INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 9 (2020), No. 1, 25 - 39

SELF-INVERSE GEMINI TRIANGLES

CLARK KIMBERLING and PETER MOSES

Abstract. In the plane of a triangle ABC, every triangle center $U = u : v : w$ (barycentric coordinates) is associated with a central triangle having A-vertex $A_U = -u : v + w : v + w$. The triangle $A_U B_U C_U$ is a self-inverse Gemini triangle. Let $m_U(X)$ be the image of a point X under the collineation that maps A, B, C, G respectively onto A_U, B_U, C_U, G, where G is the centroid of ABC. Then $m_U(m_U(X)) = X$. Properties of the self-inverse mapping m_U are presented, with attention to associated conics (e.g., Jerabek, Kiepert, Feuerbach, Nagel, Steiner), as well as cubics of the types $pK(Y, Y)$ and $pK(U * Y, Y)$.

1. Introduction

The term *Gemini triangle*, introduced in the Encyclopedia of Triangle Centers (ETC [7], just before X(24537)), applies to certain triangles defined by barycentric coordinates. In keeping with the meaning of *gemini*, these triangles tend to occur in pairs. Specifically, for a given reference triangle ABC with sidelengths a, b, c, a Germini triangle $A'B'C'$ is a central triangle ([6], pp. 53-56) with A-vertex given by

$$A' = f(a, b, c) : g(b, c, a) : g(b, a, c), \tag{1.1}$$

where f and g are center functions ([6], p. 46). Because $A'B'C'$ is a central triangle, the vertices B' and C' can be read from (1.1) as

$$B' = g(c, b, a) : f(b, c, a) : g(c, a, b)$$
$$C' = g(a, b, c) : g(a, c, b) : f(c, a, b).$$

Now suppose that $U = u : v : w$ is a triangle center, and let

$$M_U = \begin{pmatrix} -u & v+w & v+w \\ w+u & -v & w+u \\ u+v & u+v & -w \end{pmatrix}.$$

Keywords and phrases Triangle, Gemini triangle, barycentric coordinates, collineation, conic, cubic, circumcircle, Jerabek, Kiepert, Feuerbach, Nagel, Steiner
(2010)Mathematics Subject Classification: 51N15, 51N20
Received: 13.10.2019. In revised form: 18.12.2019. Accepted: 27.11.2019

Then

$$M_U^2 = \begin{pmatrix} (u+v+w)^2 & 0 & 0 \\ 0 & (u+v+w)^2 & 0 \\ 0 & 0 & (u+v+w)^2 \end{pmatrix}.$$

Thus, interpreting the rows of M_U as points—that is,

$$\begin{aligned} A_U &= -u : v+w : v+w, \\ B_U &= w+u : -v : w+u, \\ C_U &= u+v : u+v : -w, \end{aligned} \tag{1.2}$$

we regard the matrix M_U as self-inverse in the sense that row 1 of M_U^2 represents the vertex $A = 1 : 0 : 0$, and similarly for rows 2 and 3. We call $A_U B_U C_U$ the *U-Gemini triangle.* If U is on the line at infinity, given by the equation $u + v + w = 0$ and denoted by L^∞, then the triangle M_U is the degenerate triangle consisting of the single point $G = 1 : 1 : 1$ = centroid of ABC. In the sequel we assume that $U \notin L^\infty$. Certain self-inverse Gemini triangles, indexed as Gemini triangles 101-107, are introduced in ETC just before X(30738).

2. The collineation mapping m_U

Throughout this section, let U be an arbitrary point not on L^∞. Let X be a point and $m_U(X)$ the image of X under the collineation (e.g., Coxeter [1]) that maps A, B, C, G onto A_U, B_U, C_U, G, respectively. Then

$$\begin{aligned} m_U(X) = &-ux + (u+w)y + (u+v)z \\ &: (v+w)x - vy + (v+u)z \\ &: (w+v)x + (w+u)y - wz, \end{aligned} \tag{2.1}$$

and $m_U(m_U(X)) = X$.

In this section, we are mainly interested in fixed lines and fixed points, defined as follows: a line L is *fixed* if $m_U(L) = L$, meaning that $m_U(P) \in L$ for every $P \in L$; and a point P is *fixed* if $m_U(P) = P$. We are also interested in the point $L \cap m_U(L)$ and conditions under which L and $m_U(L)$ are parallel or perpendicular. We begin with a theorem on the special role of the centroid.

Theorem 2.1. *For every point X, the image $m_U(X)$ lies on the line GX.*

Proof. Writing the vertices in (2.1) as $m_1 : m_2 : m_3$, we have

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0,$$

which establishes that $X, G, m_U(X)$ are collinear. □

Corollary 2.2. *The collineation m_U maps every line through G onto itself.*

Proof. Let L be a line through G, and let X be a point on L. By Theorem 2.1, we have $m_U(X)$ on GX, which is the line L. □

By Corollary (2.2), we have, in particular,

$$m_U(\text{Nagel line}) = \text{Nagel line}$$
$$m_U(\text{Euler line}) = \text{Euler line}.$$

Corollary 2.3. *Every fixed line passes through G.*

Proof. Suppose that L is a fixed line and $X \in L$. By Theorem 2.1, $m_U(G) \in GX$, so that $G \in M_U(L)$. As $M_U(G) = G$ and $M_U(L) = L$, we have $G \in L$. □

Having thus determined the fixed lines of m_U, we turn next to fixed points.

