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ON INVERSIONS IN CENTRAL CONICS

ROOSEVELT BESSONI AND GUY GREBOT

Abstract. We use plain euclidean geometry to analyse the structure of an inversion in an ellipse or in a hyperbola. By this formulation, results on inversions in circles are directly extended to results on inversions in ellipses.

1. Introduction

Jacob Steiner is quoted in [5] as the first mathematician to formalize the bases of inversive geometry in a text dated from 1824 and published after his death by *Bützberger*. In 1965, the mathematician *Noel Childress* [1] extended the concept of circular inversion to central conics, ellipse or hyperbola. In 2014, *Ramirez* [6][7] showed other properties concerning inversion in ellipses and *Neas* [4], in 2017, looked at anallagmatic curves under inversion in hyperbolae.

Inversion in circles is a geometrical topic which is usually developped without analytic geometry [8]. Strangely enough, we could not find any work concerning inversion in central conics using plain euclidean geometry arguments. In the studied publications the basic framework is analytical geometry. Also, as far as we could reach, the structure of an inversion in a central conic is not mentioned in the cited works and it seems to us that the analytical framework tends to hide this property.

In this communication, we use plain euclidean geometry to analyse the structure of an inversion in an ellipse or in a hyperbola. We show that an inversion in a central conic is given by the composition of compressions and an inversion in a circle or in an equilateral hyperbola; in this sense, an inversion in a central conic is just an affine deformation of an inversion in a circle or in an equilateral hyperbola.

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As a byproduct, it is shown that to each circle corresponds a family of inversions in ellipses which admit this circle as an auxiliary circle. In the same way, to each equilateral hyperbola corresponds a family of inversions in hyperbolae with same auxiliary circle.

In the next section, we define inversion and compression and we state and prove an important result about compressions in parallel lines. We show the main result of this communication in section 3. Section 4 treats the immediate consequences of the main theorem as well as the proof of a result adapted from a theorem about inversion in circles.

We will use the letters a and b to refer to the major and the minor axes, respectively. We shall denote by $\overline{AB}$ the length of a segment AB and the algebraic measurement (signed length) of such segment will be indicated by $\underline{AB}$. The software *GeoGebra* was used to create the figures.

2. PRELIMINARY RESULTS

In what follows, we will consider Π to be an Euclidean plane.

Definition 2.1. *Let ϱ be a central conic in Π with center $O \notin \varrho$. The inversion in the central conic ϱ is the transformation I_ϱ of Π which associates, to each point $P \neq O$ in Π, a point P' on the ray $\overrightarrow{OP}$, such that*

$$\overline{OP}\,\overline{OP'} = \overline{OQ}^2, \tag{1}$$

where $Q \in \varrho \cap \overrightarrow{OP}$.

It follows directly from this definition that: a point $P \in \Pi \setminus \{O\}$ has an image under the inversion in ϱ, $I_\varrho(P)$, only if the ray $\overrightarrow{OP}$ intercepts ϱ; I_ϱ is a $1-1$ mapping; if ϱ is an ellipse or a circle, the inversion in ϱ, I_ϱ, is defined for all points in $\Pi \setminus \{O\}$; if ϱ is a circle of center O, $\overline{OQ}$ is its radius and the righthand side of (1) is constant; $I_\varrho(P) = P$ if, and only if, $P \in \varrho$; if $P' = I_\varrho(P)$ then $P = I_\varrho(P')$; a straight line through O that intercepts ϱ is invariant under I_ϱ.

For more properties of inversions in circles and inversions in central conics, we refer the reader to the references [8] and to [1], [4], [6] and [7] and references therein.

It is to be noted that the treatment of inversions in hyperbolae is more delicate than for inversions in ellipses. This is due to the fact that the definition allows transforming only points that lie in the region limited by the assymptots and which contains the vertices. Although this remark does not affect the results presented below, it contributes to the lack of symmetry between the properties of inversions in hyperbolae and properties of inversions in ellipses.

As will be seen in the next section, the main result of this communication expresses that inversions in ellipses and hyperbolae are related to inversions in circles and equilateral hyperbolae, respectively, via a compression.

A compression is an affine transformation defined as follows [3].

Definition 2.2. *Let $c \neq 0$ be a real number and let l be a straight line in Π. The compression $\tau_{l,c}$ with ratio c over l is the transformation that maps* *a point $M \in \Pi$ onto the point M' according to the vector equality*

$$\overrightarrow{PM'} = c\,\overrightarrow{PM},$$

where $P \in l$ is the orthogonal projection of M over l.

