersections of this two circles are the solutions of the equation:

$$\sigma_1^2 z^2-2\sigma_1\sigma_2 z+\sigma_2^2=0 \Leftrightarrow$$

$$(\sigma_1 z-\sigma_2)^2=0.$$

$z=\dfrac{\sigma_2}{\sigma_1}$ is a double solution, hence the point of tangency Φ, of the incircle and the nine point circle is just the same point of concurrency of the lines M_1S_1, M_2S_2, M_3S_3.

3. S triangles (orthopolar triangles)

In this section we will show that the triangles $T_1T_2T_3$ and $S_1S_2S_3$ from the previous two I.M.O. problems are **S triangles**.

The definition of S triangles was given by Traian Lalescu who also gave their properties in the Mathematical note with number 19, "A class of remarkable triangles", which appeared in **Gazeta Matematică**, vol. XX, 1915, p. 213 [4].

In what follows we reproduce parts of his article:

"Consider a triangle ABC and two points B' and C' on its circumcircle. Let A' be the point on the same circle whose Wallace-Simson line is perpendicular to $B'C'$. We say that the triangle $A'B'C'$ is an S triangle with respect to the triangle ABC."

Next we present a few properties of this class of triangles:

1) The algebraic sum of arcs $\widehat{AA'}$, $\widehat{BB'}$, $\widehat{CC'}$ taken in the same sense is equal to zero, i.e.:

$$\widehat{AA'}+\widehat{BB'}+\widehat{CC'}\equiv 0 \pmod{2\pi}.$$

2) The Wallace-Simson line of B' is perpendicular on $C'A'$ and the Wallace-Simson line of C' is perpendicular on $A'B'$.

3) The Wallace-Simson lines of the vertices of the triangle $A'B'C'$ with respect to the triangle ABC are concurrent.

4) The relation between the triangles ABC and $A'B'C'$ is reciprocal.

5) The concurrency point of the Wallace-Simson lines of the points A', B', C' with respect to the triangle ABC and of the Wallace-Simson lines of the points A, B, C with respect to the triangle $A'B'C'$ is the midpoint of the segment line HH', where H and H' are the orthocenters of the triangles ABC and $A'B'C'$, respectively.

In the same **volume** of **Gazeta Matematică** in the Mathematical note with number 28, "About the orthopole", p. 294, Traian Lalescu proved that the midpoint of the segment line HH' is the common orthopole of the sides of the triangle $A'B'C'$ with respect to triangle ABC and of the sides of the triangle ABC with respect to triangle $A'B'C'$ [5].

For this reason **S triangles** are known as **orthopolar triangles**.

Examples of S triangles are found in [1], [2], [6], [8], [10], [11].