Theorem 2.4. *Suppose that a point $X = x : y : z$ is a fixed point. Then either $X = G$ or else there exists a point $p : q : r$ on L^∞ such that*

$$X = \frac{p}{v+w} : \frac{q}{w+u} : \frac{r}{u+v}. \tag{2.2}$$

Conversely, if $X = G$ or (2.2) holds, then $m_U(X) = X$.

Proof. The eigenvalues of M_U are

$$-u-v-w, -u-v-w, u+v+w.$$

Let X^t denote the transpose of the 1×3 vector (x, y, z). Then the repeated eigenvalue $\lambda = -u-v-w$ in the equation $M_U X^t = \lambda X^t$ gives three equations that are all equivalent to

$$(v+w)x + (w+u)y + (u+v)z = 0, \tag{2.3}$$

for which the solutions are clearly the points (2.2) for which $p + q + r = 0$; i.e., $p : q : r$ lies on L^∞. Conversely, if $P = p : q : r$ is on L^∞, then the barycentric product $P * \hat{U}$ where $\hat{U}$ denotes the isotomic conjugate of the complement of U, satisfies $m_U(m_U(X)) = X$.

Corresponding to the eigenvalue $u + v + w$ is the single fixed point G. □

Theorem 2.4 enables us to write barycentric coordinates for infinitely many fixed points of m_U. In the following list, n represents an arbitrary positive integer.

$$u^2 - vw : v^2 - wu : w^2 - uv$$

$$\frac{v^n - w^n}{v+w} : \frac{w^n - u^n}{w+u} : \frac{u^n - v^n}{u+v},$$

$$\frac{u^n(v^n - w^n)}{v+w} : \frac{v^n(w^n - u^n)}{w+u} : \frac{w^n(u^n - v^n)}{u+v},$$

$$\frac{2u^n - v^n - w^n}{v+w} : \frac{2v^n - w^n - u^n}{w+u} : \frac{2w^n - u^n - v^n}{u+v},$$

$$\frac{u^n(2u^n - v^n - w^n)}{v+w} : \frac{v^n(2v^n - w^n - u^n)}{w+u} : \frac{w^n(2w^n - u^n - v^n)}{u+v},$$

$$\frac{(v^n - w^n)(u^n - v^n - w^n)}{v+w} : \frac{(w^n - u^n)(v^n - w^n - u^n)}{w+u} : \frac{(u^n - v^n)(w^n - u^n - v^n)}{u+v}.$$

It is clear from these examples that the second and third coordinates can be easily read from the first coordinate; a few special cases from the above families are

$$v - w ::, \quad v^2 + vw + w^2 ::, \quad (v-w)(u+v)(u+w) ::, \quad u^2(v-w) :: .$$

In addition all those already noted are the fixed points

$$\frac{b^n - c^n}{v+w} :: \text{ and } \frac{a^n(b^n - c^n)}{v+w} ::,$$

as well as

$$0 : u+v : -u-w, \quad -v-u : 0 : v+w, \quad w+u, -w-v : 0,$$

and many more.

Theorem 2.5. *Suppose that L is a line given by*

$$fx + gy + hz = 0, \tag{2.4}$$

where $f \neq 0$. Then $m_U(L)$ is the line given by

$$\begin{aligned} &\Big(-fu + (g+h)(v+w)\Big)x \\ &+ \Big(-gv + (h+f)(w+u)\Big)y \\ &+ \Big(-hw + (f+g)(u+v)\Big)z \\ &= 0. \end{aligned} \tag{2.5}$$

Proof. Two points on L are $P_1 = -h : 0 : f$ and $P_2 = -g : f : 0$. By (2.1),

$$\begin{aligned} m_U(P_1) &= hu + f(u+v) : f(u+v) - h(v+w) : -fw - h(v+w), \\ m_U(P_2) &= gu + f(u+w) : -fv - g(v+w) : f(u+w) - g(v+w). \end{aligned}$$

Writing these as $f_1 : g_1 : h_1$ and $f_2 : g_2 : h_2$, we have the line $m_U(L)$ given by

$$\begin{vmatrix} x & y & z \\ f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \end{vmatrix} = 0,$$

which equals $s(f_3+g_3+h_3)$, where $s = (u+v+w)f$, and $f_3+g_3+h_3$ represents the left-hand side of (2.5). As we have assumed that $s \neq 0$, the equation (2.5) must hold. □

Corollary 2.6. *Suppose that the line L of Theorem 2.5 is not L^∞. Then the lines L and $m_U(L)$ intersect in the point*

$$g(u+v) - h(u+w) : h(v+w) - f(v+u) : f(w+u) - g(w+v).$$

Proof. This follows routinely from (2.5). □

Corollary 2.7. *The line $m_U(L^\infty)$ is given by*

$$(u - 2v - 2w)x + (v - 2w - 2u)y + (w - 2u - 2v)z = 0. \tag{2.6}$$

Proof. This equation comes from (2.5) by putting $f : g : h = 1 : 1 : 1$. □

A dual of Corollary (2.7) comes from (2.1) by putting $x : y : z = 1 : 1 : 1$; i.e., $m_U(G) = G$.