Lemma 2.1. *There is a compression that maps an ellipse $\mathcal{E}$ with center O onto a circle centered in O.*

Proof. The given ellipse $\mathcal{E}$ is defined by $\overline{PF_1} + \overline{PF_2} = a$, for all $P \in \mathcal{E}$, where F_1 and F_2 are its foci and $\overline{F_1F_2} = c$. Let τ be the *compression* over the focal axis of $\mathcal{E}$ with coefficient k. Then, for every $P \in \mathcal{E}$, $P' = \tau(P)$ is such that $\overrightarrow{P_1P'} = k\,\overrightarrow{P_1P}$, where P_1 is the orthogonal projection of P over the focal axis of the ellipse $\mathcal{E}$. Figure 1 illustrates these settings. From

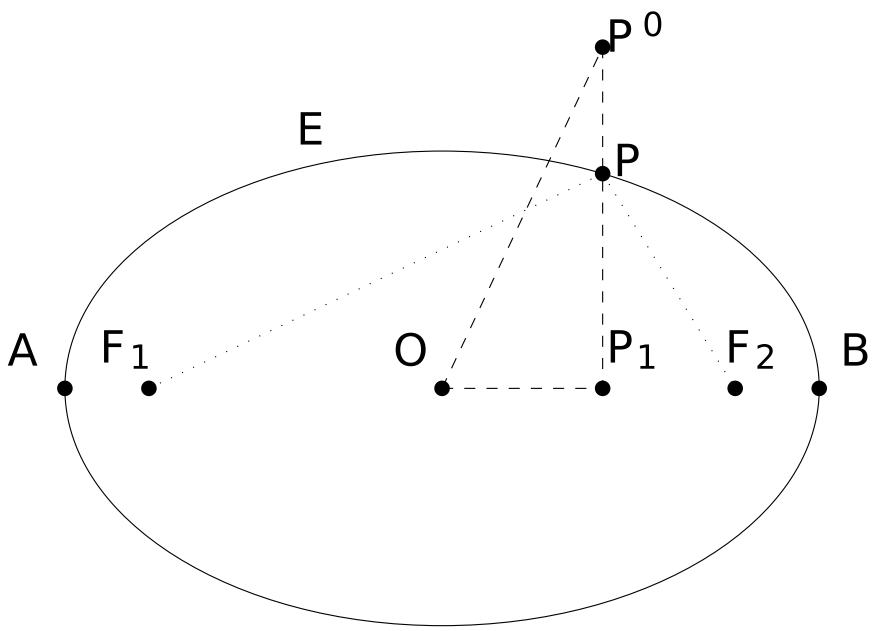

FIGURE 1. Illustration of Lemma 2.1.

$\left(\overline{PF_1} + \overline{PF_2}\right)^2 = a^2$, we get

$$\begin{aligned}
\left(2\,\overline{PF_1}\,\overline{PF_2}\right)^2 &= \left[a^2 - \left(\overline{PF_1}^2 + \overline{PF_2}^2\right)\right]^2, \\
4\,\overline{PF_1}^2\,\overline{PF_2}^2 &= a^4 - 2\,a^2\left(\overline{PF_1}^2 + \overline{PF_2}^2\right) + \left(\overline{PF_1}^2 + \overline{PF_2}^2\right)^2, \\
\overline{PF_1}^4 + \overline{PF_2}^4 - 2\,\overline{PF_1}^2\,\overline{PF_2}^2 &= 2\,a^2\left(\overline{PF_1}^2 + \overline{PF_2}^2\right) - a^4, \\
(2) \quad \left(\overline{PF_1}^2 - \overline{PF_2}^2\right)^2 &= 2\,a^2\left(\overline{PF_1}^2 + \overline{PF_2}^2\right) - a^4.
\end{aligned}$$

Since $PP_1 \perp F_1F_2$ and the points P_1, F_1 and F_2 are collinear, we have

$$\begin{aligned}
\left(\overline{PF_1}^2 - \overline{PF_2}^2\right)^2 &= \left(\overline{P_1F_1}^2 - \overline{P_1F_2}^2\right)^2 = \left(\underline{P_1F_1}^2 - \underline{P_1F_2}^2\right)^2 \\
&= \left(\underline{P_1F_1} - \underline{P_1F_2}\right)^2 \left(\underline{P_1F_1} + \underline{P_1F_2}\right)^2, \\
\underline{P_1F_1} &= \underline{P_1O} + \underline{OF_1}, \\
\underline{P_1F_2} &= \underline{P_1O} + \underline{OF_2}.
\end{aligned}$$

So equation (2) can be written as

$$4\,c^2\,\overline{OP_1}^2 = 2a^2\left(\frac{c^2}{2} + 2.\overline{OP_1}^2 + 2.\overline{P_1P}^2\right) - a^4,$$

$$4\left(c^2 - a^2\right)\overline{OP_1}^2 = 4a^2\,\overline{P_1P}^2 + a^2\left(c^2 - a^2\right). \tag{3}$$

