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# CONICS ON THE SPHERE

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Abstract. We prove that if the focus of a conic on the sphere is not the pole of the conic's directrix, then the conic can only be quadratic if it is parabolic, and it can not be symmetric.

## 1. Introduction

On the unit sphere $\mathcal{S}^{n-1}$ of $\mathbb{R}^n$ $(n \geq 3)$ a set

$$(D_1) \qquad \mathcal{C}^{\varepsilon}_{F,\mathcal{H}} := \{X \in \mathcal{S}^{n-1} : \varepsilon\hat{\delta}(X, \mathcal{H}) = \delta(F, X)\} \text{ is called a } \textit{conic},$$

where $\hat{\delta}$ is the spherical metric, $\mathcal{H}$, the *directrix*, is a great sphere of $\mathcal{S}^{n-1}$, $F \notin \mathcal{H}$ is a point, the *focus*, and $\varepsilon > 0$ is a number, the *numeric eccentricity.* A conic is said to be *elliptic*, *parabolic* and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$ and $\varepsilon > 1$, respectively.

It is proved in [2, Theorem 4.2 and Theorem 4.3] that if a conic is symmetric in a Minkowski plane, then the Minkowski plane is Euclidean, and further, in [2, Theorem 5.1], that if a conic is quadratic in a Minkowski plane, then the Minkowski plane is Euclidean. At the end of [2] Kurusa conjectures that neither quadratic, nor symmetric conic may exist in projective metric spaces other than the Euclidean one.

In this article we consider the similar problem on the sphere. We support Kurusa's conjecture by proving in Theorem 3.1 that if the focus of a conic on the sphere is not the pole of the conic's directrix, then it can only be quadratic if it is parabolic, and by proving in Theorem 4.1 that if the focus of a conic on the sphere is not the pole of the conic's directrix, then it can not be symmetric.

## 2. Preliminaries

Points of $\mathbb{R}^n$ are denoted as $A, B, \dots$, vectors are $\overrightarrow{AB}$ or $\boldsymbol{a}, \boldsymbol{b}, \dots$, but we use these latter notations also for points if the origin is fixed. The open segment with endpoints $A$ and $B$ is denoted by $\overline{AB} = (A, B)$, $\overline{A}B$ is the open ray starting from $A$ passing through $B$ and the line through $A$ and $B$ is denoted by $AB$.

Notations $\boldsymbol{u}_\varphi = (\cos\varphi, \sin\varphi)$ and $\boldsymbol{u}^{\perp}_{\varphi} := (\cos(\varphi + \pi/2), \sin(\varphi + \pi/2))$ are frequently used. A curve in the plane is called *quadratical*, if it is part of a

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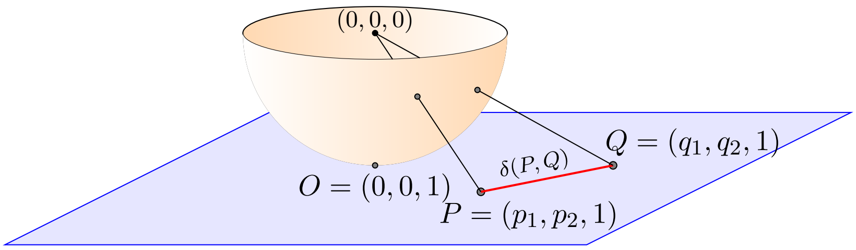
*quadric*

$$(D_q)\qquad \mathcal{Q}^{\sigma}_{\mathfrak{s}} := \left\{ (x,y) : \begin{Bmatrix} 1 = x^2 + \sigma y^2, & \text{if } \sigma \in \{-1,1\}, \\ x = y^2, & \text{if } \sigma = 0, \end{Bmatrix} \right\},$$

where $\mathfrak{s}$ is an affine coordinate system. A quadric is called *ellipse* (*affine circle*), *parabola* and *hyperbola*, if $\sigma = 1$, $\sigma = 0$ and $\sigma = -1$, respectively.

A set $\mathcal{S}$ on the sphere is called *symmetric about a point* $C$, if $X \in \mathcal{S}$ if and only if $Y \in \mathcal{S}$, where $C$ is in the shorter arc $\widehat{XY}$ of the great circle determined and divided into two arcs by $X$ and $Y$, and $S$ is the metric midpoint of the arc $\overline{XY}$.

