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ON A SIX-POINT CIRCLES FAMILY FOR THE TRIANGLE

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Abstract. The author presents three special six-point circles pertaining to the triangle geometry. Each circle passes through four feature points of the generalized Bride's Chair configurations of a given triangle [2], and the midpoints of two sides of the triangle.
In the present work, the author, first calls to mind The Six-Point Circle Theorem For The Quadrangle, then, with a reasoning based on a continous transformation of the quadrangle configuration, presents the new triangle six-point circles.
The direct synthetic proofs of the theorems, presented in section 2, are given in section 3 for completeness.

1. The Six-Point Circle Theorem For The Quadrangle

Looking at figures 1 and 2, we recall the following:

Theorem 1.1 (The Six-Point Circle Theorem For The Quadrangle)**.** *In any given quadrangle ABCD, the midpoints of the quadrangle diagonals, E and F, the first and second Van Aubel points* [5]*, V and V', and the midpoints of the Van Aubel segments* [5]*, X and Y, lie on a circle.*

Corollary 1.1. *The Newton segment* [5]*, EF, and the segment connecting the midpoints of the Van Aubel segments, XY, are two mutually orthogonal diameters of the circle.*

Remark 1.1. *The center of the circle, O, coincides with the quadrangle centroid* [1].

The circle is shown in figures 1 and 2. A synthetic proof of the theorem can be found in [5].

The following results hold true as a consequence of the Van Aubel Theorem [6][5]:

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- the Van Aubel segments O_1O_3 and O_2O_4 (intersecting at V) are of equal length and mutually orthogonal;
- the Van Aubel segments $O'_1O'_3$ and $O'_2O'_4$ (intersecting at V' on the extension of segment $O'_1O'_3$) are of equal length and mutually orthogonal (see figure 2).

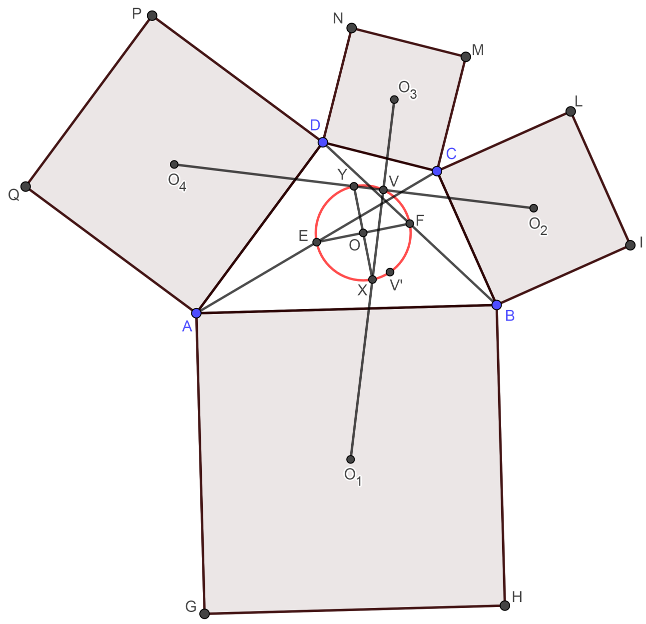

FIGURE 1. The Six Point Circle For The Quadrangle is passing through the midpoints of the quadrangle's diagonals, E and F, the midpoints of the Van Aubel segments, X and Y, and the Van Aubel points, V and V'. The definitions for the Van Aubel segments and the Van Aubel points are given in [5].

1.1. **Abuse of notation.** In reference [5], the author has abused notation regarding the concept of line segments' intersection. While two segments do not always intersect (i.e. $O'_1O'_3$ and $O'_2O'_4$ represented in figure 2), the author assumed that the intersection point of any Van Aubel segment pair is always defined.
The abuse of notation is resolved in the following way: when the two segments of a Van Aubel pair do not intersect, their extensions have to be considered. These segment extensions (also referred to as produced line segments) do intersect because they are orthogonal to each other, as stated by the Van Aubel theorem. The intersection point of the produced line segments is therefore always defined and it can be identified with a Van Aubel point.