As $\overline{OP_1}^2 = \overline{OP'}^2 - \overline{P_1P'}^2 = \overline{OP'}^2 - k^2\,\overline{P_1P}^2$, equation (3) becomes

$$\begin{aligned}
\overline{OP'}^2 - k^2\,\overline{P_1P}^2 &= \frac{a^2}{(c^2 - a^2)}\,\overline{P_1P}^2 + \frac{a^2}{4}, \\
\overline{OP'}^2 - \frac{a^2}{4} &= \overline{P_1P}^2\left(k^2 + \frac{a^2}{c^2 - a^2}\right).
\end{aligned}$$

Since $a > c$, we can put $k = \dfrac{a}{\sqrt{a^2 - c^2}}$ and we obtain $\overline{OP'} = \dfrac{a}{2}$. This is what we claimed for, i.e. that the compression over the focal axis of $\mathcal{E}$ and ratio $k = \dfrac{a}{\sqrt{a^2 - c^2}}$ takes the point $P \in \mathcal{E}$ onto the point P' on the circle of radius $\dfrac{a}{2}$ centered in O. □

An ellipse with center O has two auxiliary circles centered in O: one has radius equal to the semi-major axis and the other has radius equal to the semi-minor axis of the ellipse. In the same way as the preceeding lemma exhibits the compression that takes the ellipse onto the outer auxiliary circle, it can be shown that the compression with ratio $k = \dfrac{\sqrt{a^2 - c^2}}{a}$ and axis on the perpendicular to the focal axis through O, takes the ellipse onto the inner auxiliary circle.

Lemma 2.2. *There is a compression that maps a hyperbola $\mathcal{H}$ onto an equilateral hyperbola.*

Proof. Let t_1 and t_2 be the assymptots of $\mathcal{H}$ and let r be its focal axis. Let τ_c be the compression over r with ratio c such that t_1 and t_2 are mapped onto s_1 and s_2, respectively, where $s_1 \perp s_2$.

Let A_1 and A_2 be the vertices of $\mathcal{H}$ and let O be its center. According to [2], proposition 31, the hyperbola $\mathcal{H}'$ with vertices A_1 and A_2 and with assymptots s_1 and s_2 is uniquely defined, as illustrated in Figure 2.

If p is the parameter of $\mathcal{H}$, we have [2]

$$\overline{LL'}^2 = p\,\overline{A_1A_2},$$

where $L \in t_1$, $L' \in t_2$ and $A_1 \in LL'$.

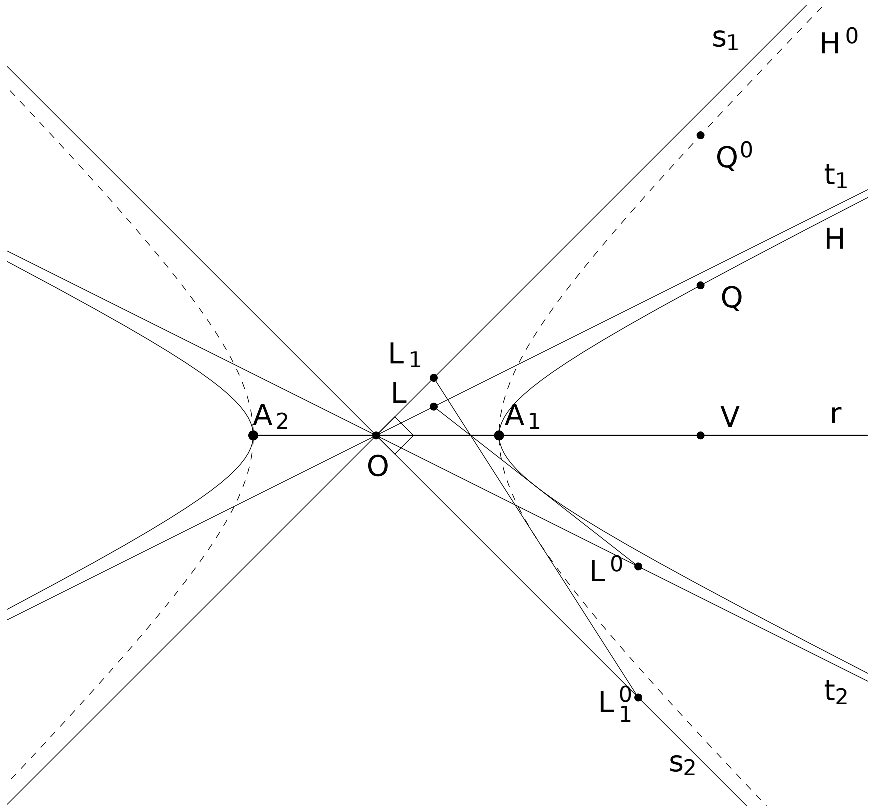

FIGURE 2. Illustration of Lemma 2.2.

