INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 8 (2019), No. 2, 70 - 83

THE NON-REDUNDANCY OF SIDE-ANGLE-SIDE AS AN AXIOM OF THE NON-CONTINUOUS NON-EUCLIDEAN PLANE

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Abstract. If we start with continuous absolute geometry, remove Side-Angle-Side as an axiom and replace it with exactly one of Angle-Side-Angle, Side-Angle-Angle, or Side-Side-Side as a new axiom, then we again end up with continuous absolute geometry. Martin has asked the question: if we start with absolute geometry, remove Side-Angle-Side as an axiom and replace it with Angle-Angle-Angle as a new axiom, then do we again end up with absolute geometry? In this paper, we give a negative answer to this question for non-continuous absolute geometry. In particular, we construct a non-continuous model $\mathbb{A}$ in which the Angle-Angle-Angle criterion for congruence of triangles holds, yet in which Side-Angle-Side does not hold.

1. Introduction

Plane geometry is a classical subject that dates back to the time of the ancient Greeks. One of the first known texts on the subject is Euclid's *Elements*. In *Elements*, Euclid develops an axiom system for plane geometry. One of the axioms that Euclid assumes in his axiomatic development is the famous Euclidean Parallel Postulate. Through the centuries, many mathematicians worked on the question of whether or not one could prove the Euclidean Parallel Postulate as a theorem of the other axioms of plane geometry. It was finally shown separately by Bolyai, Lobachevsky, and Gauss that the Euclidean Parallel Postulate is independent of the other axioms of plane geometry. While showing that the Euclidean Parallel Postulate can not be proven as a theorem, Bolyai developed the notion of Absolute Geometry [9].

Absolute Geometry is plane geometry in which we assume no parallel postulate. Absolute Geometry can be thought of as being a common ground between Euclidean Geometry and Hyperbolic Geometry, and the axioms of Absolute Geometry are satisfied by both Euclidean and Hyperbolic Geometry [6],[7],[9],[10]. There are several ways to develop an axiom system for Absolute Geometry.

Keywords and phrases: Side-Angle-Side, Axiom, Non-Euclidean Plane
(2010)Mathematics Subject Classification: 51A99
Received: 19.04.2019. In revised form: 03.09.2019. Accepted: 19.08.2019.

One way is to use the metric approach given by G.D. Birkhoff in *A Set of Postulates for Plane Geometry Based on Scale and Protractor*. The axiom system given by Birkhoff incorporates the real numbers into the axiomatic development, and in particular uses the notions of distance and angle measure when stating the axioms [1]. This approach to develop an axiom system for a absolute plane is also used by G. E. Martin in *The Foundations of Geometry and the Non-Euclidean Plane*, by R. S. Millman and G. D. Parker in *Geometry: A Metric Approach with Models*, and by E. E. Moise in *Elementary Geometry from an Advanced Standpoint* [9], [10], [11].

Another way to develop an axiom system for absolute geometry is to use the synthetic approach given by David Hilbert in *Grundlagen der Geometrie*. Unlike Birkhoff's axiomatic development, Hilbert refrains from introducing the use of the real numbers until the very end [6],[7],[8]. The axiom system given by Hilbert first develops plane geometry in a non-continuous, abstract way, and proves many results without the use of the real numbers. This approach to developing an axiom system for an absolute plane is also used by M. J. Greenberg in *Euclidean and Non-Euclidean Geometries: Development and History*, and by R. Hartshorne in *Geometry: Euclid and Beyond* [6],[7].

It can be shown that the absolute plane that one gets from the axiom system of Birkhoff is the same as the absolute plane that one gets from the axiom system of Hilbert [6],[9]. In both of these approaches, the Side-Angle-Side criterion for congruence of triangles is assumed as an axiom. After assuming Side-Angle-Side as an axiom, then one can prove the Angle-Side-Angle, Side-Angle-Angle, and Side-Side-Side criteria for congruence of triangles as theorems [6], [7], [9], [10].

It is well known that Angle-Angle-Angle is logically equivalent to the Hyperbolic Parallel Postulate, and therefore can not be proven as a theorem in Absolute geometry [9].

It is also well known that if one removes Side-Angle-Side as an axiom and replaces it with Angle-Side-Angle as a new axiom, then the resulting new axiom system is also an absolute plane. In particular, one can prove that Side-Angle-Side still holds in the new axiom system [9].

