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MORE INEQUALITIES IN QUADRILATERAL INVOLVING THE NEWTON LINE

ELLIOTT A. WEINSTEIN and JOHN D. KLEMM

Abstract. We derive new inequalities involving the length of the Newton line segment connecting the midpoints of the diagonals of a quadrilateral.

Theorem 1. *For any quadrilateral, let P be the perimeter, p and q the lengths of the diagonals, and v the length of the line segment connecting the midpoints of the diagonals. Then $P^2 \geq 8v(p+q)$, where equality holds if and only if all four vertices are collinear with a pair of adjacent vertices in the middle that coincide.*

Proof. In fact, we will show that

$$P^2 \geq (p+q+2v)^2 \geq 8v(p+q).$$

The left inequality is that in American Mathematical Monthly Problem 11841 ([2]) with both sides squared. The right inequality simplifies to $(p+q-2v)^2 \geq 0$, which obviously always holds, with equality if and only if $p+q=2v$. Since equality occurs in the left inequality if and only if two adjacent vertices of the quadrilateral coincide or else all four vertices are collinear with exactly three sides codirectional ([1]), and equality occurs in the right inequality if and only if $p+q=2v$, it is easy to verify the conditions for equality as stated. (Note that if two adjacent vertices coincide, then the length of one of the sides is zero, so at best we have a triangle. But then $p+q=2v$, where v is the length of the midline, results in the sum of the lengths of two sides of the triangle equalling the third, forcing further degeneration.) □

Theorem 1 is an improvement over Theorem 4.1 in [4], which obtained the same result for a bicentric quadrilateral.

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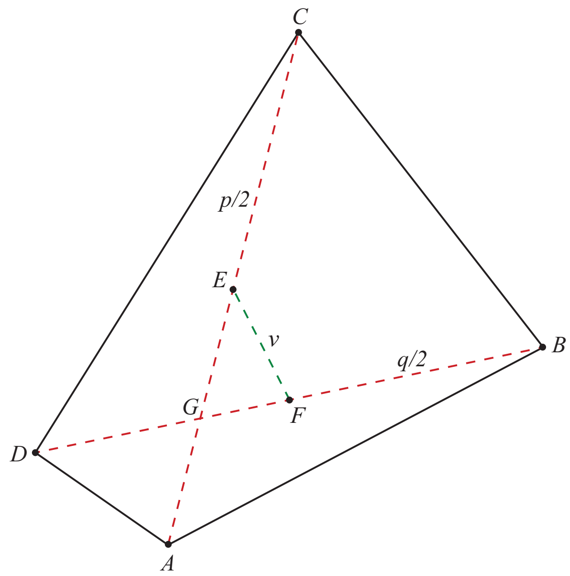

FIGURE 1. $p + q > 2v$

Theorem 2. *For any quadrilateral $ABCD$, let p and q be the lengths of the diagonals, and v the length of the line segment connecting the midpoints of the diagonals. When the quadrilateral is convex, $p + q > 2v$, except in the degenerate case when all four vertices are collinear with no other vertex lying between either pair of opposite vertices (e.g., the vertices appear in the order $ACBD$ including when $B = C$), in which case $p + q \le 2v$, with equality if and only if there are two adjacent vertices in the middle that coincide. When the quadrilateral is nonconvex, $p + q \gtreqless 2v$.*

