INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 9 (2020), No. 1, 44 - 46

ALMOST ISOGONAL

ROGER C. ALPERIN

Abstract. We define a transformation which agrees with the isogonal transformation except at points on the circumcircle.

1. Introduction

Given a triangle $\triangle = \triangle ABC$ and a point $P \notin \{A, B, C\}$ we construct the reflections P_a, P_b, P_c of P in the sides of the triangle.

Theorem 1.1. *For P on the circumcircle of $\triangle$ then the line $L = P_aP_bP_c$ passes through the orthocenter H. The line L is parallel to the Wallace-Simson line W. An affine expansion from P with a factor of 2 yields L from W.*

The reflections of the lines AP, BP, CP in the angle bisectors at A, B, C are parallel and perpendicular to L.

Proof. For the first part see [2, 3] (See Figure 1). For the second part one can easily see that the reflection of P across the angle bisector at B lies on the perpendicular bisector of P_a, P_c (passing through B), and similarly for the other vertices of the triangle.

Keywords and phrases: Isogonal, Iso-circle, Pedal circle, Triangle
(2010)Mathematics Subject Classification: 51M99
Received: 21.10.2019. In revised form: 23.02.2020. Accepted: 10.01.2020.

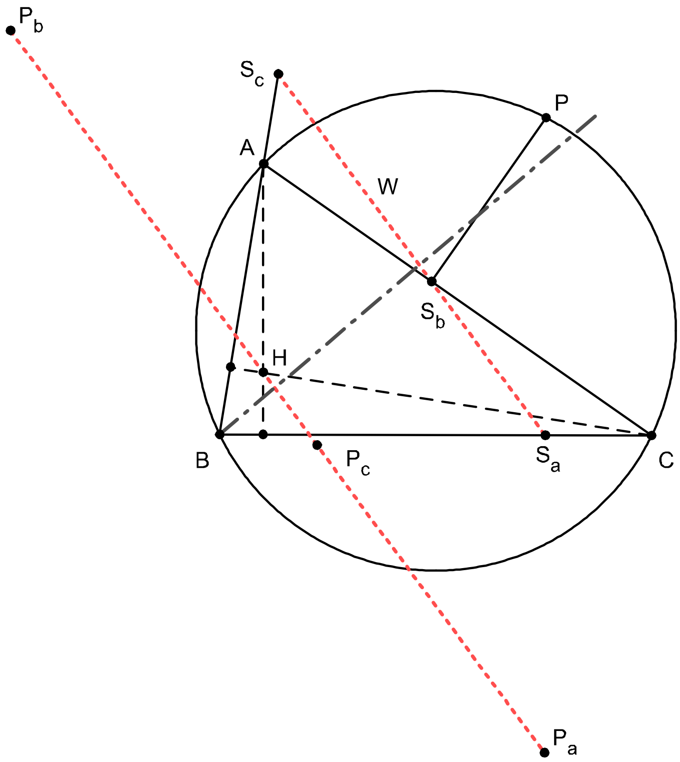

Figure 1

Theorem 1.2. *Suppose P is not on the circumcircle of $\triangle$. Let Q be the center of the circle through P_a, P_b, P_c; then Q is the isogonal transform of P.*

Proof. Suppose P is interior to $\triangle$ or on its sides; the other cases are done similarly. The angle at A is split into two parts by AP, $\beta \geq \gamma$. The angle between P_b, P_c is

$$2\beta + 2\gamma = 2\angle A,$$

$\angle PAC = \angle P'AB$ for P' the reflection of P across the angle bisector at A, and

$$\angle PAP' = \beta - \gamma.$$

Then $\angle P'AP_c = \beta + \gamma$.

Let M be the midpoint of P_bP_c on the perpendicular bisector AM of P_bP_c. Thus AM is the reflection of AP across the bisector at A. Then it follows that the center Q lies on AM.

Consequently Q is the same as the point of coincidence of AP, BP, CP reflected across the corresponding three angle bisectors; that is Q is the isogonal transform of P, [1, 3].

2. Application

The circle $P_aP_bP_c$ is called the *iso-circle* of P.

Theorem 2.1. *The radii of iso-circles corresponding to isogonal points are equal.*

Proof. Say P, Q are an isogonal pair, then reflection across the side a gives equal length P_aQ_a so that PQQ_aP_a is an isosceles trapezoid; hence the diagonals have equal length and thus the radii are equal.

The following is well-known but follows easily now from Theorem 2.1.

Corollary 2.2. *The pedal circles of isogonal points are equal.*

Proof. The pedal circle of P is the iso-circle of P scaled by $\frac{1}{2}$ from P, and similarly the pedal circle of Q. Since P, Q is an isogonal pair the centers of the iso-circles are Q, P and hence the center of the pedal circles is the midpoint of PQ. Since the radii of the the iso-circles are equal by Theorem 2.1 the pedal circles are also equal.

For example the pedal circle for the orthocenter H and circumcenter O passes through the feet of the altitudes and the midpoints of the sides, centered at the midpoint N of HO. This is six points of the famous Nine Point Circle. The iso-circle of H is the circumcircle and the iso-circle of O is the circumcircle of the circumreflective triangle, $\triangle$ rotated 180^o about N.

References

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DEPARTMENT OF MATHEMATICS
SAN JOSE STATE UNIVERSITY
SAN JOSE, CA 95192, USA
E-mail address: rcalperin@gmail.com