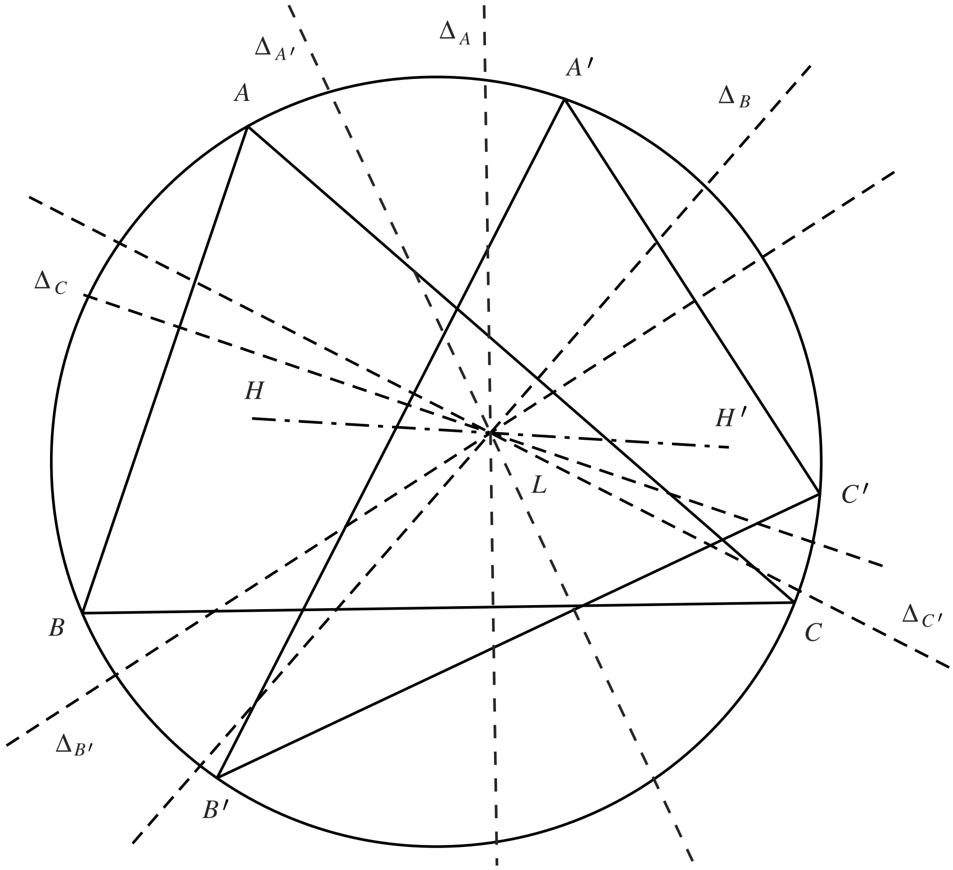

Figure 4

Let us examine the definition and the properties of S triangles from the point of view of complex coordinates.

Consider the triangles ABC and $A'B'C'$ inscribed in the unit circle and let $a, b, c,\ a', b', c'$ the complex coordinates of the points $A, B, C,\ A', B', C'$, respectively.

First of all, we will find the equation of the Wallace-Simson line $\Delta_{A'}$ of the point A' with respect to the triangle ABC.

The equation of the line BC is:

$$z + bc\overline{z} - (b + c) = 0 \Leftrightarrow$$
$$\frac{1}{\overline{b+c}}\,\overline{z} + \frac{1}{b+c}\,z - 1 = 0.$$

The complex coordinate of the projection of A' onto BC is:

$$z = \frac{\dfrac{1}{b+c}\,a' - \dfrac{1}{\overline{b+c}}\,\overline{a'} + 1}{\dfrac{2}{b+c}} \Leftrightarrow$$
$$z = \frac{a' - bc\overline{a'} + b + c}{2}.$$

Let

$$s_1 = a + b + c, \quad s_2 = ab + bc + ca, \quad s_3 = abc,$$
$$s_1' = a' + b' + c', \quad s_2' = a'b' + b'c' + c'a', \quad s_3' = a'b'c'.$$

We obtain

$$z = \frac{1}{2}\left(s_1 - a + a' - \frac{s_3}{a'a}\right)$$
$$\overline{z} = \frac{1}{2}\left(\overline{s_1} - \overline{a} + \overline{a'} - \frac{\overline{s_3}}{\overline{a'}\ \overline{a}}\right) \Leftrightarrow$$
$$\overline{z} = \frac{1}{2}\left(\frac{s_2}{s_3} - \frac{1}{a} + \frac{1}{a'} - \frac{a'a}{s_3}\right).$$

Eliminating a, we get

$$a'z - s_3\overline{z} = \frac{1}{2}\left(a'^2 + s_1 a' - s_2 - \frac{s_3}{a'}\right). \tag{4}$$

This relation contains s_1, s_2, s_3 only, and so is symmetric with respect to a, b, c.
It follows that the relation is also satisfied by the complex coordinates of the feet of the perpendiculars from A' to the sides CA and AB, respectively.

This is an equation of a straight line, hence the projections of A' onto BC, CA and AB, respectively are collinear and the equation we obtained is the equation of the Wallace-Simson line $\Delta_{A'}$ [3], [7].

The equation of $B'C'$ is

$$z + b'c'\overline{z} - (b' + c') = 0$$

and the orthogonality condition of the Wallace-Simson line $\Delta_{A'}$ and $B'C'$ is:

$$b'c' - \frac{s_3}{a'} = 0 \Leftrightarrow$$
$$a'b'c' = abc. \tag{5}$$

Remark. The symmetry of the relation $abc = a'b'c'$ allows us to conclude that the Wallace-Simson line of any vertex of triangle $A'B'C'$ with respect to ABC is orthogonal to the opposite side of the triangle $A'B'C'$ (Property 2) and the same property holds for the vertices of triangle ABC (Property 4).