We use the *gnomonic projection* [5] $\Gamma_O \colon \mathcal{S}^{n-1} \to T_O\mathcal{S}^{n-1}$, where $O \in \mathcal{S}^{n-1}$ and $T_O\mathcal{S}^{n-1}$ is the tangent hyperplane of $\mathcal{S}^{n-1}$ at point $O$.




$\Gamma_O$ projects the spherical metric $\hat{\delta}$ to the metric

$$\delta \colon \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to [0,\pi) \qquad (P,Q) \mapsto \hat{\delta}(P,Q) = \arccos\left(\frac{\langle P,Q\rangle}{|P|\,|Q|}\right). \tag{2.1}$$

Let $\ell$ be a great circle in $(\mathcal{S}^{n-1}, \hat{\delta})$ and let $P \in \mathcal{S}^{n-1} \setminus \ell$. The point $S \in \ell$ is the *$\ell$-foot of* $P$, if $\hat{\delta}(P,X) \geq \hat{\delta}(P,S)$ for every $X \in \ell$. A great circle $\ell'$ intersecting the great circle $\ell$ in a point $S$ is said to be *perpendicular to* $\ell$ if $S$ is an $\ell$-foot of $P$ for every $P \in \ell' \setminus \{S\}$. It is easy to see that this perpendicularity is symmetric. Further, $\ell'$ and $\ell$ are perpendicular to each other if and only if the 2-dimensional subspaces they span are orthogonal.

| From now on we work only in dimension 2 if not explicitly stated otherwise. |
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We usually consider a conic $\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}}$ on $\mathcal{S}^2$, and denote the foot of $\hat{F}$ on the great circle $\hat{\ell}$ by $\hat{F}^{\perp}$. If the focus $F$ of $\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}}$ is not the pole of $\hat{\ell}$, then $\hat{F}^{\perp}$ is unique. In the other case we just pick a point on $\hat{\ell}$ for $\hat{F}^{\perp}$. The correspondent elements obtained by a gnomonic projection $\Gamma$ will be denoted without the hat $\hat{}$.

## 3. Quadratic conics on the sphere

Let $\hat{O}$ be the polar of the great circle $\hat{\ell}$ on the $\mathcal{S}^2$. Let $\hat{F}$ be in the half sphere $\mathcal{S}^2_{\hat{O}}$ of $\hat{\ell}$ that contains $\hat{O}$. Let $\hat{P}$ be on the half circle $\mathcal{G}^2_{\hat{O}}$ of the great circle of $\hat{O}$ and $\hat{F}$ that is contained by $\mathcal{S}^2_{\hat{O}}$. Fix the coordinate system in the plane of $\mathcal{G}^2_{\hat{O}}$ such that $(0,0)$ is the center of $\mathcal{G}^2_{\hat{O}}$, $\hat{O} = (0,1)$, $\hat{F} = (\cos\varphi, \sin\varphi)$ and $\hat{P} = (\cos\varpi, \sin\varpi)$ for some $\varphi \in (0,\pi/2)$ and $\varpi \in (-\pi/2,\pi/2)$. Then

$\hat{P} \in \hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}}$ if and only if

$$\varepsilon = \begin{cases} \frac{\varpi-\varphi}{\pi/2-\varpi}, & \text{if } \varpi \in (\varphi, \pi/2), \\ \frac{\varphi-\varpi}{\pi/2-\varpi}, & \text{if } \varpi \in (0, \varphi), \\ \frac{\varphi-\varpi}{\pi/2+\varpi}, & \text{if } \varpi \in (-\pi/2, 0]. \end{cases}$$

If $\varpi \in (\varphi, \pi/2)$, then $\frac{\varpi-\varphi}{\pi/2-\varpi} = \frac{\pi/2-\varphi}{\pi/2-\varpi} - 1$ is a strictly monotone increasing function of $\varpi$ that vanishes when $\varpi = \varphi$ and tends to infinity when $\varpi \to \pi/2$.

So there is exactly one $\varpi \in (\varphi, \pi/2)$ for which $\hat{P} \in \hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}}$.

