Apollonius “circle” in Spherical Geometry

EUGEN J. IONĂȘCU

Abstract. We investigate the analog of the circle of Apollonius in spherical geometry. This can be viewed as the pre-image through the stereographic projection of an algebraic curve of degree three. This curve consists of two connected components each being the “reflection” of the other through the center of the sphere. We give an equivalent equation for it, which is surprisingly, this time, of degree four.

1. Introduction

It is well known that “the set of all points whose distances from two fixed points are in a constant ratio $\rho (\rho \neq 1)$” (see [1], [6]) is a circle which is one of few named after Apollonious. In Euclidean geometry this is equivalent to asking for the locus of points $P$ satisfying $\angle APB \equiv \angle BPC$, given fixed collinear points $A, B$ and $C$. This same locus is the focus of our investigation but in spherical geometry. We recall that the lines in spherical geometry are big circles (intersection of the sphere with planes passing through its center) and the angle between two lines is the dihedral angle between the planes containing them.

In Euclidean geometry, the connection between the two formulations is done by the Angle Bisector Theorem: “The angle bisector in a triangle divides the opposite sides into a ratio equal to the ratio of the adjacent sides.” Once one realizes that the statement can be equally applied to the exterior angle bisector, then the Circle of Apollonius appears naturally (Figure 1), since the two angle bisectors are perpendicular.

For instance, an easy exercise in algebra shows that the circle of equation $x^2 + y^2 = 4$ is equivalent to

$$\frac{\sqrt{x^2 + (y - 4)^2}}{\sqrt{x^2 + (y - 1)^2}} = \frac{BA}{BC} = \frac{DA}{DC} = 2,$$

taking $A(0, 4), B(0, 2), C(0, 1)$ and $D(0, -2)$.

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Its inverse has a simple form also:

We are going to work on the unit sphere in $\mathbb{R}^3$ (Cartesian coordinates $x^2 + y^2 + z^2 = 1$). The plane (Figure 3) of the stereographic projection ($SP$) is $z = 0$ and so $SP$ it is defined by

$$SP(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right), \quad (x, y, z) \neq (0, 0, 1).$$

Its inverse has a simple form also:
Figure 3. Unit sphere, stereographic projection onto \( z = 0 \)

\[
SP^{-1}(u, v) = \left( \frac{2u}{v^2 + 1 + u^2}, \frac{2v}{v^2 + 1 + u^2}, \frac{u^2 + v^2 - 1}{v^2 + 1 + u^2} \right), \quad (u, v) \in \mathbb{R}^2(z = 0).
\]

One of the interesting facts in spherical geometry is that, given three collinear points, neither one can be considered to be between the other two. This is related to one of the axioms in David Hilbert’s axiomatic system of Euclidean geometry: “given three collinear points, one is between the other two”. But what do we mean in spherical geometry by “between”? We will define this term in the following way: \textit{we say that A is between B and C if A is on the smaller arc determined by B and C on the line \( BC \).}

So, in the next section, having three collinear points \( A, B \) and \( C \) we are going to consider the locus relative to the point \( C \), under the assumption that \( A \) is not between \( C \) and \( B \) and \( B \) is not between \( C \) and \( A \). Clearly, if any of these conditions is not satisfied the locus is the empty set.

Hence, the restriction on a point \( P \) (not collinear with \( A, B \) and \( C \)) is then \( \angle APC \equiv \angle CPB \) and without loss of generality we are going to fix \( C \) as the projection point, i.e., \( C = (0, 0, 1) \). We may also assume that the points \( A \) and \( B \) are on a big circle passing through \( C \), which projects along the \( x \)-axis in the plane \( z = 0 \). So, we can take \( A = (\cos \alpha, 0, \sin \alpha) \) and \( B = (\cos \beta, 0, \sin \beta) \), with \( \alpha, \beta \in [0, 2\pi) \setminus \{\frac{\pi}{2}\} \).