Let $L_1 = \tau_c(L)$ and $L'_1 = \tau_c(L')$. Then

$$\begin{aligned} \overline{L_1L'_1}^2 &= c^2\,\overline{LL'}^2 \\ &= c^2 p\,\overline{A_1A_2} \end{aligned}$$

which means that the parameter of $\mathcal{H}'$ is given by $p' = c^2 p$. Using the notation of [2], the ordinate $Q' \in \mathcal{H}'$ at the point V on the line A_1A_2 is

$$\begin{aligned} \overline{Q'V}^2 &= p'\,\frac{\overline{A_1V}\,\overline{A_2V}}{\overline{A_1A_2}} \\ &= c^2 p\,\frac{\overline{A_1V}\,\overline{A_2V}}{\overline{A_1A_2}} \\ &= c^2\,\overline{QV}^2, \end{aligned}$$

where $Q \in \mathcal{H}$ is the ordinate relative to V. We then have

$$\overline{Q'V} = c\,\overline{QV}.$$

Since V, Q and Q' are colinear, we can state that

$$\overrightarrow{Q'V} = c\,\overrightarrow{QV},$$

which means that

$$\mathcal{H}' = \tau_c\,(\mathcal{H})$$

as we whished to prove. □

Theorem 2.1. *Let l_1 and l_2 be two parallel straight lines in Π and let $k_1, k_2 \in \mathbb{R} \setminus \{0\}$. The compressions τ_{l_1,k_1} and τ_{l_2,k_2} satisfy*

$$\tau_{l_1,k_1} \circ \tau_{l_2,k_2} = T_{\mathbf{u}} \circ \tau_{l_1,k_1 k_2}, \tag{4}$$
$$= T_{\mathbf{v}} \circ \tau_{l_2,k_1 k_2}, \tag{5}$$

where $T_{\mathbf{u}}$ and $T_{\mathbf{v}}$ are translations along the vectors $\mathbf{u} = k_1(1-k_2)\overrightarrow{P_1P_2}$ and $\mathbf{v} = (k_1 - 1)\overrightarrow{P_1P_2}$, respectively, $P_1 \in l_1$ and P_2 is the orthogonal projection of P_1 over l_2.

Proof. Let $M' = \tau_{l_2,k_2}(M)$ and $M'' = \tau_{l_1,k_1}(M')$. It follows that $\overrightarrow{P_2M'} = k_2\overrightarrow{P_2M}$ and $\overrightarrow{P_1M''} = k_1\overrightarrow{P_1M'}$, where P_1 and P_2 are the orthogonal projections of M over l_2 and of M' over l_1, respectively. From these, we obtain

$$\begin{aligned}
\overrightarrow{P_1M''} &= k_1\overrightarrow{P_1M'} \\
&= k_1\left(\overrightarrow{P_2M'} + \overrightarrow{P_1P_2}\right) \\
&= k_1\left(k_2\overrightarrow{P_2M} + \overrightarrow{P_1P_2}\right) \\
&= k_1k_2\left(\overrightarrow{P_1M} - \overrightarrow{P_1P_2}\right) + k_1\overrightarrow{P_1P_2}
\end{aligned}$$

which gives

$$\overrightarrow{P_1M''} = k_1k_2\overrightarrow{P_1M} + k_1(1-k_2)\overrightarrow{P_1P_2}.$$

We also have

$$\begin{aligned}
\overrightarrow{P_1M''} &= k_1\left(k_2\overrightarrow{P_2M} + \overrightarrow{P_1P_2}\right) \\
\overrightarrow{P_2M''} &= k_1\left(k_2\overrightarrow{P_2M} + \overrightarrow{P_1P_2}\right) - \overrightarrow{P_1P_2}
\end{aligned}$$

and so

$$\overrightarrow{P_2M''} = k_1k_2\overrightarrow{P_2M} + (k_1 - 1)\overrightarrow{P_1P_2}.$$

□

As a consequence of this last theorem, if $k_1 k_2 = 1$, we have $\tau_{l_1,k_1} = T_{\mathbf{v}} \circ \tau_{l_2,k_1}$, whenever $l_1 \parallel l_2$. From this remark and the preceeding lemmas, we can state the following corollary.

Corollary 2.1. *The compression $\tau_{l,k}$ over l with ratio k maps*

(1) *two equilateral hyperbolae with axis parallel to l onto homothetic hyperbolae;*
(2) *two circles onto homothetic ellipses.*

3. MAIN RESULT

The main result of this communication is expressed by the following theorem.

Theorem 3.1. *Let $\mathcal{V}$ be a central conic with major axis a and minor axis b. The inversion with respect to this conic is expressed by the product*

$$I_{\mathcal{V}} = \tau_{\frac{1}{c}} \circ I_{\varrho} \circ \tau_c, \tag{6}$$

where τ_c is the compression over $\mathcal{V}$'s focal axis with ratio $c = \dfrac{a}{b}$, and I_{ϱ} is either the inversion in the circle concentric to $\mathcal{V}$ with radius $\dfrac{a}{2}$, if $\mathcal{V}$ is an ellipse, or the inversion in the equilateral hyperbola with same focal axis and major axis a as $\mathcal{V}$, in case $\mathcal{V}$ is a hyperbola.