If $\varpi \in (0, \varphi)$, then $\frac{\varphi-\varpi}{\pi/2-\varpi} = 1 - \frac{\pi/2-\varphi}{\pi/2-\varpi}$ is a strictly monotone decreasing function of $\varpi$ that tends to $\frac{\varphi}{\pi/2}$ when $\varpi \to 0$ and vanishes when $\varpi = \varphi$.

If $\varpi \in (-\pi/2, 0]$, then $\frac{\varphi-\varpi}{\pi/2+\varpi} = \frac{\pi/2+\varphi}{\pi/2+\varpi} - 1$ is a strictly monotone decreasing function of $\varpi$ that tends to infinity when $\varpi \to \pi/2$ and is $\frac{\varphi}{\pi/2}$ when $\varpi \to 0$.

Thus there is exactly one $\varpi \in (-\pi/2, \varphi)$ for which $\hat{P} \in \hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}}$.

Let $\mathcal{C}^{\varepsilon}_{F,\ell} := \Gamma_{\hat{O}}(\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}})$, $O := \Gamma_{\hat{O}}(\hat{O})$, $F := \Gamma_{\hat{O}}(\hat{F})$, and $\ell := \Gamma_{\hat{O}}(\hat{\ell})$. Choose the coordinate system so that $O = (0,0,1)$ and $F = (f,0,1)$, where $f > 0$. Figure 1 shows what we have on the plane $\mathcal{P} := T_{\hat{O}}\mathcal{S}^2 = \{(x,y,z) : z = 1\}$.

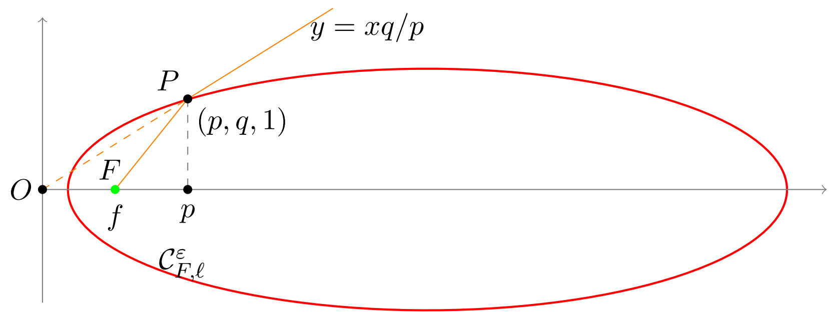


**Figure 1.** Projected conic $\mathcal{C}^{\varepsilon}_{F,\ell}$, if the directrix $\ell$ is in the infinity and the focus $F$ is at $(f,0)$, where $f > 0$.

To calculate the points $(p,q,1) = P = \Gamma_{\hat{O}}(\hat{P})$ of $\mathcal{C}^{\varepsilon}_{F,\ell}$ we have to calculate $\delta(P,\ell)$ and $\delta(F,P)$, where $P \in \mathcal{C}^{\varepsilon}_{F,\ell}$. Observe that the line through $O$ and $P$ is the gnomonic image of the great circle that is perpendicular to $\hat{\ell}$ and going through $\Gamma^{-1}_{\hat{O}}(P)$. Thus, by (2.1), we have

$$\delta(P,\ell) = \frac{\pi}{2} - \delta(P,O) = \frac{\pi}{2} - \arccos\frac{1}{\sqrt{p^2+q^2+1}}. \tag{3.1}$$

For the distance of $P$ from the focus we obtain from (2.1) that

$$\delta(P,F) = \delta(P,(f,0,1)) = \arccos\frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}}. \tag{3.2}$$

According to $D_1$ equations (3.1) and (3.2) give that

$$\varepsilon\Big(\frac{\pi}{2} - \arccos\frac{1}{\sqrt{p^2+q^2+1}}\Big) = \arccos\frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}} \tag{3.3}$$

is the equation of $\mathcal{C}^{\varepsilon}_{F,\ell}$. Figure 2 shows how $\mathcal{C}^{\varepsilon}_{F,\ell}$ looks like for different values of $\varepsilon$.