By setting $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ the arguments of the complex numbers a, b, c, a', b', c', respectively, we obtain from (5)

$$\alpha + \beta + \gamma \equiv \alpha' + \beta' + \gamma' \pmod{2\pi},$$

i.e.

$$(\alpha - \alpha') + (\beta - \beta') + (\gamma - \gamma') \equiv 0 \pmod{2\pi},$$

i.e.

$$\widehat{AA'} + \widehat{BB'} + \widehat{CC'} \equiv 0 \pmod{2\pi}, \quad \text{(Property 1)}.$$

Now let us investigate the concurrency of the Wallace-Simson lines $\Delta_{A'}$, $\Delta_{B'}$, $\Delta_{C'}$ of the points A', B', C', respectively, with respect to the triangle ABC. The equations of $\Delta_{A'}$, $\Delta_{B'}$, $\Delta_{C'}$ are:

$$a'z - s_3\overline{z} = \frac{1}{2}\left(a'^2 + s_1 a' - s_2 - \frac{s_3}{a'}\right)$$
$$b'z - s_3\overline{z} = \frac{1}{2}\left(b'^2 + s_1 b' - s_2 - \frac{s_3}{b'}\right)$$
$$c'z - s_3\overline{z} = \frac{1}{2}\left(c'^2 + s_1 c' - s_2 - \frac{s_3}{c'}\right).$$

The intersection of the first two Wallace-Simson lines is given by:

$$z = \frac{1}{2}\left(a' + b' + s_1 + \frac{s_3}{a'b'}\right)$$

and that of the last two Wallace-Simson lines is given by:

$$z = \frac{1}{2}\left(b' + c' + s_1 + \frac{s_3}{b'c'}\right).$$

Therefore, the necessary and sufficient condition for these two points to coincide is that

$$s_3 = a'b'c, \quad \text{i.e.} \quad abc = a'b'c',$$

the same condition (5) which is equivalent with the definition of S triangles. Hence, for S triangles, the Property 3 is true.
Note that the concurrency point is given by

$$z = \frac{1}{2}(a + b + c + a' + b' + c').$$

This is the midpoint of HH' (Property 5).

Theorem. *The triangles $T_1T_2T_3$ and $S_1S_2S_3$ are S triangles and the complex coordinate of the concurrency point of the six Wallace-Simson lines is:*

$$\frac{1}{2} \cdot \frac{\sigma_2^2 - \sigma_1\sigma_3}{\sigma_3}.$$

Proof. In this case, the condition (5) is true:

$$t_1t_2t_3 = \frac{t_2t_3}{t_1} \cdot \frac{t_3t_1}{t_2} \cdot \frac{t_1t_2}{t_3}.$$

The concurrency point of the Wallace-Simson lines of the points T_1, T_2, T_3 with respect to the triangle $S_1S_2S_3$ and the Wallace-Simson lines of the points S_1, S_2, S_3 with respect to the triangle $T_1T_2T_3$ has the complex coordinate

$$\frac{1}{2}\left(t_1+t_2+t_3) + \frac{t_1t_2}{t_3} + \frac{t_2t_3}{t_1} + \frac{t_3t_1}{t_2}\right) = \frac{1}{2}\left[\sigma_1 + \frac{(t_1t_2)^2 + (t_2t_3)^2 + (t_3t_1)^2}{\sigma_3}\right]$$

$$= \frac{1}{2}\left(\sigma_1 + \frac{\sigma_2^2 - 2\sigma_1\sigma_3}{\sigma_3}\right) = \frac{1}{2} \cdot \frac{\sigma_2^2 - \sigma_1\sigma_3}{\sigma_3}.$$

Theorem. *Let $\{A^*\} = T_2S_2 \cap T_3S_3$, $\{B^*\} = T_3S_3 \cap T_1S_1$, $\{C^*\} = T_1S_1 \cap T_2S_2$. Triangle $T_1T_2T_3$ is the medial triangle and triangle $S_1S_2S_3$ is the orthic triangle of the triangle $A^*B^*C^*$.*