The two important properties of the stereographic projection are:

\( (a) \) \textit{circles on the sphere (which do not contain C) are projected onto circles in the plane, (in particular, the big circles) } \\
\( (b) \) \textit{angles between curves on the sphere are preserved (it is a conformal map) } \\

2. Main Locus, Characterizations and Properties

Let us observe that any line on the sphere which passes through a point \( A \), it passes through the point \( \tilde{A} \), the opposite point on the sphere across its center. It is easy to see that, our condition on the points \( A \) and \( B \) relative
to $C$, is equivalent to require that they are on opposite arcs determined by $C$ and $\tilde{C} = (0,0, -1)$. So, we may assume that, say $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\beta \in (\frac{\pi}{2}, \frac{3\pi}{2})$.

We denote by $A' = SP(A)$, $B' = SP(B)$ and $O = SP(\tilde{C})$. The coordinates of these points are $A' = (\frac{\cos \alpha}{1 - \sin \alpha}, 0)$, $B' = (\frac{\cos \beta}{1 - \sin \beta}, 0)$, and $O = (0,0)$.

Also, let us denote by $A'' = SP(A)$ and $B'' = SP(B)$, whose coordinates are $A'' = (\frac{\cos \alpha}{1 + \sin \alpha}, 0)$, and $B'' = (\frac{\cos \beta}{1 + \sin \beta}, 0)$.

Let us consider an arbitrary point $P'$ of coordinates $(u,v)$ in the plane $z = 0$, which is the projection of point $P$ on the sphere. Then the line $\overline{PC}$ on the sphere projects into the line $\overline{PO}$.

![Figure 4. The curves in $z = 0$](image)

We can clearly rotate the big circle $\overline{AB}$ with a $90^\circ$ angle around the $x$-axis, to appear in the plane $z = 0$, as in Figure 4. This way, we have basically a plane problem. The line $\overline{AB}$ on the sphere becomes the unit circle. Since the line $\overline{PA}$ is also passing through $A$, it is projected into a circle which passing through $A'$, $A''$ and $P'$. In other words, the line $\overline{PA}$ is projected onto the circle circumscribed to the triangle $A'A''P'$. Similarly, the line $\overline{PB}$ is projected onto the circle circumscribed to the triangle $B'B''P'$. The line $\overline{PC}$ is projected onto the line $\overline{PO}$. We have included these two circles in Figure 4. Since the angles $\angle CPA$ and $\angle CPB$ must be congruent, it follows that, this
is equivalent to asking that the line $OP'$ makes congruent angles with the two circles mentioned earlier. On the other hand, the point $P'' = SP(P)$ is on these two circles and so the line $OP'P''$ is the line determined by the intersection of these two circles.

Clearly, the line determined by the intersection of two (distinct) circles makes the same angle with the two circles if and only if the two circles are congruent. So, we have a metric description of our locus.

**Proposition 2.1.** Under our setup, and assuming that $A$ is not $\bar{B}$, the set of all points $P$ such that $\angle CPA \equiv \angle CPB$ is the set of points $P$ such that the two circles $SP(PA)$ and $SP(PB)$ are congruent.

In general, for a triangle $ABC$ in the Euclidean plane, the radius $R$ of its circle circumscribed, is satisfies

$$2R = \frac{a}{\sin A} = \frac{a}{h/b} = \frac{ab}{h}$$

where $h$ is the height corresponding to the side $BC$.

Then the equation that $u$ and $v$ must satisfy is

$$P'_A \cdot P''_A = P'_B \cdot P''_B$$

or

$$\left( u - \frac{\cos \alpha}{1 - \sin \alpha} \right)^2 + v^2 \left( u + \frac{\cos \alpha}{1 - \sin \alpha} \right)^2 = \left( u - \frac{\cos \beta}{1 - \sin \beta} \right)^2 + v^2 \left( u + \frac{\cos \beta}{1 - \sin \beta} \right)^2.$$

**Theorem 2.2.** Under our setup, the set of all points $P$ on the sphere such that $\angle CPA \equiv \angle CPB$, not on the line $\overline{AB}$, is the set of points $P$ whose stereographic from $C$ onto the plane has coordinates $(u, v)$ ($|v| > 0$) satisfying

$$(2) \quad (u - k)v^2 + u(u^2 - ku - 1) = 0,$$

where $k = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}$ if $\beta \neq \alpha + \pi$, or

$$(3) \quad k = 2 \tan \alpha, \quad \text{if} \quad \beta = \alpha + \pi.$$

**Proof.** It is not difficult to argue that (3) follows from (2) using a limiting argument. From (1) expanding the left hand side, we obtain

$$v^4 + 2v^2(u^2 - 2u \tan \alpha + 1 + 2 \tan^2 \alpha) + (u^2 - 2u \tan \alpha - 1)^2.$$