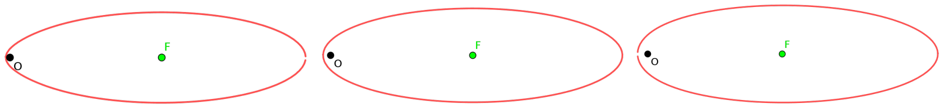


**Figure 2.** An elliptic ($\varepsilon = 0.90$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conic in the projected model of the sphere.

We say that *a conic is quadratic* if it fits on a quadric ($D_q$), hence satisfies an equation of the form $\bar{a}x^2+\bar{b}xy+\bar{c}y^2+\bar{d}x+\bar{e}y+\bar{f}=0$, where the coefficients are real and $\bar{a}\geq 0$.

The parabolic conics (i.e. $\varepsilon = 1$) are quadratic, because taking the cosine of (3.3) results in

$$\sqrt{1-\frac{1}{p^2+q^2+1}} = \Big|\frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}}\Big|,$$

hence by squaring we obtain the clearly quadratic equation $(p^2+q^2)(f^2+1) = pf+1$. To find all the quadratic conics,

$$\boxed{\text{from now on we assume that } \mathcal{C}^{\varepsilon}_{F,\ell} \text{ is quadratic.}}$$

As every conic $\mathcal{C}^{\varepsilon}_{F,\ell}$ is symmetric in the $x$-axis, the quadratic equation should be invariant under changing $y$ to $-y$, so $\bar{b}=\bar{e}=0$ follows. Then the equation is of the form $\bar{a}x^2+\bar{c}y^2+\bar{d}x+\bar{g}=0$, hence $\bar{c}\neq 0$, because otherwise the curve will degenerate into straight lines. So the quadratic equation simplifies to

$$ax^2+y^2+bx+c=0, \quad a\geq 0. \tag{3.4}$$

As this is an ellipse, because it is bounded and intersect line $OP$ in exactly two point, we deduce that

$$a>0 \quad\text{and}\quad b^2>4ac. \tag{3.5}$$

Thus, for a point $P$ of $\mathcal{C}^{\varepsilon}_{F,\ell}$ we have $q^2=-ap^2-bp-c$. Putting this into (3.3) gives the identity

$$\varepsilon\Big(\frac{\pi}{2}-\arccos\frac{1}{\sqrt{(1-a)p^2-bp+1-c}}\Big) = \arccos\frac{(pf+1)/\sqrt{f^2+1}}{\sqrt{(1-a)p^2-bp+1-c}}. \tag{3.6}$$

Differentiating this with respect to $p$ gives

$$\begin{aligned}\varepsilon\frac{-\frac{2(1-a)p-b}{2((1-a)p^2-bp+1-c)^{3/2}}}{\sqrt{1-\frac{1}{(1-a)p^2-bp+1-c}}}\\ = -\frac{-\frac{(2(1-a)p-b)(1+fp)}{2\sqrt{1+f^2}((1-a)p^2-bp+1-c)^{3/2}}+\frac{f}{\sqrt{1+f^2}\sqrt{(1-a)p^2-bp+1-c}}}{\sqrt{1-\frac{(1+fp)^2}{(1+f^2)((1-a)p^2-bp+1-c)}}}.\end{aligned} \tag{3.7}$$

Simplifying the ratios and multiplying both sides by $2(1-c-bp+(1-a)p^2)$ give the equivalent form

$$\begin{aligned}\varepsilon\frac{2(1-a)p-b}{\sqrt{(1-a)p^2-bp-c}}\\ =\frac{-(2(1-a)p-b)(1+fp)+2f((1-a)p^2-bp+1-c)}{\sqrt{((1-a)p^2-bp+1-c)(1+f^2)-(1+fp)^2}}.\end{aligned} \tag{3.8}$$

After the additions and multiplications are completed this becomes

$$\varepsilon\frac{2(1-a)p-b}{\sqrt{(1-a)p^2-bp-c}}=\frac{-(fb+2(1-a))p+2f(1-c)+b}{\sqrt{((1-a)p^2-bp+1-c)(1+f^2)-(1+fp)^2}}. \tag{3.9}$$

Multiplying both sides of (3.9) by the product of the denominators and squaring gives

$$\begin{aligned}\varepsilon^2(2(1-a)p-b)^2((1-a(1+f^2))p^2-(2f+b(1+f^2))p+(f^2-c(1+f^2)))\\ =((fb+2(1-a))p-(b+2f(1-c)))^2((1-a)p^2-bp-c).\end{aligned} \tag{3.10}$$