We shall call ϱ, such that $I_{\mathcal{V}} = \tau_{\frac{1}{c}} \circ I_{\varrho} \circ \tau_c$, the *associated curve* to $\mathcal{V}$.

Proof. Let $P' = I_{\mathcal{V}}(P)$, $Q' = \tau_c(P')$ e $P'' = \tau_c(P)$ and ϱ be the image of $\mathcal{V}$ by the compression τ_c.

By definition, we have

$$\overline{OP}\,\overline{OP'} = \overline{OH}^2$$

where H is the point of intersection between the ray $\overrightarrow{OP}$ and $\mathcal{V}$. Let $\tau_c(H) = H' \in \varrho$. Figure 3 illustrates these settings.

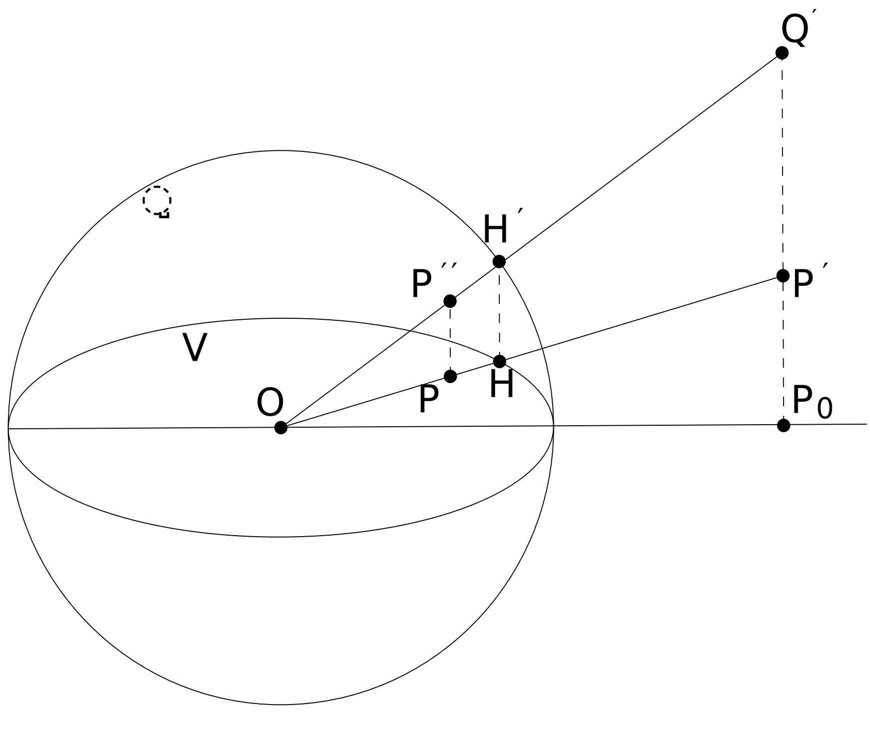

FIGURE 3. Illustration of Theorem 3.1 in the case $\mathcal{V}$ is an ellipse.

Since a compression is an affine transformation, it preserves ratios of colinear segments. Thus

$$\frac{\overline{OP}}{\overline{OP'}} = \frac{\overline{OP''}}{\overline{OQ'}} \text{ and } \frac{\overline{OH}}{\overline{OP'}} = \frac{\overline{OH'}}{\overline{OQ'}}.$$

From these equations, we obtain

$$\overline{OP''}\,\overline{OQ'} = \frac{\overline{OP}\,\overline{OQ'}^2}{\overline{OP'}} = \overline{OH'}^2. \tag{7}$$

Again, using the definition of inverse in a central conic, equation (7) means that $Q' = I_{\varrho}(P'')$ and so

$$\begin{aligned} \tau_{c^{-1}} \circ I_{\varrho} \circ \tau_c(P) &= \tau_{c^{-1}} \circ I_{\varrho}(P'') \\ &= \tau_{c^{-1}}(Q') \\ &= P' = I_{\mathcal{V}}(P) \end{aligned}$$

which completes the proof that

$$I_{\mathcal{V}} = \tau_{c^{-1}} \circ I_{\varrho} \circ \tau_c.$$

□

4. CONSEQUENCES OF THE MAIN THEOREM

From Theorem 3.1 we see that each circle ϱ corresponds to a familly of inversions in concentric ellipses $\mathcal{V}_i$. Likewise, each equilateral hyperbola ϱ corresponds to a familly of inversions in concentric hyperbolae $\mathcal{V}_i$. Two members of such a familly, which are inversions in concentric ellipses/hyperbolae $\mathcal{V}_1$ and $\mathcal{V}_2$, satisfy

$$\tau_{l_1,c_1} \circ I_{\mathcal{V}_1} \circ \tau_{l_1,\frac{1}{c_1}} = I_{\varrho} = \tau_{l_2,c_2} \circ I_{\mathcal{V}_2} \circ \tau_{l_2,\frac{1}{c_2}},$$

where $\tau_{l_i,c_i}, i = 1,2$ is a compression of ratio c_i over the axis l_i. Consequently, we have

$$I_{\mathcal{V}_1} = \tau_{l_1,\frac{1}{c_1}} \circ \tau_{l_2,c_2} \circ I_{\mathcal{V}_2} \circ \tau_{l_2,\frac{1}{c_2}} \circ \tau_{l_1,c_1}. \tag{8}$$