Proof. For a nondegenerate convex quadrilateral, let E be the midpoint of AC, F the midpoint of BD, and G the point where AC and BD intersect. Let $AC = p$ and $BD = q$ (depending on the context, we will use two vertices to indicate either a line segment or its length). When $v = 0$, obviously $p+q > 2v$, so assume $v > 0$. Then as in Figure 1, $EG < \frac{p}{2}$ and $FG < \frac{q}{2}$, but $EG+FG > EF$, and so $p+q > 2v$. Now consider the degenerate convex cases. When all four vertices are collinear, it is straightforward to test the cases to verify the conditions as stated. When exactly three vertices are collinear, an argument similar to that for the nondegenerate case demonstrates that when no two vertices coincide, $p + q > 2v$, and when two adjacent vertices coincide, say $A = B$, $p+q > CD = 2v$ since EF is the midline parallel to CD. For nonconvex quadrilaterals, see Figure 2 for examples where a concave and a crossing quadrilateral can be "shrunk" or "stretched" so that p and q remain fixed but $v = EF$ can be made small enough so that $p + q > 2v$, or else arbitrarily large so as to reverse the inequality. Similar examples can be constructed for the degenerate nonconvex cases. So any of $p+q > 2v$, $p+q = 2v$, and $p + q < 2v$ are possible. $\square$

Combining Theorem 1 (or the inequality of [2]) and Theorem 2, we can deduce that for a nondegenerate convex quadrilateral, $P > 4v$. But we can say more.

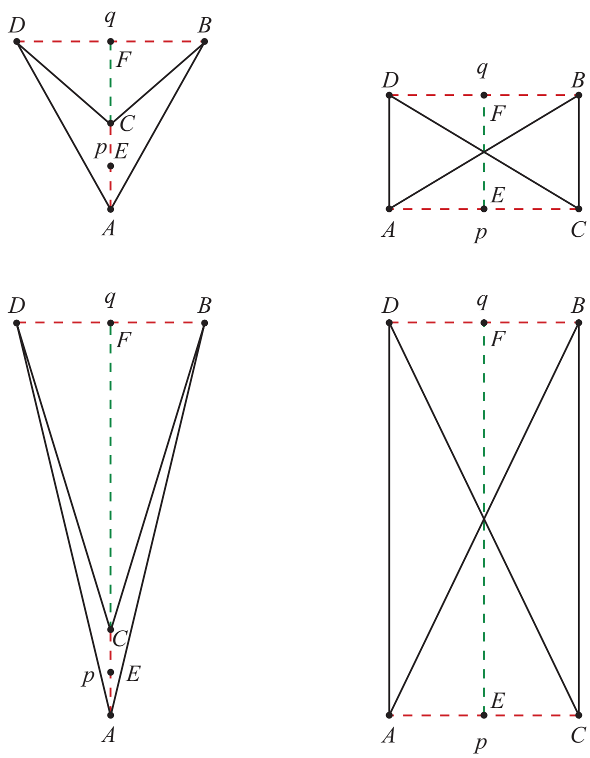

FIGURE 2. $p + q \gtreqless 2v$

Theorem 3. *For any quadrilateral, let P be its perimeter and v the length of the line segment connecting the midpoints of the diagonals. Then $P \geq 4v$, with equality if and only if all four vertices are collinear with no other vertex lying between either pair of opposite vertices.*

Proof. According to Theorem 2 in [3], the sum of the lengths of a pair of opposite sides of any nondegenerate quadrilateral is at least equal to $2v$, with equality only for a pair of parallel opposite sides of a crossing quadrilateral. So for all nondegenerate quadrilaterals the perimeter exceeds $4v$. We now address the degenerate cases. When all four vertices are collinear, it can be verified directly that each pair of opposite sides sums to $2v$ if and only if no other vertex lies between either pair of opposite vertices, and otherwise, one pair sums at least to $2v$ and the other pair exceeds it. When exactly three vertices are collinear, the first proof of Theorem 2 in [3] extends to these cases; alternatively in the convex case, Theorem 2 above tells us that $p + q > 2v$, and inserting this into $P \geq p + q + 2v$ (see the proof of Theorem 1) yields $P > 4v$. $\square$

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OFFICE OF THE ACTUARY
CENTERS FOR MEDICARE & MEDICAID SERVICES
BALTIMORE, 21244 MD, US
E-mail address: elliott.weinstein@cms.hhs.gov

BALTIMORE, MD, US
E-mail address: johnklemm@verizon.net