Corollary 2.8. *The lines L and $m_U(L)$ are parallel if and only if there exists a point $P = p : q : r$ on L^∞ satisfying*

$$\frac{p}{(g-h)u} : \frac{q}{(h-f)v} : \frac{r}{(f-g)w} \in L^\infty. \tag{2.7}$$

Proof. The line L in (2.4) meets L^∞ in the point

$$g-h : h-f : f-g. \tag{2.8}$$

For L to meet $m_U(L)$ in (2.8) it is necessary and sufficient that

$$\frac{g(u+v)-h(u+w)}{h(v+w)-f(v+u)} = \frac{g-h}{h-f} \text{ and } \frac{f(w+u)-g(w+v)}{g(u+v)-h(u+w)} = \frac{f-g}{g-h},$$

and likewise for $\frac{h-f}{f-g}$. The three equations are equivalent to

$$(g-h)u + (h-f)v + (f-g)w = 0,$$

which is equivalent to the existence of a point P as in (2.7). As the two lines meet in the same point on L^∞, they must be parallel. □

An example of a point P as in (2.7) is $(g-h)^2u : (h-f)^2v : (f-g)^2w$.

Corollary 2.9. *The lines L and $m_U(L)$ are perpendicular if and only if*

$$\begin{aligned} &S_A f(g(u+v)-h(u+w)) \\ &+ S_B g(h(v+w)-f(v+u)) \\ &+ S_C h(f(w+u)-g(w+v)) \\ &= 0, \end{aligned} \tag{2.9}$$

where S_A, S_B, S_C are Conway symbols (e.g., $S_A = bc\cos A$). In particular, if L is given by

$$(\tan A)x + (\tan B)y + (\tan C)z = 0,$$

then $m_U(L)$ is perpendicular to L.

Proof. The equation (2.9) is an immediate consequence of a general equation for perpendicular lines (e.g., [10], p. 54). Putting

$$f : g : h = a\sec A : b\sec B : c\sec C = \tan A : \tan B : \tan C$$

reduces the left-hand side of (2.9) to 0. □

3. Conics

As a collineation, m_U maps conics to conics, and in particular, m_U maps a circumconic (one that passes through the points A, B, C) onto a conic that passes though the vertices A_U, B_U, C_U of the U-Gemini triangle (1.2). In this section and the next, we continue to assume that U is an arbitrary point not on L^∞.

Theorem 3.1. *Let Ψ_P be the circumconic with perspector $P = p : q : r$, given by*

$$pyz + qzx + rxy = 0. \tag{3.1}$$

The conic $m_U(\Psi_P)$ *is given by*

$$\begin{aligned}(3.2)\qquad &\sum_{cyclic}(v+w)(pv+pw-qu-ru)x^2\\&+\sum_{cyclic}\Big((p+q+r)(u^2+uv+uw+2vw)-(qw+rv)(u+v+w)\Big)yz=0.\end{aligned}$$

Proof. An equation for the general conic through five distinct points $P_i = p_i : q_i : r_i$ is the following:

$$(3.3)\qquad \begin{vmatrix} x^2 & y^2 & z^2 & yz & zx & xy\\ p_1^2 & q_1^2 & r_1^2 & q_1r_1 & r_1p_1 & p_1q_1\\ p_2^2 & q_2^2 & r_2^2 & q_2r_2 & r_2p_2 & p_2q_2\\ p_3^2 & q_3^2 & r_3^2 & q_3r_3 & r_3p_3 & p_3q_3\\ p_4^2 & q_4^2 & r_4^2 & q_4r_4 & r_4p_4 & p_4q_4\\ p_5^2 & q_5^2 & r_5^2 & q_5r_5 & r_5p_5 & p_5q_5 \end{vmatrix} = 0.$$

The points $J_1 = p/(b-c)$ :: and $J_2 = bcp/(b-c)$ :: are on Ψ_P, as are A, B, C. Substituting

$$(P_1, P_2, P_3, P_4, P_5) = (A_U, B_U, C_U, m_U(J_1), m_U(J_2))$$

into (3.3) and simplifying yields (3.2). □

Corollary 3.2. *The center of the conic (3.2) is given by*

$$(3.4)\qquad f(a,b,c,u,v,w) : f(b,c,a,v,w,u) : f(c,a,b,w,u,v),$$

where

$$\begin{aligned} f(a,b,c,u,v,w) = {} & (p^2+2q^2+2r^2+3pq+3pr-4qr)u^2\\ &+r(2r-2p+q)v^2\\ &+q(2q-2p+r)w^2\\ &-(q^2+r^2+2pq+2pr+4qr)vw\\ &+(-2p^2+4q^2-r^2-3qr-3rp+pq)wu\\ &+(-2p^2-q^2+4r^2-3qr-3pq+pr)uv; \end{aligned}$$

or, equivalently, by

$$g(a,b,c,p,q,r) : g(b,c,a,q,r,p) : g(c,a,b,r,p,q),$$

where

$$\begin{aligned} g(a,b,c,p,q,r) = {} & u(u-2v-2w)p^2\\ &+(2w^2+2u^2-uv-vw+4uw)q^2\\ &+(2u^2+2v^2-uw-vw+4uv)r^2\\ &+(v^2+w^2-4u^2-3uv-3uw-4vw)qr\\ &+(3u^2-2v^2+uv-2vw-3uw)rp\\ &+(3u^2-2w^2+uw-2vw-3uv)pq. \end{aligned}$$