This equation is valid on an interval of $p$, so the coefficients of the polynomials on the sides are equal, hence

$(p^4)$ $$4\varepsilon^2(1-a)^2(1-a(1+f^2))=(1-a)(fb+2(1-a))^2$$

$(p^3)$ $$\begin{aligned}&4\varepsilon^2\big((1-a)^2(2f+b(1+f^2))+b(1-a)(1-a(1+f^2))\big)\\ &=b(fb+2(1-a))^2+2(1-a)(b+2f(1-c))(fb+2(1-a))\end{aligned}$$

$(p^2)$ $$\begin{aligned}&\varepsilon^2\big(b^2(1-a(1+f^2))+4b(1-a)(2f+b(1+f^2))+\\ &\qquad+4(1-a)^2(f^2-c(1+f^2))\big)\\ &=-c(fb+2(1-a))^2+2b(b+2f(1-c))(fb+2(1-a))+\\ &\quad+(1-a)(b+2f(1-c))^2\end{aligned}$$

$(p^1)$ $$\begin{aligned}&4\varepsilon^2\big(b(1-a)(f^2-c(1+f^2))+b^2(2f+b(1+f^2))\big)\\ &=b(b+2f(1-c))^2-2c(b+2f(1-c))(fb+2(1-a))\end{aligned}$$

$(p^0)$ $$\varepsilon^2b^2(f^2-c(1+f^2))=-c(b+2f(1-c))^2,$$

where $\varepsilon, f>0$ are fixed, and $a>0$, $b^2>4ac$ by (3.5).

If $b=0$, then ($p^4$) implies $a=1$, and ($p^0$) implies $c=0$ or $c=1$. In both cases $\mathcal{C}^{\varepsilon}_{F,\ell}$ is empty by (3.4), and this contradiction proves $b\neq 0$.

If $c=0$, then ($p^0$) implies $b=0$, a contradiction, so $c\neq 0$.

If $a=1$, then ($p^3$) gives $b=0$, a contradiction, so $a\neq 1$.

Dividing ($p^4$) with $1-a$ gives

$$4\varepsilon^2(1-a)(1-a(1+f^2))=(fb+2(1-a))^2, \tag{3.11}$$

Multiplying the sides of (3.11) with the opposite sides of ($p^0$), respectively, we obtain

$$\begin{aligned}-4c(1-a)(b+2f(1-c))^2(1-a(1+f^2))\\ =b^2(f^2-c(1+f^2))(fb+2(1-a))^2.\end{aligned} \tag{3.12}$$

Multiplying (3.11) with $b$ and subtracting the result from ($p^3$) give, after a light simplification, that

$$2\varepsilon^2(1-a)(2f+b(1+f^2)) = (b+2f(1-c))(fb+2(1-a)). \tag{3.13}$$

The right-hand side of the square of (3.13) multiplied by $-c$ is the product of the right-hand sides of ($p^0$) and (3.11), so we get

$$-4c(1-a)(2f+b(1+f^2))^2 = b^2(f^2-c(1+f^2))(1-a(1+f^2)). \tag{3.14}$$

Multiplying (3.13) by $2b$ and subtracting the product from ($p^2$) give

$$\begin{aligned}\varepsilon^2\big(b^2(1-a(1+f^2)) &+ 4(1-a)^2(f^2-c(1+f^2))\big)\\ &= (1-a)(b+2f(1-c))^2 - c(fb+2(1-a))^2.\end{aligned} \tag{3.15}$$

Multiplying (3.11) by $c$ and adding to this give

$$\begin{aligned}\varepsilon^2\big((b^2+4c(1-a))(1-a(1+f^2)) &+ 4(1-a)^2(f^2-c(1+f^2))\big)\\ &= (1-a)(b+2f(1-c))^2.\end{aligned} \tag{3.16}$$