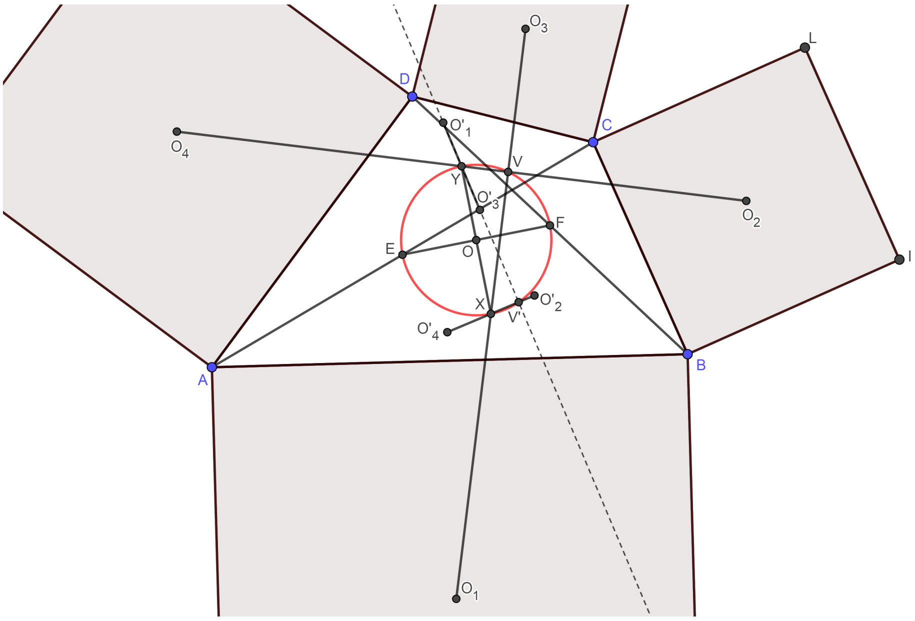

FIGURE 2. The Six Point Circle For The Quadrangle - enlargement showing the construction for the second Van Aubel point. The internal squares are not shown for the sake of clarity, their centers are represented by the points O'_i, with i=1,2,3,4.

2. Three New Six-Point Circles For The Triangle

Definition 1. *In any given triangle, on each side construct a square external to the triangle. The obtained configuration will be named as the generalized external Bride's Chair configuration* [2].

Definition 2. *In any given triangle, on each side construct a square internal to the triangle. The obtained configuration will be named as the generalized internal Bride's Chair configuration.*

Theorem 2.1. *In any generalized Bride's Chair configuration, the segment obtained connecting the centers of two squares constructed on any two sides of the triangle is of equal length and orthogonal to the segment obtained connecting the center of the square constructed on the other side with the triangle vertex opposite to this side.*

For instance, looking at figure 3, which portrays a generalized external Bride's Chair configuration, segments O_2O_3 and O_1C are of equal length and mutually orthogonal. Considering the other two possible combinations, the following results hold true:

- segments O_1O_2 and O_3B, represented in figure 3 in green, are of equal length and mutually orthogonal and,
- segments O_1O_3 and O_2A, represented in figure 3 in red, are of equal length and mutually orthogonal.

Incidentally, it is worth noticing that the three segments O_1C, O_2A and O_3B concurr at a point, represented as K in figure 3, as they superimpose to the

altitudes of triangle $O_1O_2O_3$.
A synthetic proof of theorem 2.1 for the generalized external Bride's Chair configuration can be found in [3]. A proof of the same theorem for the generalized internal Bride's Chair configuration can be developed in the same way as the one for the generalized external Bride's Chair configuration. The author presents another synthetic proof in section 3.
The theorem can also be inferred after applying a continous transformation to any given quadrangle. For instance, starting from a given quadrangle ABCD, move the vertex C continuosly in the plane until it superimposes to the vertex D. A generic triangle ABC is obtained at the end of this process with the full square DCMN (see figure 1) reducing and turning into point C. It can be noticed that the Van Aubel segments O_1O_3 and O_2O_4 end up into the Bride's Chair segments O_1C and O_2O_3, respectively, shown in figure 3 and figure 4. As these Van Aubel segments, during the transformation, will always be of equal length and mutually orthogonal, as dictated by the Van Aubel Theorem, the generalized Bride's Chair configuration segments O_1C and O_2O_3 will retain the same mutual relation.
Incidentally, notice that the diagonals AC and BD of the quadrangle end up into the triangle sides AC and BC respectively.
By symmetry, the same reasoning can be applied focusing on the A vertex of the quadrangle: letting the vertex A superimpose to the vertex D, the equality and orthogonality relation between segments O_1O_3 and O_2A is deduced. In the same way, it can be shown that the same relation holds true for segments O_1O_2 and O_3B.
Regarding the generalized internal Bride's Chair configuration an analogous reasoning can be applied, so that, for instance, segments O'_1C and $O'_2O'_3$, shown in figure 4, are of equal length and mutually orthogonal.