Proof. This is an application of straightforward methods (e.g., [10], p. 124). □

Corollary 3.3. *Let P' denote the center of the circumconic Ψ_P in (3.1). The center of the conic $m_U(\Psi_P)$ lies on the line GP' if and only if U lies on the line given by*

$$(q-r)x+(r-p)y+(p-q)z=0.$$

Proof. Substituting the point (3.4) for x, y, z in the equation

$$(q-r)(-p+q+r)x+(r-p)(p-q+r)y+(p-q)(p+q-r)z=0$$

for the line GP' yields

$$6pqr(u+v+w)\Big((q-r)u+(r-p)v+(p-q)w\Big)=0. \qquad \square$$

Corollary 3.4. *If $U=P$, then the conic $m_U(\Psi_P)$ is a circumconic.*

Proof. The equation (3.2) reduces to

$$\Big(2qr+(p+q+r)p\Big)yz+\Big(2rp+(p+q+r)\Big)zx+\Big(2pq+(p+q+r)\Big)xy=0. \qquad \square$$

Certain choices of P yield families of conics whose members, for various choices of U, are of individual interest. Sections 4-11 give examples of such conics.

4. **Examples: $\Psi_P =$ circumcircle, $P = X(6)$**

Here, the circumconic Ψ_P is the circumcircle, denoted by Γ, given by $P = X(6)$, the symmedian point. For each point U, the center of the conic $m_U(\Psi_P)$ lies on the Euler line.

Example 4.1. *Let $U=G=1:1:1$. Then the conic (3.3) is given by*

$$2\Big((2a^2-b^2-c^2)x^2+(2b^2-c^2-a^2)y^2+(2c^2-a^2-b^2)z^2\Big)$$
$$+(5a^2+2b^2+2c^2)yz+(5b^2+2c^2+2a^2)zx+(5c^2+2a^2+2b^2)xy=0.$$

Actually, the mapping m_U is simply reflection in G, so that this conic is the circle with center $X(381)$ that passes through $X(i)$ for $i = 671, 6054, 9140$, and 10706-10720.

Example 4.2. *Let $U=a^2:b^2:c^2=X(6)$. In this case, $m_U(\Gamma)=\Gamma$. By Theorem 2.1, if $X\in\Gamma$, then $m_U(X)$ is the point, other than X, where the line GX meets Γ.*

Example 4.3. *Let $U=-a^2+b^2+c^2:a^2-b^2+c^2:a^2+b^2-c^2=X(69)$. Here, $m_U(\Gamma)$ is the conic given by*

$$\sum_{cyclic}\Big(2a^2(2a^2-b^2-c^2)x^2+((a^4-a^2(b^2+c^2)-2(b^4-4b^2c^2+c^4))yz\Big)=0,$$

with center $f(a,b,c):f(b,c,a):f(c,a,b)$, where

$$f(a,b,c)=5(a^6+2b^6+2c^6)-10a^4(b^2+c^2)-a^2(5b^4-22b^2c^2+5c^4)-10b^2c^2(b^2+c^2).$$

Example 4.4. *Let $U=b^2+c^2:c^2+a^2:a^2+b^2=X(141)$. Here, $m_U(\Gamma)$ is the conic given by*

$$\sum_{cyclic}\Big((4a^4-(b^2+c^2)^2)x^2+2(a^2(a^2+b^2+c^2)+3b^2c^2)yz\Big)=0,$$

with center $g(a,b,c):g(b,c,a):g(c,a,b)$, where

$$g(a,b,c)=2b^6+2c^6-2(b^2+c^2)(a^4+b^2c^2)+a^2b^2c^2.$$

5. **Examples:** $\Psi_P =$ **Jerabek hyperbola,** $P = X(647)$

The Jerabek hyperbola ([8], [9]), given by

$$a^2(b^2-c^2)(b^2+c^2-a^2)yz + b^2(c^2-a^2)(c^2+a^2-b^2)zx + c^2(a^2-b^2)(a^2+b^2-c^2)xy = 0,$$

is the isogonal conjugate of the Euler line. As a first example, $m_P(\Psi_P) = \Psi_P$. Another example follows:

Example 5.1. *Let* $U = 1:1:1 = X(2)$*. Here,* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic} (b^2-c^2)(b^2+c^2-a^2)(2x^2+yz) = 0,$$

The conic $m_U(\Psi_P)$ passes through the points $X(i)$ for these i: 376, 381, 599, 1992, 2574, 2575, 10706, 24473, 31165, 31166.