Multiplying this with $c$ and adding to the product of ($p^0$) and $(1-a)$ result in

$$c(b^2+4c(1-a))(1-a(1+f^2)) + (1-a)(b^2+4c(1-a))(f^2-c(1+f^2))) = 0,$$

hence

$$b^2+4c(1-a) = 0 \quad\text{or}\quad 1 = a+c. \tag{3.17}$$

Add ($p^1$) times $1-a$, (3.16) times $-b$ and (3.13) times $2c(1-a)$. The result is

$$4(b^2+c(1-a))(1-a)(2f+b(1+f^2)) = b(b^2+4c(1-a))(1-a(1+f^2)). \tag{3.18}$$

• *Assume that* $1 = a+c$ *fulfills in* (3.17).
Then $b^2+4a(1-c) = b^2+4a^2 > 0$, and it is also easy to show that $f^2-c(1+f^2) = -(1-a(1+f^2))$. Further, (3.12) gives $4c^2(b+2f(1-c))^2 = b^2(fb+2c)^2$, hence

$$\begin{aligned}0 &= b^2(fb+2c)^2 - 4c^2(b+2f(1-c))^2\\ &= (b(fb+2c)-2c(b+2f(1-c)))(b(fb+2c)+2c(b+2f(1-c)))\\ &= f(b^2-4c(1-c))(b(fb+2c)+2c(b+2f(1-c))).\end{aligned}$$

Since $b^2 > 4c(1-c)$ by (3.5), we obtain

$$-b(fb+2c) = 2c(b+2f(1-c)). \tag{3.19}$$

From ($p^3$) we obtain

$$\begin{aligned}4\varepsilon^2 c\big(c(2f+b(1+f^2)) &- b(f^2-c(1+f^2))\big)\\ &= b(fb+2c)^2 + 2c(fb+2c)(b+2f(1-c))).\end{aligned} \tag{3.20}$$

The right-hand side of this equation vanishes by (3.19), so we arrive at

$$c(2f+b(1+f^2)) = b(f^2-c(1+f^2)). \tag{3.21}$$

Since $1-a=c$, (3.13) reads

$$2\varepsilon^2 c(2f+b(1+f^2))=(b+2f(1-c))(fb+2c). \tag{3.22}$$

Multiplying this with $2c$ and using (3.19) results in

$$4\varepsilon^2 c^2(2f+b(1+f^2))=2c(b+2f(1-c))(fb+2c)=-b(fb+2c)^2. \tag{3.23}$$

Using (3.21) and then (3.11) this gives

$$4\varepsilon^2 cb(f^2-c(1+f^2))=-(fb+2c)^2 b=4\varepsilon^2 c(1-(1-c)(1+f^2))b. \tag{3.24}$$

i.e. $f^2-c(1+f^2)=1-(1-c)(1+f^2)$, so, by (3.17) and (3.21), we have

$$c=\frac{f^2}{1+f^2},\quad a=\frac{1}{1+f^2},\quad \text{and}\quad b=\frac{-2f}{1+f^2}. \tag{3.25}$$

This, however gives $b^2=4ac$ that contradicts (3.5).
• *Assume that* $1\neq a+c$ *and* $b^2+4c(1-a)=0$ *fulfills in* (3.17).
As a first consequence, we get from (3.18) that

$$b^2+c(1-a)=0 \ \text{ or }\ 2f+b(1+f^2)=0. \tag{3.26}$$

In the former case the assumption implies $c(1-a)=0$, so either $c=0$ or $a=1$, which was already closed out, so we deduce $b=\frac{-2f}{1+f^2}$. From this (3.14) implies immediately that $0=(f^2-c(1+f^2))(1-a(1+f^2))$, so

$$c=\frac{f^2}{1+f^2}\ \text{ or }\ a=\frac{1}{1+f^2}. \tag{3.27}$$

Then, by the assumption we respectively obtain that

$$a=\frac{4c+b^2}{4c}=\frac{2+f^2}{1+f^2}\ \text{ and }\ c=\frac{-b^2}{4(1-a)}=\frac{-1}{1+f^2}. \tag{3.28}$$

In the first case we get

$$b^2-4ac=\frac{4f^2}{(1+f^2)^2}-4\frac{2+f^2}{1+f^2}\frac{f^2}{1+f^2}=\frac{4f^2}{(1+f^2)^2}(1-(2+f^2))=\frac{-4f^2}{1+f^2}<0$$

that contradicts (3.5), so we deduce

$$c=\frac{-1}{1+f^2},\quad a=\frac{1}{1+f^2},\quad \text{and}\quad b=\frac{-2f}{1+f^2}. \tag{3.29}$$

With these values ($p^0$) gives $\varepsilon=1$. ($p^1$) gives also $\varepsilon=1$.