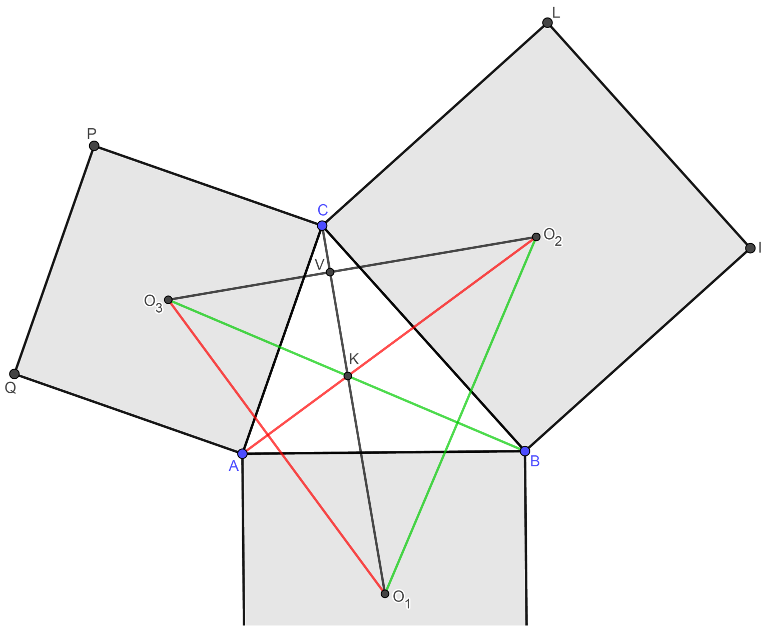

FIGURE 3. The generalized external Bride's Chair configuration relative to the given triangle ABC.

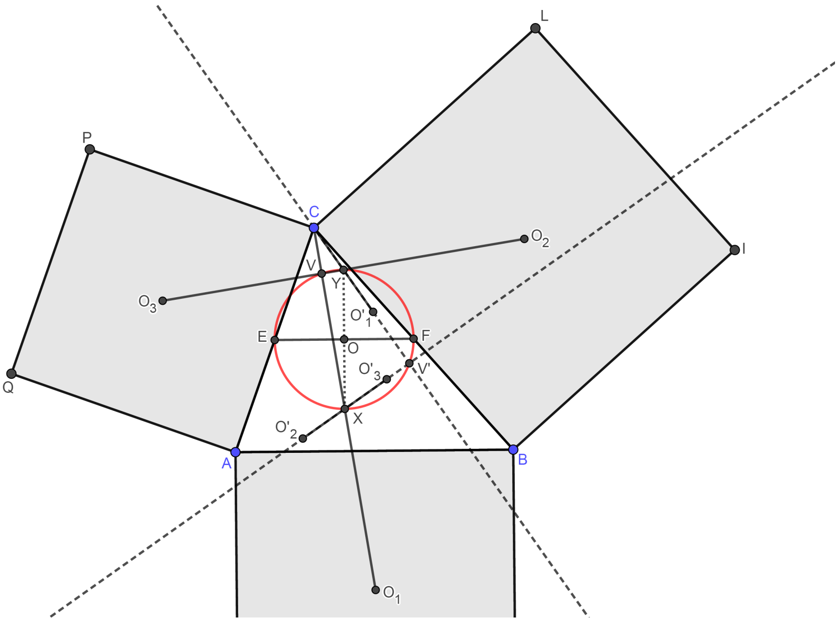

FIGURE 4. A Triangle Six Point Circle - The internal squares are not shown for the sake of clarity, their centers are represented by the points O'$_\mathrm{i}$, with i=1,2,3.

The following nomenclature is introduced to simplify the exposition of the following results:

- considering the generalized external Bride's Chair configuration, the three couples of equal and orthogonal segments will be referred to as the external Bride's Chair segment pairs;
- considering the generalized internal Bride's Chair configuration, the three couples of equal and orthogonal segments will be referred to as the internal Bride's Chair segment pairs.