6. **Examples:** $\Psi_P, P = X(525)$

The circumconic with perspector $X(525)$ is given by

$$(b^2-c^2)(b^2+c^2-a^2)yz + (c^2-a^2)(c^2+a^2-b^2)zx + (a^2-b^2)(a^2+b^2-c^2)xy = 0.$$

Example 6.1. *Let* $U = 1:1:1 = X(2)$*, so that* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic} (b^4-c^4)(b^2+c^2-a^2)(2x^2+yz) = 0,$$

This conic passes through no points $X(i)$ for $1 \le i \le 32000$.

Example 6.2. *Let* $U = X(6)$*, so that* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic} (b^2-c^2)\Big((b^2+c^2)(b^2+c^2-a^2)x^2 + (b^4+c^4-a^4)yz\Big) = 0.$$

This conic passes through the vertices of the circum-medial triangle (Gemini triangle 44) and the points $X(i)$ for these i: 2, 112, 251, 6032, 7735, 8105, 8106, 8879, 15479, 15437, 22240.

7. **Examples:** $\Psi_P =$ **Kiepert hyperbola,** $P = X(523)$

The Kiepert hyperbola [2] is given by

$$(b^2-c^2)yz + (c^2-a^2)zx + (a^2-b^2)xy = 0.$$

The conic $m_U(\Psi_P)$ is described in the next examples for three choices of U.

Example 7.1. *Let* $U = 1:1:1$*, so that* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic} (b^2-c^2)(2x^2+yz) = 0,$$

which passes through $X(i)$ for these i: 2, 99, 376, 551, 3413, 3414, 5465, 5464, 6054, 7757, 8592, 9168, 9741, 13712, 13835, 14482, 22712, 31168.

Example 7.2. *Let* $U = a^2 : b^2 : c^2 = X(6)$*, so that* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic} \Big((b^4-c^4)x^2 + (b^2-c^2)(b^2+c^2-a^2)yz\Big) = 0,$$

which passes through $X(i)$ for these i: 2, 110, 7493, 8793, 9168, 9465, 9829, 10130, 26230.

Example 7.3. *Let* $U = X(3569) = a^2(b^2-c^2)(b^4+c^4-a^2b^2-a^2c^2) :: $, *so that* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic}\Big((b^2-c^2)^2(a^4-b^2c^2)(b^2+c^2-a^2)x^2+a^2(a^2-b^2)(c^2-a^2)(b^4+c^4-a^2b^2-a^2c^2)yz\Big)=0,$$

which passes through $X(i)$ for these i: 2, 4, 39, 114, 511, 543, 626, 1916, 5978, 5979, 31173.

Example 7.4. *Let* $U = X(9979) = (b^2-c^2)(b^4+c^4-a^4-b^2c^2) :: $, *so that* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic}(b^2+c^2-2a^2)\Big((b^2-c^2)^2(b^2+c^2-a^2)x^2+a^2(a^2-b^2)(a^2-c^2)yz\Big)=0,$$

which passes through $X(i)$ for these i: 2, 4, 316, 530, 531, 671, 6054, 7827, 7883, 31862, 31863.

8. Examples: Ψ_P = Feuerbach hyperbola, $P = X(650)$

The Feuerbach hyperbola is given by

$$a(b-c)(b+c-a)yz+b(c-a)(c+a-b)zx+c(a-b)(a+b-c)=0.$$

As a first example, $m_P(\Psi_P)=\Psi_P$. The conic $m_U(\Psi_P)$ is described in the next examples for two other choices of U.

Example 8.1. *Let* $U = 1:1:1$, *so that* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic}(b-c)\Big(2(a^2-ab-ac-2bc)x^2+(7a^2-7ab-7ac+4bc)yz\Big)=0, \tag{8.1}$$

which passes through $X(i)$ for these i: 376, 3241, 3307, 3308, 3679, 6172, 6173, 6175, 10711.

Example 8.2. *Let* $U = X(4025) = (b-c)(b^2+c^2-a^2) ::$.

The conic $m_U(\Psi_P)$ passes through $X(i)$ for these i: 1, 3, 7, 142, 377, 1387, 3616, 6224, 10427, 31637.

9. Example: Ψ_P = Steiner circumellipse, $P = X(1)$

The Steiner circumellipse is given by $yz+zx+xy=0$.

Example 9.1. *Let* $U = X(1)$, *so that* $m_U(\Psi_P)$ *is the conic given by*

$$\sum_{cyclic}\Big((b+c)(b+c-2a)x^2+(3a^2-b^2-c^2+2ab+2ac+4bc)yz\Big)=0,$$

which passes through $X(i)$ for these i: 115, 543, 2479, 2480, 6103, 23967.

10. Examples: Ψ_P = Nagel hyperbola, $P = X(8)$

The term *Nagel hyperbola* is introduced here for the circumhyperbola having as perspector the Nagel point, $X(8) = b - c : c - a : a - b$. The Nagel hyperbola, given by

$$(b-c)yz + (c-a)zx + (a-b)xy = 0,$$

passes through $X(i)$ for these i:

$$2, 7, 27, 75, 86, 234, 272, 273, 310, 335, 554, 673, 675, 871, 903, 1081, 1088, 1223,$$
$$1240, 1246, 1268, 1440, 1659, 2296, 2400, 2989, 4373, 5936, 6384, 6548, 6650, 7249,$$
$$7318, 8049, 13390, 14621, 15467, 16078, 16099, 18815, 18884, 19975, 20028, 20527,$$
$$21453, 24154, 24155, 27447, 27475, 27483, 27494, 27498, 28626, 30598, 30712, 31002.$$

Example 10.1. *Let $U = X(1)$, so that $m_U(\Psi_P)$ is the conic given by*

$$\sum_{cyclic} (b-c)\Big((b+c)x^2 + (b+c-a)yz\Big) = 0,$$

with center $X(32043)$. The conic $m_U(\Psi_P)$ passes through the vertices of Gemini triangle 2 and $X(i)$ for these i: 2, 100, 1817, 4850, 5235, 5744, 25057.