Thus the second case in (3.17) implies contradiction, while it follows from the first case of (3.17) that the conic is parabolic.

The contradiction means that the system of equations ($p^0$)–($p^4$) does not have solution, so the polynomials of the sides in (3.10) are different, hence the conics in this case are not quadratic.

We conclude our first theorem:

**Theorem 3.1.** *A conic on the sphere is quadratic if and only if either the focus is the pole of the directrix, or the focus is not the pole of the directrix, but the conic is parabolic, i.e.* $\varepsilon=1$.

## 4. Symmetric conics on the sphere

Firstly we notice that the conic on the sphere is a hypersphere, hence symmetric if the focus is the pole of the directrix, so we assume for the sake of a later contradiction that

$$\boxed{\hat{F} \text{ is not the pole of } \hat{\ell}, \text{ and } \hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}} \text{ is symmetric in a point } \hat{C}.}$$

Such a point of symmetry $\hat{C}$ clearly is on the great circle of $\hat{F}\hat{F}^{\perp}$, where $\hat{F}^{\perp}$ is the unique foot of $\hat{F}$ on the great circle $\hat{\ell}$.

Take the gnomonic projection $\Gamma_{\hat{C}}$. Let $\mathcal{C}^{\varepsilon}_{F,\ell} := \Gamma_{\hat{C}}(\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}})$, $P := \Gamma_{\hat{C}}(\hat{P})$ and $P^{\perp} := \Gamma_{\hat{C}}(\hat{P}^{\perp})$ for any point $P$, and $\ell := \Gamma_{\hat{C}}(\hat{\ell})$. Choose the coordinate system so that $C = (0,0,1)$, $F = (f,0,1)$, and $\ell = \{(x,y,z) : x = m \wedge z = 1\}$. Figure 3 shows what we have on the plane $\mathcal{P} := T_{\hat{C}}\mathcal{S}^2 = \{(x,y,z) : z = 1\}$.

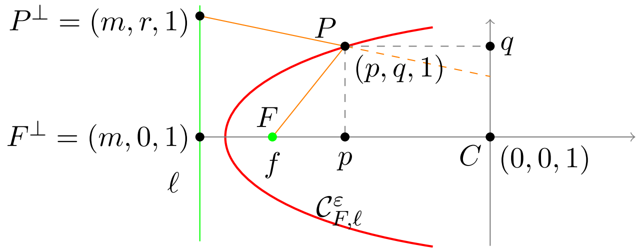


**Figure 3.** Projected conic $\mathcal{C}^{\varepsilon}_{F,\ell}$, if the directrix $\ell$ is parallel to the $y$-axis and the focus $F$ is at $(f,0)$, where $f < 0$.

The advantage of taking the gnomonic projection $\Gamma_{\hat{C}}$ is that $\hat{\mathcal{C}}^{\varepsilon}_{\hat{\delta};\hat{F},\hat{\ell}}$ is symmetric about $\hat{C}$ in the spherical meaning if and only if $\mathcal{C}^{\varepsilon}_{F,\ell}$ is symmetric about $C$ in the Euclidean meaning.

Since $\hat{F}^{\perp}\hat{P}^{\perp}\hat{P}$ is a right triangle on the sphere, the cosine rule for the spherical triangle [6] gives $\cos\hat{\delta}(\hat{F}^{\perp},\hat{P}) = \cos\hat{\delta}(\hat{F}^{\perp},\hat{P}^{\perp})\cos\hat{\delta}(\hat{P}^{\perp},\hat{P})$. Then (2.1) gives

$$\cos(\hat{\delta}(\hat{F}^{\perp},\hat{P})) = \frac{\langle \hat{F}^{\perp},\hat{P}\rangle}{|\hat{F}^{\perp}|\,|\hat{P}|},\ \cos(\hat{\delta}(\hat{F}^{\perp},\hat{P}^{\perp})) = \frac{\langle \hat{F}^{\perp},\hat{P}^{\perp}\rangle}{|\hat{F}^{\perp}|\,|\hat{P}^{\perp}|},\ \cos(\hat{\delta}(\hat{P}^{\perp},\hat{P})) = \frac{\langle \hat{P}^{\perp},\hat{P}\rangle}{|\hat{P}^{\perp}|\,|\hat{P}|},$$

so we obtain $\langle \hat{F}^{\perp},\hat{P}\rangle\,|\hat{P}^{\perp}|^2 = \langle \hat{F}^{\perp},\hat{P}^{\perp}\rangle\,\langle \hat{P}^{\perp},\hat{P}\rangle$, i.e.