Since $\tau_{l_2,\frac{1}{c_2}} \circ \tau_{l_1,c_1}(\mathcal{V}_1) = \mathcal{V}_2$, it is clear that the composition of compressions $\phi = \tau_{l_2,\frac{1}{c_2}} \circ \tau_{l_1,c_1}$ is the composition of a rotation about the center and a compression of ratio $\dfrac{c_1}{c_2}$.

In the case of inversions in hyperbolae, the families are disjoint. But, in order to have disjoint families of inversions in ellipses, we must consider the circles to be only circunscribed auxiliary circles or only inscribed auxiliary circles.

The properties of inversions in ellipses cited in [1] and [6] are easily seen to be valid, since a compression preserves straight lines and maps circles onto ellipses. For example, the image of a straight line l by the inversion in an ellipse $\mathcal{V}_1$ would be the image, under the compression $\tau_{\frac{1}{c}}$, of the image of $\tau_c(l)$ by the inversion in the associated circle to $\mathcal{V}_1$. As such, it has to be an homothetic ellipse to $\mathcal{V}_1$ in the case where l does not pass through the centre, or a straight line otherwise.

Another interesting example, cited in [7], concerns the transformation of a pair of perpendicular straight lines under inversion in an ellipse $\mathcal{V}_1$ with center O. Suppose the point of intersection A, between the perpendicular lines s and t, is distinct from the center O of inversion and that $O \notin s$. Then the images of $\tau_c(s)$ and $\tau_c(t)$ by the inversion in the associated circle to $\mathcal{V}_1$ will be either

(1) one circle with tangent at O parallel to $\tau_c(s)$ which intercepts $\tau_c(t)$;
(2) two circles through O with tangents at O parallel to $\tau_c(s)$ and $\tau_c(t)$,

depending on whether t passes through O or not, respectively. Upon transformation under $\tau_{\frac{1}{c}}$, the tangents at O are mapped onto perpendicular lines through O and circles are mapped onto ellipses homothetic to $\mathcal{V}_1$. As the result of the inversion we get an ellipse and a straight line or two homothetic ellipses that intercept perpendicularly at O.

Another consequence of Theorem 3.1 concerns anallagmatic curves under inversion in a central conic $\mathcal{V}$.

Corollary 4.1. *A curve $\mathcal{C}$ is invariant by the inversion in a central conic $\mathcal{V}$ if, and only if, $\tau_{l,c}(\mathcal{C})$ is invariant by the inversion in ϱ, the associated curve to $\mathcal{V}$.*

Proof.

$$I_{\mathcal{V}}(\mathcal{C}) = \tau_{l,\frac{1}{c}} \circ I_{\varrho} \circ \tau_{l,c}(\mathcal{C}) = \mathcal{C} \Leftrightarrow \tau_{l,c}(\mathcal{C}) = I_{\varrho} \circ \tau_{l,c}(\mathcal{C}).$$

□

We end this communication by proving a result about inversion in ellipses which is a new complement to the properties already published in the literature. It is adapted from a theorem cited in [8] that expresses that lines and circles can be mapped into lines or concentric circles by an inversion in a circle.

Theorem 4.1. *Under an inversion in an ellipse $\mathcal{E}$, any two ellipses homothetic to $\mathcal{E}$, or a line and an ellipse homothetic to $\mathcal{E}$, can be transformed into concurrent lines, parallel lines, or two concentric ellipses.*

Proof. The proof will be developped by considering three cases, depending on whether the given objects (i.e. two ellipses or one ellipse and a line) have a single point in common, two points in common or no point in common.

Case 1 - a single point in common: Let $\mathcal{E}_1$ be an ellipse and let $\mathcal{E}_2$ be either an ellipse homothetic to $\mathcal{E}_1$ or a straight line such that $\mathcal{E}_1 \cap \mathcal{E}_2 = \{O\}$. Figure 4 illustrates this case when $\mathcal{E}_2$ is a straight line.