Example 10.2. *Let $U = X(2)$, so that $m_U(\Psi_P)$ is the conic given by*

$$\sum_{cyclic} (b-c)(2x^2 + yz) = 0,$$

with center $X(4370)$. The conic $m_U(\Psi_P)$ passes through the vertices of Gemini triangle 107 and $X(i)$ for these i: 2, 190, 4664, 6172, 31144, 31153, 31332, 31349, 31992.

Example 10.3. *Let $U = X(6)$, so that $m_U(\Psi_P)$ is the conic given by*

$$\sum_{cyclic} (b-c)\Big((b^2+c^2)x^2 + (b^2+c^2+bc-ab-ac)yz\Big) = 0,$$

which passes through the vertices of the circum-medial triangle (Gemini triangle 44) and $X(i)$ for these i: 2, 101, 26242, 26244, 26252, 26258.

Example 10.4. *Let $U = X(10)$, so that $m_U(\Psi_P)$ is the conic given by*

$$\sum_{cyclic} (b-c)\Big((2a+b+c)x^2 + 2ayz\Big) = 0,$$

with center $X(32045)$. The conic $m_U(\Psi_P)$ passes through the vertices of Gemini triangle 105 and $X(i)$ for these i: 2, 3952, 27081, 31018, 31035, 31045, 31992.

Example 10.5. *Let $U = X(75)$, so that $m_U(\Psi_P)$ is the conic given by*

$$\sum_{cyclic} a(b-c)\Big((b+c)x^2 + ayz\Big) = 0,$$

with center $X(32044)$. The conic $m_U(\Psi_P)$ passes through the vertices of Gemini triangle 104 and $X(i)$ for these i: 2, 668, 17149, 30946, 30963, 30966, 30973, 31008, 31341.

Example 10.6. *Let $U = X(4010) = (b^2-c^2)(a^2-bc) :: .$*

The conic $m_U(\Psi_P)$ passes through $X(i)$ for these i: 2, 75, 545, 740, 1213, 6650, 17045, 17257, 17755, 26582.

Example 10.7. *Let* $U = X(4784) = a(b-c)(a^2+2ab+2ac+bc) :: .$

The conic $m_U(\Psi_P)$ passes through $X(i)$ for these i: 2, 37, 86, 4649, 17239, 24325, 27494, 31306, 31336.

11. Example: Ψ_P = Kiepert circumparabola, $P = X(115)$

The term *Kiepert circumparabola* (not to be confused with the Kiepert parabola [2]) is introduced here as the circumparabola having as perspector the center of the Kiepert hyperbola, which is $X(115) = (b^2-c^2)^2 ::$, so that an equation for Ψ_P is

$$(b^2-c^2)^2yz + (c^2-a^2)^2zx + (a^2-b^2)^2xy = 0.$$

(It is easy to see ([10], p. 127), that in general, a circumconic $pyz + qzx + rxy = 0$ is a parabola if and only if $p : q : r$ lies on the Steiner inellipse. The Kiepert circumparabola can thus be regarded as representative of a class of circumconics that seems under-represented in the existing lore of triangle geometry.) The Kiepert circumparabola passes through $X(i)$ for these i:

$$476, 523, 685, 850, 892, 2395, 2501, 4024, 4036, 4581, 4608, 5466, 8599, 10412,$$
$$12065, 12079, 13636, 13722, 14775, 15328, 18808, 20578, 20579, 30508, 30509, 31065.$$

As an example, for $U = X(141) = b^2+c^2 : c^2+a^2 : a^2+b^2$, the conic $m_U(\Psi_P)$ is a hyperbola with center

$$X(31949) = (b^2-c^2)(7a^4 + 11a^2b^2 + 10b^4 + 11a^2c^2 + 23b^2c^2 + 10c^4) ::$$

This hyperbola passes through $X(i)$ for these i: 523, 3005, 9168, 31065, 31296, 31950.

12. Cubics of type $pK(W, W)$

In this section and the next, we show that if U is a triangle center, then the U-Gemini triangle $A_UB_UC_U$ is perspective to several central triangles. These triangles are associated with classes of cubic curves whose geometric properties are described by Ehrmann and Gibert ([3], [4], [5]).