A similar expression can be written for the right hand side of (1), and subtracting from the one above we obtain:

$$4v^2[(\tan \beta - \tan \alpha)u - (\tan^2 \beta - \tan^2 \alpha)]$$

$$+ 4u(\tan \beta - \tan \alpha) [u^2 - u(\tan \alpha + \tan \beta) - 1] = 0.$$

If we observe that $k = \tan \alpha + \tan \beta$, and under the assumption that $\beta \neq \alpha + \pi$, the above equality can be simplified by a factor of $\tan \beta - \tan \alpha$, then (2) follows.

The particular case $\beta = \alpha + \pi$ is special because the two curves on the sphere pass through the points $C, \bar{C}, A$ and $B$ (see Figure 5 (b)). The cases $\beta = \pi \pm \alpha$ are essentially different from analogous situations in the Euclidean geometry.
Corollary 2.3. If the point \( C \) is equally distant to \( A \) and \( B \) then the locus of all points \( P \) on the sphere but not on the line \( \overline{AB} \), such that \( \angle CPA = \angle CPB \), is the set of points \( P \) whose preimage of the stereographic projection is the set \( u = 0 \) and the unit circle \( u^2 + v^2 = 1 \) \((|v| > 0)\) (Figure 5 (a)).

**Proof.** In this case \( \beta = \pi - \alpha \) and so \( k = 0 \). The equation of the locus (2) becomes \( u(u^2 + v^2 - 1) = 0 \), and so \( u = 0 \) or \( u^2 + v^2 = 1 \).

The plane curve \( C \) in (2) looks like the one in Figure 6). Its preimage through \( SP \) consists of two connected curves on the sphere passing through \( C \) and \( \tilde{C} \) and intersecting \( \overline{AB} \) at the points \( D := SP^{-1}(\frac{k}{2} - \sqrt{1 + \frac{k^2}{4}}, 0) \) and \( E := SP^{-1}(\frac{k}{2} + \sqrt{1 + \frac{k^2}{4}}, 0) \). Assuming that \( k > 0 \), let us denote by

\[
u_1 = \frac{k}{2} - \sqrt{1 + \frac{k^2}{4}}, \quad \text{and} \quad \nu_2 = \frac{k}{2} + \sqrt{1 + \frac{k^2}{4}}.
\]

Because \( \nu_1 \nu_2 = -1 \) then the points \( D \) and \( E \) are opposite to each other across the center of the sphere, i.e. \( E = \tilde{D} \). This property is more general.
Proposition 2.4. The curve $SP^{-1}(C)$ is symmetric with respect to the center of the sphere. The curve $C$ is invariant under the inversion $X \sim Y$ (Y is defined by $OY \cdot OX = -1$).

Proof. The proof follows from Theorem 2.2 and the simple observation that $P$ is on the locus if and only if $P$ is. The second statement is a consequence of the fact that $P$ is opposite to $Q$ on the sphere, if and only if $SP(P) = Inv(SP(Q))$.

Let us introduce the angle $\gamma \in [0, \frac{\pi}{2})$ by $\tan(\gamma) = \frac{k}{2}$ ($k \geq 0$).

Proposition 2.5. The topological closure of the curve $SP^{-1}(C)$ intersects $AB$ at the points $\pm (\cos \gamma, 0, \sin \gamma)$, $C$, and $\tilde{C}$.

Proof. From what have done earlier, it is enough to check that $SP^{-1}(u_2) = (\cos \gamma, 0, \sin \gamma)$. Since $u_2 = \tan \gamma + \sec \gamma$, we can calculate

\[1 + u_2^2 = 2 \sec^2 \gamma + 2 \sec \gamma \tan \gamma = 2u_2 \sec \gamma \Rightarrow SP^{-1}(u_2) = (\frac{2u_2}{1 + u_2^2}, 0, \frac{u_2^2 - 1}{1 + u_2^2}) = (\cos \gamma, 0, 1 + u_1 \cos \gamma) = (\cos \gamma, 0, \sin \gamma).