In [2], [3], and [4], the author shows that if one uses the metric approach of Birkhoff to develop an axiom system for Absolute Geometry, removes Side-Angle-Side as an axiom of absolute geometry, and then replaces it with exactly one of either Side-Angle-Angle or Side-Side-Side as a new axiom, then each of the resulting new axiom systems is an absolute plane. Again, one can show that Side-Angle-Side still holds in either of the new axiom systems.

In [9], Martin asks the question: if we start with an axiom system for Absolute Geometry, remove Side-Angle-Side as an axiom, and replace it with Angle-Angle-Angle as a new axiom, then is the resulting new axiom system an absolute plane? If the answer to this question were yes, then we would be able to replace the two axioms of Side-Angle-Side and the Hyperbolic Parallel Postulate with the single axiom of Angle-Angle-Angle.

In this paper, we construct a model $\mathbb{A}$ of non-continuous plane geometry in which all of the Incidence Axioms, Betweenness Axioms, and Congruence Axioms (with the single exception of Side-Angle-Side) from Hilbert's axiom system (as communicated in [7]) hold. In addition, we show that Angle-Angle-Angle holds in $\mathbb{A}$, but that Side-Angle-Side does not hold in $\mathbb{A}$. To construct $\mathbb{A}$, we start with the set $\mathbb{Q} \times \mathbb{Q}$, and define distance using the taxicab metric, so that all distances are rational. We initially define angle measure to be usual Euclidean angle measure. We then proceed to alter angle measure is such a way that Angle-Angle-Angle holds, yet in which Side-Angle-Side does not hold. We use the fact that $\mathbb{Q} \times \mathbb{Q}$ is countable to alter distance and angle measure when necessary. We alter distance and angle measure so that the new distances are also rational, and so that the axioms stated in Section 2 hold in the model $\mathbb{A}$. We alter distance and angle measure so that no two distinct triangles in $\mathbb{A}$ satisfy the hypothesis of Angle-Angle-Angle. Moreover, we define $\mathbb{A}$ so that the only time the hypothesis of Angle-Angle-Angle is satisfied is by a single triangle $\triangle ABC$ two of whose angles are congruent to each other. Essentially, Angle-Angle-Angle holds true in the model $\mathbb{A}$ vacuously.

2. THE AXIOMS

In this section we state the axioms of absolute geometry as given in [7]. It is shown in [5] that all of the axioms (I1)-(I3) and (B1)-(B4), hold in $\mathbb{A}$. In later sections we will show that axioms (C1)-(C5), and Angle-Angle-Angle all hold in $\mathbb{A}$.

The Incidence Axioms:

(I1) Given any two distinct points A and B, then there exists a unique line l containing A and B.

(I2) Every line contains at least two distinct points.

(I3) There exist three distinct noncollinear points.

The Betweenness Axioms:

(B1) If B is between A and C, (written $A-B-C$), then A, B, and C are three distinct collinear points, and also $C-B-A$.

(B2) Given any two distinct points A and B, then there exists a point C such that $A-B-C$.

(B3) Given three distinct points on a line, then one and only one of them is between the other two.

(B4) (PASCH) Let A, B, and C be three distinct noncollinear points, and let l be a line not containing any of A, B, or C. If l contains a point D such that $A-D-B$, then l also contains a point lying between A and C or else a point lying between B and C, but not both.

It is shown in [6],[7],[9] that PASCH is equivalent to the *Plane Separation Postulate*:

Let $\mathcal{P}$ denote the set of all points. For each line l, there exist two nonempty convex sets $\mathcal{H}_1$ and $\mathcal{H}_2$ such that

(1) $\mathcal{P} \backslash l = \mathcal{H}_1 \cup \mathcal{H}_2$

(2) If $P \in \mathcal{H}_1$, $Q \in \mathcal{H}_2$, and $P \neq Q$, then $\overline{PQ} \cap l \neq \varnothing$.

In [5], it is shown that PASCH holds in $\mathbb{A}$ by showing that the Plane Separation Postulate holds in $\mathbb{A}$. The sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are called the *sides* of l. If $A, B \in \mathcal{H}_i$ (i=1,2), then we say that A and B are on the *same side* of l. If $A \in \mathcal{H}_1$ and $B \in \mathcal{H}_2$, then we say that A and B are on *opposite sides* of l. When referring to the Plane Separation Postulate, we will abbreviate it as PSP.