Theorem 12.1. *The locus of a point* $X = x : y : z$ *such that the cevian triangle of* X *is perspective to* $A_UB_UC_U$ *is the cubic given by*

$$vw(y-z)(yz-xy-xz) + wu(z-x)(zx-yz-yx) + uv(x-y)(xy-zx-zy) = 0. \tag{12.1}$$

Proof. The A-vertex of the cevian triangle of X is $A' = 0 : y : z$, so that the coefficients for the line A_UA' are given by row 1 of the determinant

$$\begin{vmatrix} (v+w)(y-z) & -uz & uy \\ vz & (w+u)(z-x) & -vx \\ -wy & wx & (u+v)(x-y) \end{vmatrix}.$$

Rows 2 and 3 represent the lines B_UB' and C_UC'. The three lines concur if and only if the determinant is 0, which yields (12.1). □

The cubic (12.1) is of the type $pK(W,W)$ in Gibert's classification [4], where

$$W = \frac{1}{u(v+w)} : \frac{1}{v(w+u)} : \frac{1}{w(u+v)}$$

is the isotomic conjugate of the crosspoint of U and G.

Here are a few examples of the pairs U, W:

U:	$X(4)$	$X(330)$	$X(2992)$	$X(2993)$	$X(7261)$
W:	$X(76)$	$X(1)$	$X(300)$	$X(301)$	$X(334)$

Points on the cubic (12.1) include A, B, C, G, and these:

$$vw : wu : uv = \text{isotomic conjugate of } U$$

$$\frac{1}{v+w} : \frac{1}{w+u} : \frac{1}{u+v}$$

$$\frac{vw}{v+w} : \frac{wu}{w+u} : \frac{uv}{u+v}$$

$$\frac{-u+v+w}{v+w} : \frac{u-v+w}{w+u} : \frac{u+v-w}{u+v}$$

$$u \pm d : v \pm d : w \pm d, \text{where } d = \sqrt{vw+wu+uv}.$$

With regard to the points involving d, note that

$$(u+v+w)^2 = u^2 + v^2 + w^2 + 2(vw + wu + uv). \tag{12.2}$$

Now suppose that U lies on the Steiner inellipse, which is given by

$$u^2 + v^2 + w^2 - 2(vw + wu + uv) = 0.$$

Then by (12.2),

$$(u+v+w)^2 = 4(vw + wu + uv),$$

so that $d = (u+v+w)/2$. To summarize, the points $u \pm d : v \pm d : w \pm d$ are radical-free if U is on the Steiner inellipse.

13. Cubics of type $pK(U * Y, U)$

Theorem 13.1. *The locus of a point $X = x : y : z$ such that the anticevian triangle of X is perspective to $A_U B_U C_U$ is the cubic*

$$(y-z)(vwx^2 + u^2yz) + (z-x)(wuy^2 + v^2zx) + (x-y)(uvz^2 + w^2xy) = 0. \tag{13.1}$$

Proof. The A-vertex of the anticevian triangle of X is $-x : y : z$. The method of proof of (12.1) gives (13.1) as a simplification of

$$\begin{vmatrix} (v+w)(y-z) & -uz+xv+xw & uy-xv-xw \\ vz-yw-yu & (w+u)(z-x) & -vx+yw+yu \\ -wy+zu+zv & wx-zu-zv & (u+v)(x-y)) \end{vmatrix} = 0.$$

□

The cubic (13.1) is of type $pK(U*Y,Y)$, where * denotes barycentric product and

$$Y=\frac{1}{v+w}:\frac{1}{w+u}:\frac{1}{u+v},$$

this being the cevapoint of G and U, as well as the isotomic conjugate of the complement of U. Among the points on (13.1) are U, Y, and $U*Y$.

14. Construction of a self-inverse Gemini triangle

Here, we return to the cubic (12.1) in order to determine the perspector of the U-Gemini triangle $A_UB_UC_U$ with a known constructible triangle, and then we use that point to construct the vertex A_U.

Theorem 14.1. *The perspector of $A_UB_UC_U$ and the cevian triangle of the isotomic conjugate of U is the point*

$$P=u(-u^2+v^2+w^2):v(u^2-v^2+w^2):w(u^2+v^2-w^2). \tag{14.1}$$

Proof. The isotomic conjugate of U is the point $U'=1/u:1/v:1/w$. As U' lies on the cubic (12.1), its cevian triangle $A'B'C'$ is perspective to $A_UB_UC_U$. The lines $B'B_U$ and $C'C_U$ are given by

$$-vux+(w^2-u^2)y+vwz=0,$$
$$wux-xvy+(u^2-v^2)z=0,$$

respectively, so that $B'B_U\cap C'C_U$ is the point (14.1), which also happens to be the U'-Ceva conjugate of U; i.e., the perspector P of the cevian triangle of U and the anticevian triangle of U'. □

Now, to construct the self-inverse Gemini triangle $A_UB_UC_U$, we begin with U, and then U', and then the U'-Ceva conjugate of U. We then have $A_U=AG\cap A'P$, where $A'=0:w:v$ is the A-vertex of the cevian triangle of U'. Likewise, $B_U=BG\cap B'P$ and $C_U=CG\cap C'P$, as in Figure 1.

15. A special case

If $U=a^2:b^2:c^2$, the symmedian point of ABC, then the U-Gemini triangle is given by

$$A_U=-a^2:b^2+c^2:b^2+c^2$$
$$B_U=c^2+a^2:-b^2:c^2+a^2$$
$$C_U=a^2+b^2:a^2+b^2:-c^2,$$

this being the circum-medial triangle of ABC ([6], p. 162).