The rest of the statement follows from Proposition 2.4 and the fact that $(0, 0) \in C$.

It is important to write $\gamma$ just in terms of the angles $\angle COA = \hat{\alpha}$ and $\angle COB = \hat{\beta}$. We just have to substitute, $\alpha = \frac{\pi}{2} - \hat{\alpha}$ and $\beta = \frac{\pi}{2} + \hat{\beta}$. Then, we get

\[\tan \gamma = \frac{1 \sin(\hat{\beta} - \hat{\alpha})}{2 \sin \hat{\alpha} \sin \hat{\beta}}.

Also, if $\hat{\gamma} = \frac{\pi}{2} - \gamma = \angle COE$, we have an intrinsic formula which defines $E$:

\[\cot \hat{\gamma} = \frac{1 \sin(\hat{\beta} - \hat{\alpha})}{2 \sin \hat{\alpha} \sin \hat{\beta}}.

3. The same locus, but different angle

In this section, given the three collinear points $A$, $B$, and $C$, we are going to consider the locus relative to the point $B$, under the assumption that $C$ is not between $A$ and $B$ and $B$ is not between $C$ and $A$.

Hence, the restriction on a point $P$ (not collinear with $A$, $B$ and $C$) is then $\angle CPA \equiv \angle APB$. $C$ is still the projection point. As before, we assume that the points $A$ and $B$ are on a big circle passing through $C$, which projects along the $x$-axis in the plane $z = 0$. In this case we can take $A = (\cos \alpha, 0, \sin \alpha)$ and $B = (\cos \beta, 0, \sin \beta)$, with $\beta \in (-\pi/2, \pi/2)$ and $\alpha \in (\beta, 2\pi)$.

As before, we are going to start with a point $P' = (u, v)$ in the plane $z = 0$. Using the same notations, $A' = (\frac{\cos \alpha}{1 - \sin \alpha}, 0)$, $B' = (\frac{\cos \beta}{1 - \sin \beta}, 0)$, $A'' = (\frac{-\cos \alpha}{1 + \sin \alpha}, 0)$, and $B'' = (\frac{-\cos \beta}{1 + \sin \beta}, 0)$. We are going to use an analytic approach this time. The condition $\angle CPA \equiv \angle APB$ translates into (Figure 7) $\angle C'P'A' \equiv \angle A''P'B''$. If $O_1$ and $O_2$ are the centers of the circles circumscribed to the triangles $P'A'A''$ and $P'B'B''$ respectively, the angle
\( \angle \hat{A}'P'P''_1B' \) is the same as the angle \( \angle O_1P''_1O_2 \). In order to calculate the coordinates of \( O_1 \), one has to compute the equations of the perpendicular bisectors of \( \overline{AA''} \) and \( \overline{PP''} \). Let us denote by \( d = u^2 + v^2 \). The coordinates of \( P''_1 \) are \((-\frac{u}{a}, -\frac{v}{a}) \) (note that \( \overline{OP''_1OP''} = -1 \)). The perpendicular bisector of \( \overline{AA''} \) has equation \( X = \tan \alpha \), and the perpendicular bisector of \( \overline{PP''} \) is given by \( Y - \frac{(d-1)v}{2d} = -\frac{v}{u} \left[ X - \frac{d-1)u}{2d} \right] \). This means that the point \( O_1 \) has coordinates \((\tan \alpha, \frac{d-1-2u\tan \alpha}{2v})\), and similarly the point \( O_2 \) has coordinates \((\tan \beta, \frac{d-1-2u\tan \beta}{2v})\).

**Proposition 3.1.** Under the above setting, the locus of the points \( P \), non-collinear with \( \overline{AB} \), is characterized by the equation

\[
(u^2 + v^2)^2 + 4 \left[ (2 \tan(t) - u) \tan s - \tan^2 t \right] (u^2 + v^2) + 2v^2 + 1 - 2u^2 + 4u \tan s = 0.
\]

Figure 7. Another angle

It is somewhat surprising that (5) and (3) represent the same curve on the sphere. This second version allows us to plot these curves that appear in terms of four points.