We next state the congruence axioms. Given two line segments $\overline{AB}$ and $\overline{CD}$, then we use the notation $\overline{AB} \cong \overline{CD}$ to denote when $\overline{AB}$ and $\overline{CD}$ are congruent to each other. Similarly, given two angles $\angle ABC$ and $\angle DEF$, then we use the notation $\angle ABC \cong \angle DEF$ to denote when $\angle ABC$ and $\angle DEF$ are congruent to each other. Finally, given two triangles $\triangle ABC$ and $\triangle DEF$, then we use the notation $\triangle ABC \cong \triangle DEF$ to denote when $\triangle ABC$ and $\triangle DEF$ are congruent to each other.

The Congruence Axioms:

(C1) Given a line segment $\overline{AB}$, and given a ray r originating at a point C, then there exists a unique point D on the ray r such that $\overline{AB} \cong \overline{CD}$.

(C2) If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Every line segment is congruent to itself.

(C3) (Addition of Segments) Given three points A, B, and C satisfying $A - B - C$, and three other points D, E, and F satisfying $D - E - F$, if $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

(C4) Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, then there exists a unique ray $\overrightarrow{DE}$ on a given side of line $\overleftrightarrow{DF}$ such that $\angle BAC \cong \angle EDF$.

(C5) Given any three angles α, β, and γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

We do not include the sixth congruence axiom *Side-Angle-Side* given in [7] in our list of axioms to be satisfied by the model $\mathbb{A}$. Instead we will show that it does not hold in the model $\mathbb{A}$. However, we state it here for the sake of completeness.

Side-Angle-Side

Given triangles $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

We will show that the following criterion for triangle congruence does hold in $\mathbb{A}$.

Angle-Angle-Angle

Given triangles $\triangle ABC$ and $\triangle DEF$, if $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$, then $\triangle ABC \cong \triangle DEF$.

3. The Model $\mathbb{A}$

In this section, we define the model $\mathbb{A}$.

The set of points in the model $\mathbb{A}$ is the set $\mathbb{Q} \times \mathbb{Q}$ of points in $\mathbb{R}^2$ both of whose coordinates are rational numbers. Since $\mathbb{Q}^2$ is countably infinite, then we can enumerate the points $P_1, P_2, \ldots, P_j, \ldots$ in $\mathbb{Q}^2$. Thus, we often will treat the points in $\mathbb{Q}^2$ as a sequence (P_n).

To define lines in the models $\mathbb{A}$, we let $\mathcal{L}$ be the set of all lines in $\mathbb{R}^2$ whose equations are $y = mx + b$ or $x = k$, where $m, b, k \in \mathbb{Q}$. We define a line in $\mathbb{A}$ to be the intersection of a line $l \in \mathcal{L}$ with $\mathbb{Q} \times \mathbb{Q}$. We denote the distance in $\mathbb{A}$ between two points P and Q by $\hat{t}(P, Q)$. We denote the angle measure in $\mathbb{A}$ of an angle $\angle ABC$ by $\hat{m}\angle ABC$.

We define distance and angle measure in the model $\mathbb{A}$ by using the enumeration on the points $P_1, P_2, \ldots, P_j, \ldots$ in $\mathbb{Q}^2$. Even though we are constructing a noncontinuous model, we will use this distance and angle measure to determine when two segments or two angles are congruent. We initially define distance in $\mathbb{A}$ using the taxicab metric t. More specifically, given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, then we define $t(P_1, P_2) = |x_2 - x_1| + |y_2 - y_1|$. Since $x_1, x_2, y_1, y_2 \in \mathbb{Q}$, then $t(P_1, P_2) \in \mathbb{Q}$ as well. We initially define angle measure in $\mathbb{A}$ to be usual Euclidean angle measure.

If we are given a point $P_j = (a, b)$ in $\mathbb{A}$, where $a, b \in \mathbb{Q}$, then the lines $x = a$ and $y = b$ divide up the model $\mathbb{A}$ into four "quadrants" similar to the way that the x-axis and y-axis divide up the xy-plane into four quadrants. We will use a variation of this terminology when defining and altering angle measure to construct the model $\mathbb{A}$. We refer to the set $\{(x, y) \in \mathbb{A} \mid x > a \text{ and } y > b\}$ as P_j-quadrant I. We refer to the set $\{(x, y) \in \mathbb{A} \mid x < a \text{ and } y > b\}$ as P_j-quadrant II. We refer to the set $\{(x, y) \in \mathbb{A} \mid x < a \text{ and } y < b\}$ as P_j-quadrant III. Finally, we refer to the set $\{(x, y) \in \mathbb{A} \mid x > a \text{ and } y < b\}$ as P_j-quadrant IV.