In order to distinguish an external or internal Bride's Chair segment pair from another one of the same kind, it is convenient to consider the triangle vertexes as references: the external and internal Bride's Chair segment pairs, represented in figure 4, will be referred to as the C-vertex Bride's Chair segment pairs or the Bride's Chair segment pairs relative to the vertex C.

Taking into consideration all the geometrical elements pertaining to the Six-Point Circle Theorem For The Quadrangle, and applying the forementioned *transformation reasoning*, the following results can be stated:

Theorem 2.2. *Each segment of any external (internal) Bride's Chair segment pair which connects a vertex of the triangle to the center of the square constructed on the opposite side, is bisected by the segment of the internal (external) Bride's Chair segment pair that connects the centers of the squares constructed on the other two sides.*

For instance, looking at figure 4, segment O_1C (obtained connecting the triangle vertex C with the center of the square constructed external to the triangle over the opposite side AB) and segment O'$_2$O'$_3$ (obtained connecting

the centers of the squares constructed internal to the triangle over the sides BC and AC) bisect each other at point X, as shown in figure 4.
Also, segment O'$_1$C (obtained connecting the triangle vertex C with the center of the square constructed internal to the triangle over the opposite side AB) and segment O_2O_3 (obtained connecting the centers of the squares constructed external to the triangle over the sides BC and AC) bisect each other at point Y, as shown in figure 4.

In order to simplify the exposition of the following theorem, the midpoints of the segments which belong to a given Bride's Chair segment pair will be referred to as the midpoints of the given Bride's Chair segment pair. As a consequence of theorem 2.2, the midpoints of the external Bride's Chair segment pair, relative to a given vertex, coincide with the midpoints of the internal Bride's Chair segment pair, relative to the same vertex. The midpoints of the Bride's Chair segment pairs relative to the vertex C are represented in figure 4 by points X and Y.
The point of intersection of the two mutually orthogonal segments of a given Bride's Chair segment pair will be simply referred to as the point of intersection of the segment pair. For instance, the intersection point of the external Bride's Chair segment pair relative to the vertex C is represented by point V in figures 3 and 4. When the segments do not intersect, the lines through their end points (briefly the segments' extensions) do intersect. The intersection point of the segments' extensions will still be referred to as the intersection point of the Bride's Chair segment pair. For instance, the intersection point of the internal Bride's Chair segment pair relative to the vertex C is represented by point V' in figure 4. Taking the other Bride's Chair segment pairs into consideration, there are six points of this kind in total for the internal and external Bride's Chair configurations of the given triangle.

Theorem 2.3 (The Three Six-Point Circles Theorem For The Triangle)**.** *In any given triangle the midpoints of any two sides, the midpoints of both the internal and the external Bride's Chair segment pairs relative to the common vertex of the sides, and the intersection points of the very same internal and external Bride's Chair segment pairs lie on a circle.*

Corollary 2.1. *The segment connecting the midpoints of the triangle sides and the segment connecting the midpoints of the Bride's Chair segment pairs are two mutually orthogonal diameters of the circle.*

For instance, looking at figure 4, the midpoints of the triangle sides AC and BC, E and F, the midpoints of both the internal and the external Bride's Chair segment pairs relative to the vertex C of the triangle, X and Y, and the intersection points of the internal and external Bride's Chair segment pairs relative to the vertex C, V and V', lie on a circle.
The segments EF and XY are two mutually orthogonal diameters of the circle.

The same reasoning applies considering the midpoints of the triangle sides AB and BC and the Bride's Chair segment pairs relative to the vertex B, and, the midpoints of the triangle sides AC and AB and the Bride's Chair segment pairs relative to the vertex A, leading to the two remaining circles of the *family*.