$$(mp+1)(m^2+r^2+1) = (m^2+1)\,(mp+rq+1).$$

This equation is equivalent to equation $r(r(mp+1) - q(m^2+1)) = 0$ that gives

$$r = \frac{q(m^2+1)}{mp+1}. \tag{4.1}$$

Thus, by (2.1), we have

$$\begin{aligned}\delta(P,\ell) &= \delta(P,(m,r,1)) \\ &= \arccos \frac{mp+rq+1}{\sqrt{m^2+r^2+1}\sqrt{p^2+q^2+1}} = \arccos \frac{(mp+1)\sqrt{m^2+r^2+1}}{(m^2+1)\sqrt{p^2+q^2+1}} \\ &= \arccos \frac{\sqrt{(mp+1)^2+q^2(m^2+1)}}{\sqrt{m^2+1}\sqrt{p^2+q^2+1}},\end{aligned} \tag{4.2}$$

where we used (4.1) and its predecessor. For the distance of $P$ from the focus we have (3.2).

According to $(D_1)$ equations (4.2) and (3.2) give

$$\varepsilon \arccos \frac{\sqrt{(mp+1)^2+q^2(m^2+1)}}{\sqrt{m^2+1}\sqrt{p^2+q^2+1}} = \arccos \frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}}. \tag{4.3}$$

Figure 4 shows how these conics look like by (3.3).

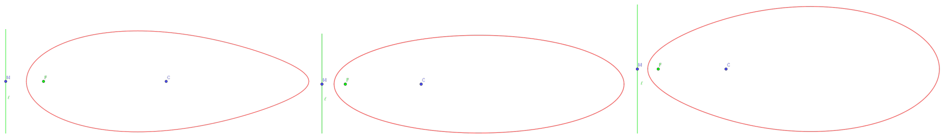

**Figure 4.** An elliptic ($\varepsilon = 0.90$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conic in projected model of the sphere.

We now that there exist exactly two solutions of (4.3) for $q = 0$, and by the symmetry these are $\pm p_0$. Thus $\pm p_0$ satisfies

$$\varepsilon \arccos \frac{|mp+1|}{\sqrt{m^2+1}\sqrt{p^2+1}} = \arccos \frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+1}}. \tag{4.4}$$

Let $m = \tan\mu$, $f = \tan\varphi$, and $p_0 = \tan\varpi$. Substituting these values into (4.4) results in

$$\varepsilon \arccos \frac{|1 \pm \tan\mu \tan\varpi|}{\sqrt{1+\tan^2\mu}\sqrt{1+\tan^2\varpi}} = \arccos \frac{1 \pm \tan\varphi \tan\varpi}{\sqrt{1+\tan^2\varphi}\sqrt{1+\tan^2\varpi}}, \tag{4.5}$$

i.e.

$$\varepsilon \arccos |\cos\mu\cos\varpi \pm \sin\mu\sin\varpi| = \arccos(\cos\varphi\cos\varpi \pm \sin\varphi\sin\varpi), \tag{4.6}$$

hence by the angle sum and difference identities [7] we get

$$\varepsilon \arccos |\cos(\mu \mp \varpi)| = \arccos(\cos(\varphi \mp \varpi)), \tag{4.7}$$

Thus, $\varepsilon\mu - \varphi = \pm\varpi(1-\varepsilon)$, hence $\varpi(1-\varepsilon) = 0$. Since $\varpi \neq 0$, we get $\varepsilon = 1$, so $\mu = \varphi$ that is a contradiction.

In sum, we have proved the following theorem.

**Theorem 4.1.** *A conic on the sphere is symmetric if and only if the focus is the pole of the directrix.*

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