In Figure 8, we plotted these two curves calculated for angles \( \frac{5\pi}{18} \), \( \frac{\pi}{2} \), and \( \frac{\pi}{3} \) the curve (3) (in blue), and points \( \frac{\pi}{2} \), \( \frac{\pi}{3} \) and \( \frac{\pi}{4} \) for the curve in red, given by
Figure 8. Two curves

(5). The two curves appear to be tangent to one another and because of the concavity we can conclude that in general two such curves (determined by collinear points $D, C, A$ and $B$—first three and the last three) are intersecting at four points, tangent at two points, or no intersection.

4. FOUR POINTS “EQUALLY” SPACED AND MOTIVATION

Our interest in this locus was motivated by the Problem 11915 in this Monthly ([5]). This problem stated: Given four (distinct) points $A, B, C$ and $D$ in (this) order on a line in Euclidean space, under what conditions will there be a point $P$ off the line such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure?

It is not difficult to show, using two Apollonius circles, that the existence of such a point $P$ is characterized by the inequality involving the cross-ratio

(6) 
$$[A, B; C, D] = \frac{BC}{DC} \frac{BA}{DA} < 3.$$ 

We were interested in finding a similar description for the same question in Hyperbolic space. In [4], we showed that given four points $A, B, C$ and $D$ in (this) order on a line in the Hyperbolic space, we can use an isometry to transform them on the line $x = 0$ and having coordinates $A(0, a)$, $B(0, b)$, $C(0, c)$ and $D(0, d)$ with $a > b > c > d$. The existence of a point $P$ off the line $x = 0$, such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure in the Hyperbolic space is equivalent to the existence of $P$ in Euclidean space corresponding to the points $A', B', C'$ and $D'$ as constructed in the proof of Theorem 1.2. It turns out that a simple answer may be formulated using a similar inequality

(7) 
$$[A', B'; C', D'] = \frac{B'C'}{B'A'} \frac{D'C'}{D'A'} < 3 \Leftrightarrow \frac{(b^2 - c^2)(a^2 - d^2)}{(a^2 - b^2)(c^2 - d^2)} < 3.$$ 

To have a different take of what (6) means we will translate it into a geometric probability which is not difficult to compute: if two points are
randomly selected (uniform distribution) on the segment $AD$, then the probability that a point $P$ off the line $AD$ such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure exists (where the two points are denoted by $B$ and $C$, $B$ being the closest to $A$), is equal to

$$P_e = \frac{15 - 16 \ln 2}{9} \approx 0.4345$$

The inequality (7) gives us the similar probability in the Hyperbolic space:

$$P_h = \frac{2\sqrt{5} \ln(2 + \sqrt{5}) - 5}{5 \ln 2} \approx 0.420154924$$

where the uniform distribution here means, it is calculated with respect to the measure $\frac{1}{y} dy$ along the $y$-axis.

In spherical geometry we can formulate a similar probability question: “Given four collinear points (uniform distribution over the big circle on the sphere), what is the probability that there exists a point from which the four points can be seen as equally spaced?”

To be more precise, when we are given four collinear points on a sphere, say $A$, $B$, $C$, and $D$ in clockwise order, we consider here all four possibilities

$$\angle APB \equiv \angle BPC \equiv \angle CPD \quad \text{or} \quad \angle BPC \equiv \angle CPD \equiv \angle DPA \quad \text{or}$$

$$\angle CPD \equiv \angle DPA \equiv \angle APB \quad \text{or} \quad \angle DPA \equiv \angle APB \equiv \angle BPC,$$

the point $P$ may or may not be the same in these equalities. The probability here is certainly not that difficult to compute, but with so many cases that one needs to consider, it is definitely a challenge which we leave for a future study or the interested reader.

References


EUGEN J. IONASCU

DEPARTMENT OF MATHEMATICS
COLUMBUS STATE UNIVERSITY
COLUMBUS, 31907 GA, US
E-mail address: math@ionascu.ro