Let r be the straight line through O parallel to $\mathcal{E}_1$ major axis. By lemma 2.1, the compression $\tau_{r,c}$ over r with ratio c maps $\mathcal{E}_1$ onto a circle $\mathcal{C}'_1$ and $\tau_{r,c}(\mathcal{E}_2) = \mathcal{C}'_2$ is either a circle or a straight line such that $\mathcal{C}'_1 \cap \mathcal{C}'_2 = \{O\}$. Let Γ be a circle with center O and let I_Γ be the inversion in this circle. I_Γ maps $\mathcal{C}'_1$ and $\mathcal{C}'_2$ onto two parallel lines $\mathcal{C}''_1$ and $\mathcal{C}''_2$, respectively, according to [8]. The compression $\tau_{r,c^{-1}}$ maps parallel lines onto parallel lines and it maps Γ onto an ellipse $\mathcal{E}$ homothetic to $\mathcal{E}_1$, from corollary 2.1.

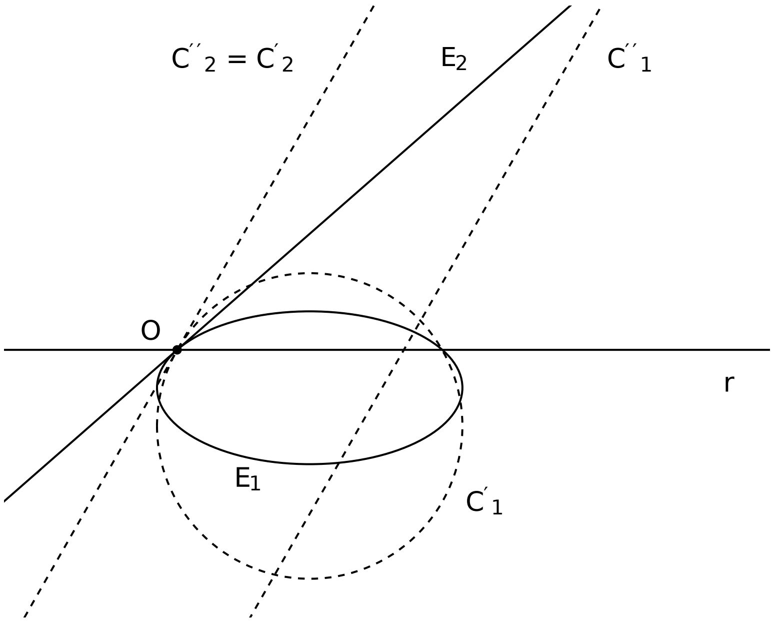

FIGURE 4. Ellipse $\mathcal{E}_1$ and tangent line $\mathcal{E}_2$.

Since we have $I_{\mathcal{E}} = \tau_{r,c^{-1}} \circ I_\Gamma \circ \tau_{r,c}$, the theorem is valid in this case.

Case 2 - two points in common: Let $\mathcal{E}_1$ be an ellipse and let $\mathcal{E}_2$ be either an ellipse homothetic to $\mathcal{E}_1$ or a straight line such that $\mathcal{E}_1 \cap \mathcal{E}_2 = \{O, A\}$. Figure 5 illustrates this case when $\mathcal{E}_2$ is a straight line.

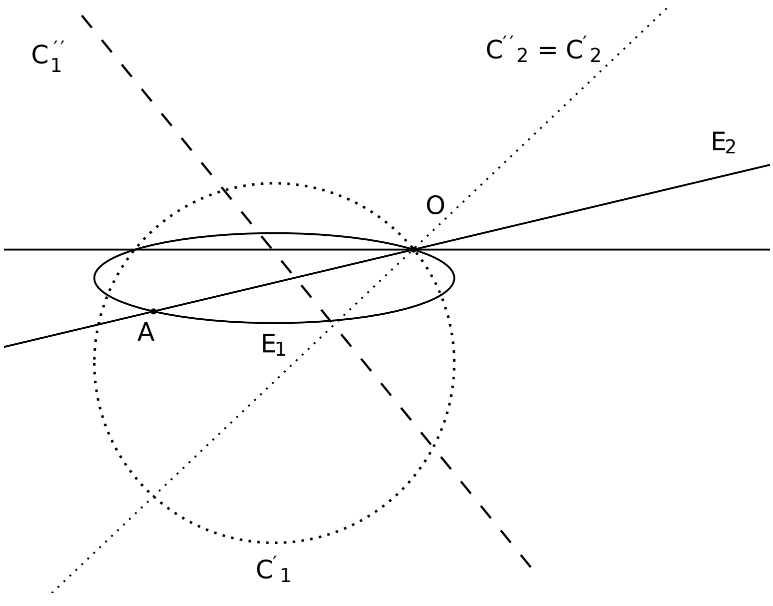

FIGURE 5. Ellipse $\mathcal{E}_1$ and secant line $\mathcal{E}_2$.