Theorem 15.1. *The locus of a point $X=x:y:z$ such that the antipedal triangle of X is perspective to the circum-medial triangle is the union of the Euler line, the circumcircle, and the line at infinity.*

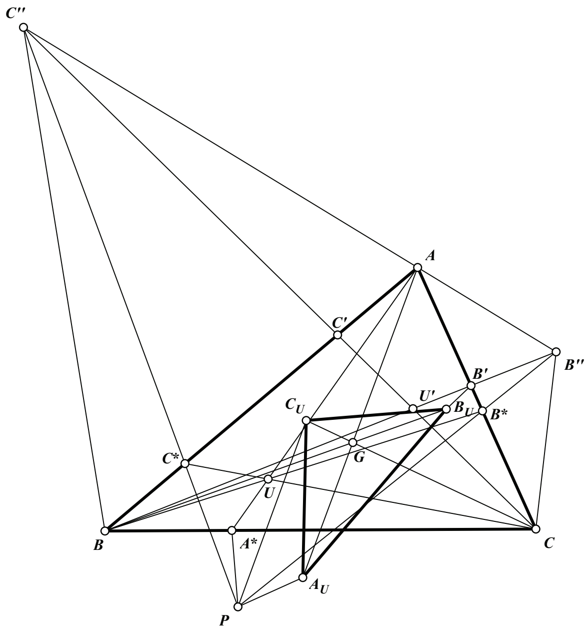

FIGURE 1. A point U and the U-Gemini triangle $A_UB_UC_U$

Proof. Let $A'B'C'$ be the antipedal triangle of X; then

$$\begin{aligned} A' &= -a(ay + bx\cos C)(az + cx\cos B) \\ &: b(az + cx\cos B)(bx + ay\cos C) \\ &: c(ay + b\cos C)(cx + az\cos B). \end{aligned}$$

The lines $A'A_U, B'B_U, C'C_U$ are represented as $a_ix + b_iy + c_iz = 0$ for $i = 1, 2, 3$, as the actual coefficients are too long to be shown here. The equation for perspectivity is then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

The determinant expands as

$$S(a, b, c)(x + y + z)(a^2yz + b^2zx + c^2xy)^2(E_1x + E_2y + E_3z),$$

where $S(a, b, c)$ is a function symmetric in a, b, c, and E_i are coefficients for the Euler line; e.g.,

$$E_1 = (b^2 - c^2)(a^2 - b^2 - c^2).$$

The proof is finished, as L^∞ is given by $x + y + z = 0$ and Γ by $a^2yz + b^2zx + c^2xy = 0$. □

The loci of $m_U(X)$ as X ranges through Γ and L^∞ are easily described. First, let L_A be the line through A perpendicular to line AX. Let X_A be the point of intersection, other than A, of L_A and Γ. The perpendicularity implies that X_A is the antipode of X. Define X_B and X_C cyclically. They, too, are antipodes of X, so that the antipedal triangle of X is simply the point X_A. Trivially, this degenerate triangle is perspective to the circum-medial triangle, with perspector X_A. Thus, the first locus mentioned is Γ.

Next, suppose that X is on L^∞. Then the lines AX, BX, CX are parallel, so that their perpendiculars through A, B, C are also parallel and therefore meet on L^∞, which is the second locus. The interested reader may wish to investigate the locus of the perspector as X ranges through the Euler line.

References

[1] H. S. M. Coxeter, *The Real Projective Plane*, Third Edition, Springer-Verlag **(1993)**.

[2] R. H. Eddy and R. Fritsch, The Conics of Ludwig Kieprt: A Comprehensive Lesson in the Geometry of the Triangle, *Math. Mag.* **67 (1994)** 188-205.

[3] Bernard Gibert, *Cubics in the Triangle Plane*, `https://bernard-gibert.pagesperso-orange.fr`.

[4] Bernard Gibert, *Parallel Tripolar Cubics, $pK(W,W)$*, `https://bernard-gibert.pagesperso-orange.fr/Classes/cl007.html`.

[5] Jean-Pierre Ehrmann and Bernard Gibert, *Special Isocubics in the Triangle Plane*, `https://bernard-gibert.pagesperso-orange.fr/files/Resources/SITP.pdf`.

[6] Clark Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium* **129 (1998)** *i-xxv*, 1-295.

[7] Clark Kimberling, *Encyclopedia of Triangle Centers*, `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[8] Paris Pamfilos, On the Jerabek Hyperbola, *International Journal of Geometry*, **7 (2018)** 5-11.

[9] Paris Pamfilos, Second Note on Jerabek's Hyperbola, *International Journal of Geometry*, **8 (2019)** 32-37.

[10] Paul Yiu, *Introduction to the Geometry of the Triangle* (2012) `http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry121226.pdf`.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF EVANSVILLE
1800 LINCOLN AVENUE
EVANSVILLE, INDIANA 47722 USA
E-mail address: ck6@evansville.edu

PETER J. C. MOSES
ENGINEERING DIVISION
MOPARMATIC CO.
1154 EVESHAM ROAD
ASTWOOD BANK, NR REDDITCH
WORCESTERSHIRE, B96 6DT, UK
E-mail address: moparmatic@gmail.com