Remark 2.1. *The three circles correspond to the circles constructed with the sides of the medial triangle of triangle ABC as diameters. It is quite straightforward to deduce that they meet, two at a time, at a vertex of the medial triangle and at the foot of the altitude drawn from that vertex to the opposite side (see figure 5). The proof of this fact is omitted as it is a simple application of the converse of Thales theorem* [7] *and can be left to the reader. It follows that the radical center of the three circles coincides with the orthocenter of the medial triangle, which, incidentally, coincides with the circumcenter of the given triangle ABC (point R in figure 5). A deeper study of the circles configuration and the properties of triangles $O_1O_2O_3$ (see figure 3) and O'_1 $O'_2O'_3$ (vertexes shown in figure 4) goes beyond the scope of this paper and can be carried out in a future work.*

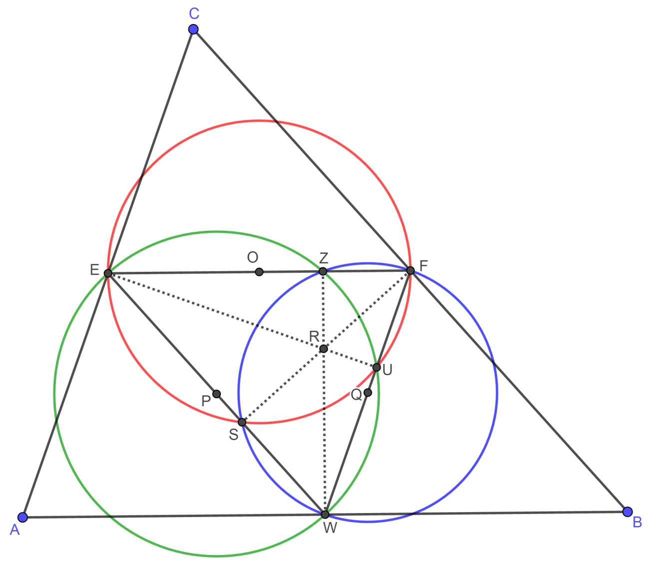

FIGURE 5. The radical center of the three six-point circles coincides with the circumcenter of the given triangle ABC.

3. Proofs

This section might appear a bit pedantic as it is quite similar to the Proofs section of reference [5]. It has been included for completeness as the deductions of the theorems, presented in section 2, cannot be considered rigorous proofs.
Theorem 2.1 will be proved below for the generalized external Bride's Chair segment pair constituted by segments O_2O_3 and O_1C. For the other pairs, including the generalized internal Bride's Chair segment pairs, the proof is essentially the same.

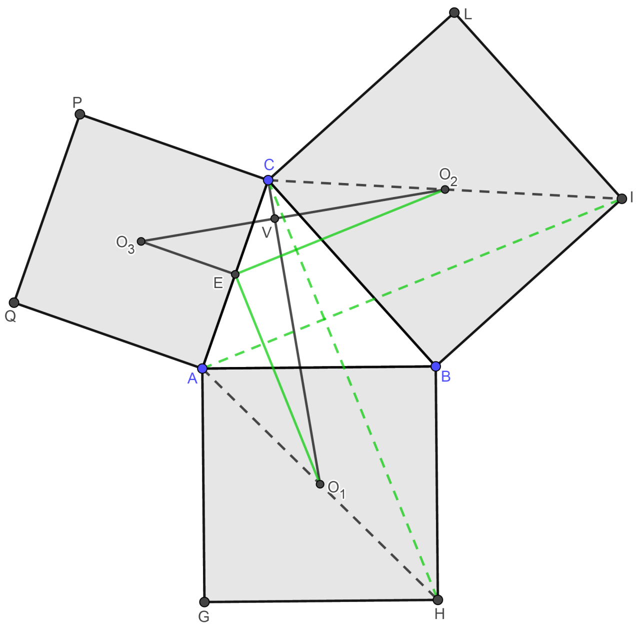

FIGURE 6. Diagram for the theorem 2.1 proof.

3.1. **Theorem 2.1. Proof.** Looking at figure 6, we deduce that segments HC and AI are of equal length and orthogonal to each other. This fact follows from the relation between the congruent triangles HBC and ABI: this latest triangle can be obtained from a right angle clockwise rotation around point B of triangle HBC.
If we apply the Midsegment Theorem [8] to triangles CAH and AIC, we deduce that segments O_1E and O_2E are of equal length and orthogonal to each other at point E, the midpoint of the triangle side AC. In addition, segments EC and EO_3 are of equal length and orthogonal to each other.
We deduce that triangles O_1EC and O_2EO_3 are congruent according to the SAS (Side-Angle-Side) rule: the included angles O_1EC and O_2EO_3 are equal, as they both measure the right angle plus the shared angle O_2EC.
Moreover, triangle O_1EC can be obtained from a right angle clockwise rotation around point E of triangle O_2EO_3.
It follows that segments O_1C and O_2O_3 are of equal length and orthogonal to each other.