Let r be the straight line through O parallel to $\mathcal{E}_1$ major axis. By lemma 2.1, the compression $\tau_{r,c}$ over r with ratio c maps $\mathcal{E}_1$ onto a circle $\mathcal{C}'_1$ and $\tau_{r,c}(\mathcal{E}_2) = \mathcal{C}'_2$ is either a circle or a straight line such that $\mathcal{C}'_1 \cap \mathcal{C}'_2 = \{O, \tau_{r,c}(A)\}$. Let Γ be a circle with center O and let I_Γ be the inversion in this circle. I_Γ maps $\mathcal{C}'_1$ and $\mathcal{C}'_2$ onto two concurrent lines according to [8]. The compression $\tau_{r,c^{-1}}$ maps concurrent lines onto concurrent lines and it maps Γ onto an ellipse $\mathcal{E}$ homothetic to $\mathcal{E}_1$, as stated by corollary 2.1.
Since we have $I_{\mathcal{E}} = \tau_{r,c^{-1}} \circ I_\Gamma \circ \tau_{r,c}$, the theorem is valid in this case.

Case 3 - no point in common: Let $\mathcal{E}_1$ be an ellipse and let $\mathcal{E}_2$ be either an ellipse homothetic to $\mathcal{E}_1$ or a straight line such that $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$. Figure 6 illustrates this case when $\mathcal{E}_2$ is a straight line.
Let r be a straight line parallel to $\mathcal{E}_1$ major axis. By lemma 2.1, the compression $\tau_{r,c}$ over r with ratio c maps $\mathcal{E}_1$ onto a circle $\mathcal{C}'_1$ and $\tau_{r,c}(\mathcal{E}_2) = \mathcal{C}'_2$ is either a circle or a straight line such that $\mathcal{C}'_1 \cap \mathcal{C}'_2 = \emptyset$. According to [8], there is a circle Γ such that $I_\Gamma(\mathcal{C}'_2) = \mathcal{C}''_2$ and $I_\Gamma(\mathcal{C}'_1) = \mathcal{C}''_1$ are concentric circles. The images of the circles $\mathcal{C}''_2$ and $\mathcal{C}''_1$ under the compression $\tau_{r,c^{-1}}$ are concentric ellipses. The image of the circle Γ under the compression $\tau_{r,c^{-1}}$ is the desired ellipse of inversion $\mathcal{E}$, homothetic to $\mathcal{E}_1$. □

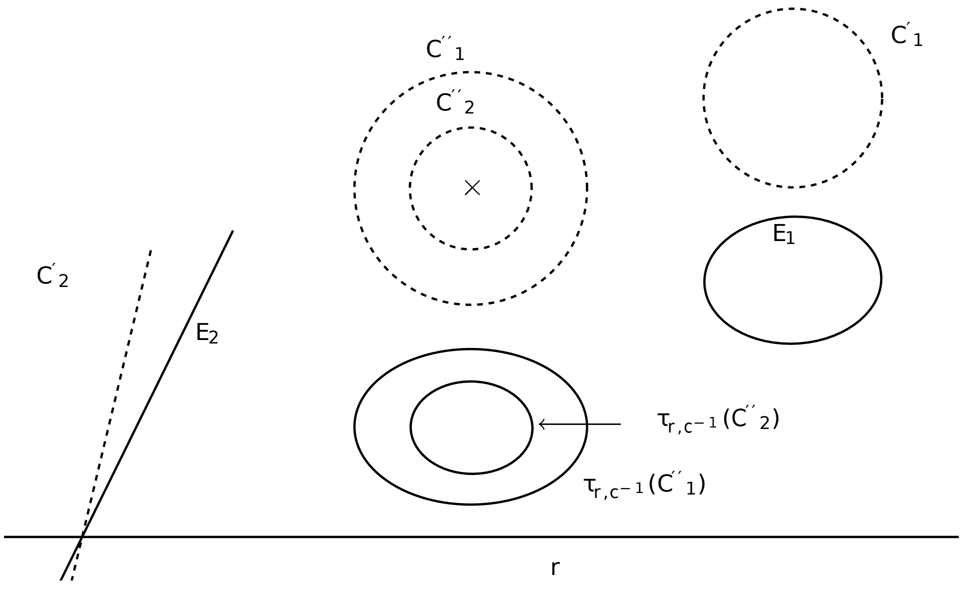

FIGURE 6. Ellipse $\mathcal{E}_1$ and line $\mathcal{E}_2$ with $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$.

5. CONCLUDING REMARKS

The use of plain euclidean geometry arguments has shed light upon the structure of inversions in central conics, which are compositions of compressions and inversions in circles or inversions in equilateral hyperbolae.

From this structure, the properties of inversions in these central conics follow directly from the corresponding properties of inversions in circles and equilateral hyperbolae.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRASÍLIA
BRASÍLIA, 70910-900 DF, BRAZIL
E-mail address: bessoni@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRASÍLIA
BRASÍLIA, 70910-900 DF, BRAZIL
E-mail address: guy@mat.unb.br