Let $E_j = (d, b)$, where $d, b \in \mathbb{Q}$ and $d = a + 1$. Let $W_j = (h, b)$, where $h, b \in \mathbb{Q}$ and $h = a - 1$. Let $N_j = (a, n)$, where $a, n \in \mathbb{Q}$ and $n = b + 1$. Let $S_j = (a, z)$, where $a, z \in \mathbb{Q}$ and $z = b - 1$.

4. ALTERING ANGLE MEASURE

In this section we show how we alter angle measure at a point P_j in the sequence (P_n), if it is necessary. We alter angle measure at P_j is to ensure that no two distinct triangles satisfy the hypotheses of Angle-Angle-Angle and also to ensure that distance in the model $\mathbb{A}$ is well defined.

Let $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ be those points from $P_1, \ldots, P_{j-1}$ that are in P_j-quadrant I. Let $P_{l_1}, P_{l_2}, \ldots, P_{l_h}$ be those points from $P_1, \ldots, P_{j-1}$ that are in P_j-quadrant III. Note that if $P_{i_a} - P_{i_b} - P_j$, then any two angles $\angle P_{i_a} P_j P_z$ and $\angle P_{i_b} P_j P_z$ (for some $P_z \in (P_n)$) are the same. However, we will still need to take both of the points P_{i_a} and P_{i_b} into account when determining the measure of angle $\angle P_{i_a} P_j P_z$ in $\mathbb{A}$, and in particular, when stating the conditions for angle measure below. This will be important later in the construction when altering angle measure in $\mathbb{A}$. Since there exist only a finite number of points $P_1, \ldots, P_{j-1}$ that come before P_j in the sequence (P_n), then the process will eventually stop and the measure of angle $\angle P_{i_a} P_j P_z$ in $\mathbb{A}$ is well-defined. For each $d = 1, \ldots, h$, let $T_{l_d} \in \mathbb{Q}^2$ be a point in P_j-quadrant I such that $T_{l_d} - P_j - P_{l_d}$. As we construct $\mathbb{A}$, we always assign the same slope-values to the rays $\overrightarrow{P_j P_{l_d}}$ and $\overrightarrow{P_j T_{l_d}}$ so that $\angle E_j P_j T_{l_d}$ is congruent to $\angle W_j P_j P_{l_d}$ in $\mathbb{A}$.

Let $\mathcal{S}$ denote the set $\{P_{i_1}, \ldots, P_{i_k}, T_{l_1}, \ldots, T_{l_h}\}$. Let $B_1, B_2, \ldots, B_{k+h} \in \mathcal{S}$ be such that:

1. $B_a \neq B_c$ whenever $a \neq c$. Consequently, $\mathcal{S} = \{B_1, B_2, \ldots, B_{k+h}\}$
2. None of the points $B_2, \ldots, B_{k+h}$ are in the interior of $\angle E_j P_j B_1$
3. B_1 is in the interior of $\angle E_j P_j B_2$ and none of $B_3, \ldots, B_{k+h}$ are in the interior of $\angle E_j P_j B_2$
4. For each $z = 3, \ldots, k + h$, all of $B_1, B_2, \ldots, B_{z-1}$ are in the interior of $\angle E_j P_j B_z$, and if $z < k + h$, then none of $B_{z+1}, \ldots, B_{k+h}$ are in the interior of $\angle E_j P_j B_z$

Note that we can make the assumption in statement (1) above due to the fact that if $P_{i_a} - P_{i_b} - P_j$, then two angles $\angle P_{i_a}P_jP_z$ and $\angle P_{i_b}P_jP_z$ are the same. For each $t = 1, \dots, k+h$, denote the slope of ray $\overrightarrow{P_jB_t}$ by s_t. Note that each $t = 1, \dots, k+h$, $s_t \in \mathbb{Q}$, and moreover that $s_1 < s_2 < \cdots < s_{h+k}$. For each $t = 1, \dots, k+h$, Let $\theta_t = \hat{m}\angle E_jP_jB_t$. Thus, $\mathrm{Tan}(\theta_t) = s_t$ and $\theta_t = \mathrm{Tan}^{-1}(s_t)$.