For theorem 2.2, We take into consideration segments O_2O_3 and O'_1C, and prove that they bisect each other at point Y. An analogous proof can be elaborated to prove that segments $O'_2O'_3$ and O_1C bisect each other at point X (see figure 4).

3.2. **Theorem 2.2. Proof.** Looking at figure 7, if we apply the Midsegment Theorem to triangle AH'P, we deduce that segment PH' is parallel to the $O_3O'_1$ segment and its length is twice the length of the $O_3O'_1$ segment. We further deduce that segments AC and H'I are of equal length and orthogonal to each other. This fact follows from the relation between the congruent triangles ABC and H'BI: This latest triangle can be obtained from a right

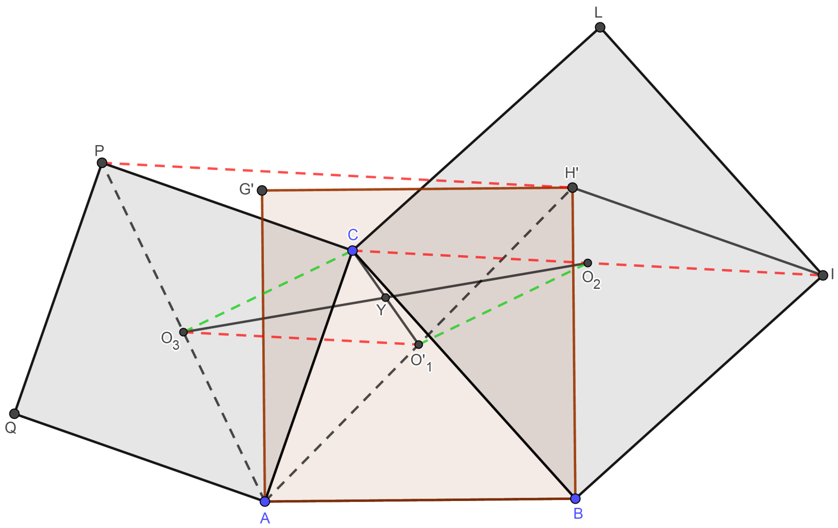

FIGURE 7. Diagram for the theorem 2.2 proof. The diagram shows the necessary geometrical elements to prove that quadrangle $O_3O'_1O_2C$ is a parallelogram.

angle clockwise rotation around point B of triangle ABC. As segment PC is of equal length and orthogonal to AC, we have proven that CIH'P is a parallelogram (opposites sides CP and IH' are of equal length and parallel). So, segment PH' is of equal length and parallel to segment CI. But segment CI superimposes segment CO_2 and its length is twice the length of segment CO_2. We can conclude that quadrangle $O_3O'_1O_2C$ is a parallelogram (opposite sides $O_3O'_1$ and O_2C are of equal length and parallel) and its diagonals bisect each other at point Y.

Theorem 2.3 will be proven for the generalized Bride's Chair configurations relative to the vertex C, showing that the points E, X, V', F, Y and V are concyclic. For the other generalized Bride's Chair configurations, we would proceed exactly in the same way.
The proof is equivalent to the proof of the Six-Point Circle Theorem For The Quadrangle presented in reference [5]: the reasoning that revolved around the iso-ortho-diagonal quadrangle $O_1O_2O_3O_4$ applies now to the iso-ortho-diagonal quadrangle $O_1O_2CO_3$, represented in figure 8.
According to the Four Concurring Circles' Lemma [5], we have that point V is the common point of circles (C_i, C_iV), for i = 1,2,3,4 and C_i a given midpoint of a given side of quadrangle $O_1O_2CO_3$ (we adopt the same convention for representing circles as in reference [5], providing the center and a segment radius).
Moreover, it can be easily deduced (in the same way as in [5]) that segments EV and VF are the common chords of the circles constructed with the opposite sides of the iso-ortho-diagonal quadrangle $O_1O_2CO_3$ as diameters (see figure 8).