Proof. The equation of the line S_1T_1 is

$$\begin{vmatrix} z & \overline{z} & 1 \\ \dfrac{t_2t_3}{t_1} & \dfrac{\overline{t_2}\,\overline{t_3}}{\overline{t_1}} & 1 \\ t_1 & \overline{t_1} & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$t_1z + t_1t_2t_3\overline{z} - (t_1^2 + t_2t_3) = 0.$$

Similarly the equations of S_2T_2 and S_3T_3 are:

$$t_2z + t_1t_2t_3\overline{z} - (t_2^2 + t_3t_1) = 0$$

and

$$t_3z + t_1t_2t_3\overline{z} - (t_3^2 + t_1t_2) = 0.$$

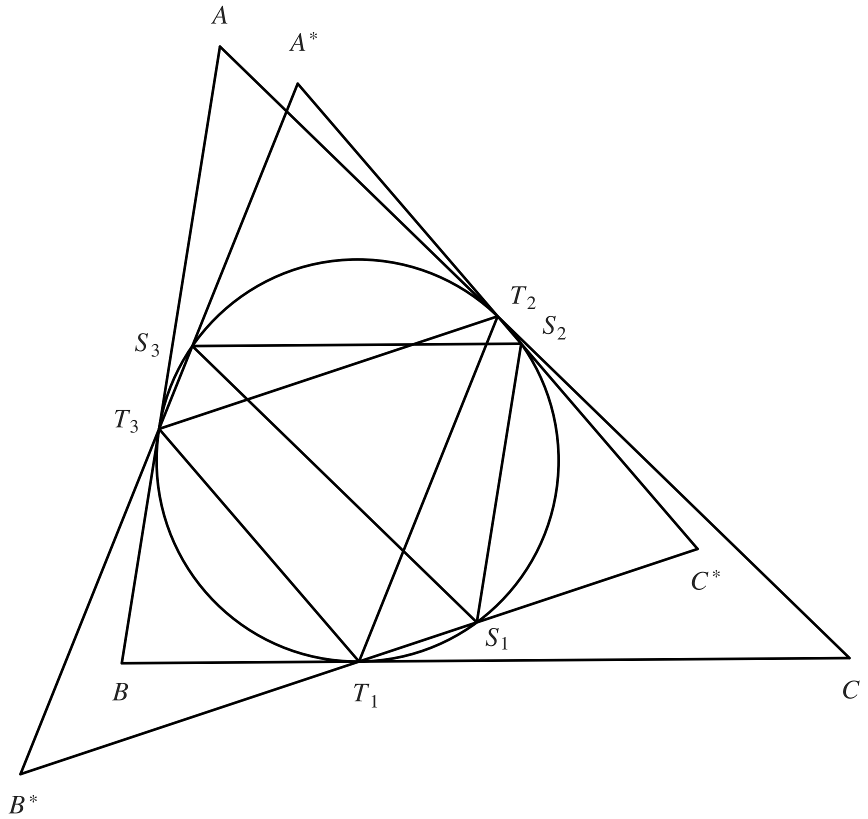

Figure 5

Subtracting the last two equations we obtain the complex coordinate of the point A^*:

$$a^* = t_2 + t_3 - t_1.$$

Similarly, the complex coordinates of B^* and C^* are

$$b^* = t_3 + t_1 - t_2$$

$$c^* = t_1 + t_2 - t_3.$$

The midpoint of the segment line B^*C^* has the complex coordinate

$$\frac{1}{2}(b^* + c^*) = t_1,$$

hence this point is T_1.
Similarly, T_2 and T_3 are the midpoints of the segment lines C^*A^* and A^*B^*. Therefore, the incircle of the triangle ABC is the nine point circle of the triangle $A^*B^*C^*$, and $S_1S_2S_3$ is the orthic triangle of the triangle $A^*B^*C^*$.

Remark. From this theorem we obtain again the statement that $T_1T_2T_3$ and $S_1S_2S_3$ are S triangles, because it is well-known that the medial triangle and the orthic triangle are S triangles in the nine point circle [1], [2], [3], [6].

The common point of the six Wallace-Simson lines is the incenter of the medial triangle of the orthic triangle i.e. the Spiecker point of the orthic triangle $S_1S_2S_3$.

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