Let $b_1, b_2, \dots b_{k+h} \in \mathbb{Q}^+$ be such that:

(1) For each $u = 1, \dots, k+h-1$, $s_u < b_u < s_{u+1}$
(2) $b_{k+h} > s_{k+h}$

Define $f_{j,0} : \mathbb{Q}^+ \cap [0, s_1] \to \mathbb{Q}^+ \cap [0, b_1]$ by $f_{j,0}(x) = \left(\frac{b_1}{s_1}\right)(x)$. We leave it to the reader to check that $f_{j,0}(0) = 0$, $f_{j,0}(s_1) = b_1$, $f_{j,0}$ is a bijection, $f_{j,0}$ is increasing, and that $f_{j,0}$ is continuous on it's domain.

Similarly, define $f_{j,1} : \mathbb{Q}^+ \cap [s_1, s_2] \to \mathbb{Q}^+ \cap [b_1, b_2]$ by $f_{j,1}(x) = \left(\frac{b_2 - b_1}{s_2 - s_1}\right)(x - s_1) + b_1$. Again, we leave it to the reader to check that $f_{j,1}(s_1) = b_1$, $f_{j,1}(s_2) = b_2$, $f_{j,1}$ is a bijection, $f_{j,1}$ is increasing, and that $f_{j,1}$ is continuous on it's domain.

In general, for $m = 1, \dots, h+k-1$, we define $f_{j,m} : \mathbb{Q}^+ \cap [s_m, s_{m+1}] \to \mathbb{Q}^+ \cap [b_m, b_{m+1}]$ by $f_{j,m}(x) = \left(\frac{b_{m+1} - b_m}{s_{m+1} - s_m}\right)(x - s_m) + b_m$. Again, we leave it to the reader to check that $f_{j,m}(s_m) = b_m$, $f_{j,m}(s_{m+1}) = b_{m+1}$, $f_{j,m}$ is a bijection, $f_{j,m}$ is increasing, and that $f_{j,m}$ is continuous on it's domain.

Finally, we define $f_{j,h+k} : \mathbb{Q}^+ \cap [s_{h+k}, \infty) \to \mathbb{Q}^+ \cap [b_{h+k}, \infty)$ by $f_{j,h+k}(x) = x + (b_{k+h} - s_{k+h})$. Again, we leave it to the reader to check that $f_{j,h+k}(s_{h+k}) = b_{h+k}$, $f_{j,h+k}$ is a bijection, $f_{j,h+k}$ is increasing, and that $f_{j,h+k}$ is continuous on it's domain.

We assign to each ray $\overrightarrow{P_jB_m}$ the value $f_{j,m}(s_m)$. In general, if Y is a point in the interior of $\angle B_mP_jB_{m+1}$, and s denotes the slope of ray $\overrightarrow{P_jY}$, then we assign the value $f_{j,m}(s)$ to ray $\overrightarrow{P_jY}$. We call $f_{j,m}(s)$ the *slope-value* of ray $\overrightarrow{P_jY}$. If θ is the slope-value of $\overrightarrow{P_jY}$, then $\hat{m}\angle E_jP_jY = \mathrm{Tan}^{-1}(\theta)$. Essentially, we replace the euclidean slope s of ray $\overrightarrow{P_jY}$ with it's slope-value and then compute angle measure using the relationship between slope and the Tangent function.

In this way, we assign slope-values to all of the points from $P_1, P_2, \dots, P_{j-1}$ that are in P_j-quadrants I and III. We then use the relationship between slope and the Tangent function to compute the angle measures in $\mathbb{A}$ of angles in the form $\angle E_jP_jP_{i_t}$ (if P_{i_t} is in P_j-quadrant I) and $\angle W_jP_jP_{i_t}$ (if P_{i_t} is in P_j-quadrant III), first replacing the actual slope of a ray $\overrightarrow{P_jB_m}$ with its slope-value. Using complimentary angles, we also compute the angle measures in $\mathbb{A}$ of angles in the form $\angle N_jP_jP_{i_t}$ (if P_{i_t} is in P_j-quadrant I) and $\angle S_jP_jP_{i_t}$ (if P_{i_t} is in P_j-quadrant III). We use a similar method to compute the measure of angles involving rays in P_j-quadrants II and IV. Finally, to find the measure of any angle whose vertex is P_j, regardless of which P_j-quadrant its sides are in, we apply angle addition and angle subtraction to types of angles indicated in the above procedure.