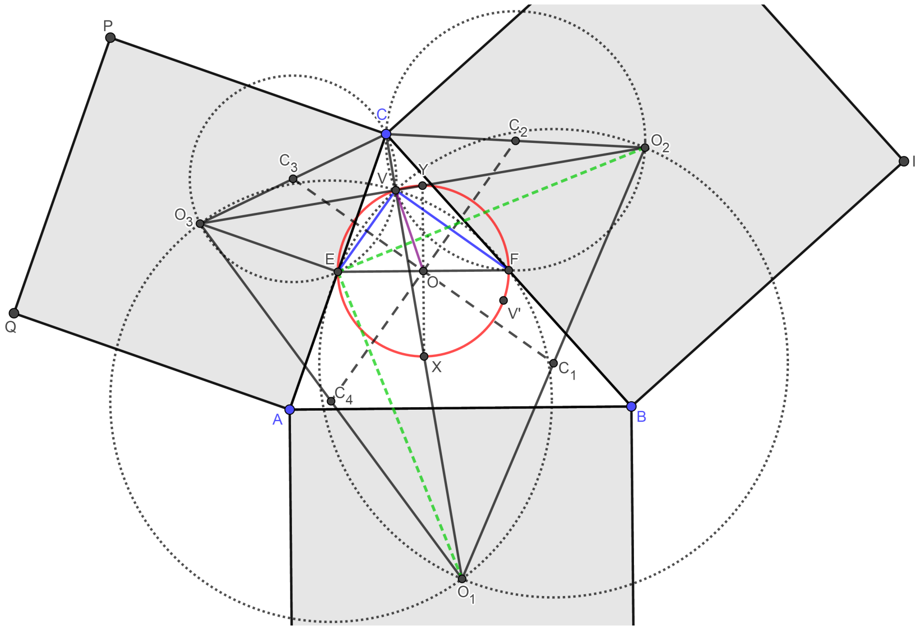

FIGURE 8. Diagram for the theorem 2.3 proof.

3.3. **Theorem 2.3. Proof.** According to theorem 2.1 and the converse of Thales Theorem, it follows that quadrangle XV'YV (vertexes represented in figures 4 and 8) is cyclic, and XY is a diameter of its circumscribed circle. The center of the circle, O, is the midpoint of segment XY. By definition, point O is the centroid of the iso-ortho-diagonal quadrangle $O_1O_2CO_3$ [1]. Segment OV, also represented in figure 8, is a radius of the circle.
Segments EV and VF are the common chords of the circles constructed with the opposite sides of the iso-ortho-diagonal quadrangle as diameters (see figure 8). Therefore, segment EV is bisected orthogonally by segment C_1C_3, and segment VF is bisected orthogonally by segment C_2C_4. Segments C_1C_3 and C_2C_4 are the segments which connect the midpoints of the opposite sides of the iso-ortho-diagonal quadrangle $O_1O_2CO_3$.
They bisect each other at point O forming a right angle.
The fact that they bisect each other forming a right angle is a very well known property of any iso-ortho-diagonal quadrangle: quadrangle $C_1C_2C_3C_4$ is indeed a square, so its diagonals bisect each other orthogonally.
The fact that the segments which connect the midpoints of the opposite sides of any quadrangle bisect each other at the quadrangle centroid (point O for the $O_1O_2CO_3$ quadrangle) is another quite known property [1].
With segments C_1C_3 and C_2C_4 being the segment bisectors of the EV and FV segments, respectively, we deduce that:
– EVF is a right angle. Again, according to the converse of Thales Theorem, segment EF is a diameter of the circumscribed circle of triangle EVF, also shown in red in figure 8.
– The center of the circle coincides with point O. OV is a radius of the circle. Thus, all six points E, X, V', F, Y and V lie on the same circle (O, OV) and segments EF and XY are two diameters of the circle.

The corollary proof follows hereafter.

3.4. **Corollary 2.1. Proof.** This proof can be carried out via angles inspection. Looking at figure 8 we notice that angle EVX measures half of a right angle.
Indeed, angle EVX is equal to angle O_1O_2E which measures half of a right angle as both angles subtend arc O_1E of circle (C_1, C_1E).
As a consequence of theorem 2.3, we know that angle EVX subtend the EX arc of circle (O, OV) and thus angle EOX is a right angle according to the Inscribed Angle Theorem [4].

4. ACKNOWLEDGEMENT

The author wishes to thank his friend Jackie Clayton once again for a careful proof reading of the document. The author is also grateful to Mr. William James Seamore who wrote the following short poem.

Just like The Almighty
from single to threefold grows
so does the Six Point Circle
at the Quadrangle's fall
and the new backdrop is acknowledged in the tumble
not a Segment nor a trifle. A Triangle.

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