There are several cases to look at when defining the measure of an angle $\angle P_xP_jP_y$, where $x, y \in \{1, \dots, j-1\}$, in $\mathbb{A}$. These cases are not only determined by the points P_x, P_j, and P_y, but also by all of the points $P_1, P_2, \ldots, P_{j-1}$ that come before P_j in the sequence (P_n). We look at triangles in the form $\triangle P_x P_j P_y$ and $\triangle P_a P_b P_c$, where $a, b, c \leq j-1$. In particular, we look at the cases where the triangles $\triangle P_x P_j P_y$ and $\triangle P_a P_b P_c$ have either a vertex, or even an entire side in common. We also look at the cases where at least one of the triangles $\triangle P_x P_j P_y$ or $\triangle P_a P_b P_c$ has both a vertical side and horizontal side, and consequently a right angle. In this last case, the right angle can not be altered, and remains as a right angle in the new model $\mathbb{A}$. In all of these cases, we use the fact that there are only a finite number of points $P_1, P_2, \ldots, P_{j-1}$ that come before P_j in the sequence (P_n), and therefore only a finite number of pairs of triangles in the form $\triangle P_x P_j P_y$ and $\triangle P_a P_b P_c$, yet an infinite number of possible values for $b_1, b_2, \ldots, b_{k+h}$. This allows us to choose $b_1, b_2, \ldots, b_{k+h}$ so that the desired conditions in each case are satisfied.

Case 1: First assume that the triangles $\triangle P_x P_j P_y$ and $\triangle P_a P_b P_c$ have neither a vertex nor a side in common. Furthermore, assume that at most one side of each triangle is vertical or horizontal. In this case, we choose $b_1, b_2, \ldots, b_{k+h}$ such that $\angle P_x P_j P_y$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_x P_j P_y$ is not equal to any of $\hat{m}\angle P_a P_b P_c$, $\hat{m}\angle P_b P_a P_c$, or $\hat{m}\angle P_b P_c P_a$.

Case 2: Next assume that the triangles $\triangle P_x P_j P_y$ and $\triangle P_a P_b P_c$ have a vertex P_a in common, where $a < j$, but that they do not have a side in common. As in Case 1 above, we assume that at most one side of each triangle is vertical or horizontal. In this case, we again choose $b_1, b_2, \ldots, b_{k+h}$ such that $\angle P_x P_j P_y$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_x P_j P_y$ is not equal to any of $\hat{m}\angle P_a P_b P_c$, $\hat{m}\angle P_b P_a P_c$, or $\hat{m}\angle P_b P_c P_a$.

Case 3: Next assume that the triangles $\triangle P_x P_j P_y$ and $\triangle P_a P_b P_c$ have a side in common, but that P_j is not an endpoint of that common side. As in Cases 1 and 2 above, we assume that at most one side of each triangle is vertical or horizontal. In this case, we again choose $b_1, b_2, \ldots, b_{k+h}$ such that $\angle P_x P_j P_y$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_x P_j P_y$ is not equal to any of $\hat{m}\angle P_a P_b P_c$, $\hat{m}\angle P_b P_a P_c$, or $\hat{m}\angle P_b P_c P_a$.

Case 4: Next assume that the triangles $\triangle P_x P_j P_y$ and $\triangle P_a P_j P_c$ have the vertex P_j in common, but that they do not have a side in common. As above, we assume that at most one side of each triangle is vertical or horizontal. In this case, we choose $b_1, b_2, \ldots, b_{k+h}$ such that neither $\angle P_x P_j P_y$ nor $\angle P_a P_j P_c$ are right angles in $\mathbb{A}$ and such that $\hat{m}\angle P_x P_j P_y$ is not equal to either of $\hat{m}\angle P_j P_a P_c$ or $\hat{m}\angle P_j P_c P_a$, and $\hat{m}\angle P_a P_j P_c$ is not equal to either of $\hat{m}\angle P_j P_x P_y$ or $\hat{m}\angle P_j P_y P_x$.

Case 5: Next assume that the triangles $\triangle P_x P_j P_y$ and $\triangle P_x P_j P_c$ have a side in common, and moreover that P_j is an endpoint of that common side. As above, we assume that at most one side of each triangle is vertical or horizontal. As in Case 4, we choose $b_1, b_2, \ldots, b_{k+h}$ such that neither $\angle P_x P_j P_y$ nor $\angle P_x P_j P_c$ are right angles in $\mathbb{A}$ and such that $\hat{m}\angle P_x P_j P_y$ is not equal to either of $\hat{m}\angle P_j P_x P_c$ or $\hat{m}\angle P_j P_c P_x$, and $\hat{m}\angle P_x P_j P_c$ is not equal to either of $\hat{m}\angle P_j P_x P_y$ or $\hat{m}\angle P_j P_y P_x$.

In the remaining cases, we will look at the situation where at least one of the triangles has both a vertical side and horizontal side, and consequently a right angle. We denote the vertices at these right angles by R_1 and R_2.

We note that the points R_1 and R_2 are points in $\mathbb{Q}^2$, but that they might actually come after the point P_j in the sequence (P_n). Since the points R_1 and R_2 are determined by the points $P_1, P_2, \ldots, P_j$, then there are only a finite number of such possible points R_1 and R_2. As above, this allows us to choose the values $b_1, b_2, \ldots, b_{k+h}$ so that the desired conditions are satisfied.

Case 6: Assume that the triangles $\triangle P_jR_1P_x$ and $\triangle P_aR_2P_c$ have neither a vertex nor a side in common. Assume that one of the following holds:

1. $\overrightarrow{R_1P_x}$ is horizontal and $\overrightarrow{R_1P_j}$ is vertical
2. $\overrightarrow{R_1P_j}$ is horizontal and $\overrightarrow{R_1P_x}$ is vertical

and one of the following holds:

1. $\overrightarrow{R_2P_a}$ is horizontal and $\overrightarrow{R_2P_c}$ is vertical
2. $\overrightarrow{R_2P_c}$ is horizontal and $\overrightarrow{R_2P_a}$ is vertical

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that $\angle P_xP_jR_1$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_xP_jR_1$ is not equal to either of $\hat{m}\angle R_2P_aP_c$ or $\hat{m}\angle R_2P_cP_a$. The case where only one of the triangles has a right angle with vertical and horizontal sides is similar, and we leave it to the reader to check the details.

Case 7: Assume that the triangles $\triangle P_jR_1P_x$ and $\triangle P_aR_1P_c$ have the vertex R_1 in common. Assume that one of the following holds:

1. $\overrightarrow{R_1P_x}$ is horizontal and $\overrightarrow{R_1P_j}$ is vertical
2. $\overrightarrow{R_1P_j}$ is horizontal and $\overrightarrow{R_1P_x}$ is vertical

and one of the following holds:

1. $\overrightarrow{R_1P_a}$ is horizontal and $\overrightarrow{R_1P_c}$ is vertical
2. $\overrightarrow{R_1P_c}$ is horizontal and $\overrightarrow{R_1P_a}$ is vertical

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that $\angle P_xP_jR_1$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_xP_jR_1$ is not equal to either of $\hat{m}\angle R_1P_aP_c$ or $\hat{m}\angle R_1P_cP_a$.

Case 8: Assume that the triangles $\triangle P_jR_1R_2$ and $\triangle P_aR_2P_c$ have the vertex R_2 in common. Assume that one of the following holds:

1. $\overrightarrow{R_1R_2}$ is horizontal and $\overrightarrow{R_1P_j}$ is vertical
2. $\overrightarrow{R_1P_j}$ is horizontal and $\overrightarrow{R_1R_2}$ is vertical

and one of the following holds:

1. $\overrightarrow{R_2P_a}$ is horizontal and $\overrightarrow{R_2P_c}$ is vertical
2. $\overrightarrow{R_2P_c}$ is horizontal and $\overrightarrow{R_2P_a}$ is vertical

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that $\angle R_1P_jR_2$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle R_1P_jR_2$ is not equal to either of $\hat{m}\angle R_2P_aP_c$ or $\hat{m}\angle R_2P_cP_a$.

Case 9: Assume that the triangles $\triangle P_jR_1P_x$ and $\triangle P_xR_2P_c$ have a vertex P_x in common, where $x < j$, but do not have a common side. Assume that one of the following holds:

1. $\overrightarrow{R_1P_x}$ is horizontal and $\overrightarrow{R_1P_j}$ is vertical
2. $\overrightarrow{R_1P_j}$ is horizontal and $\overrightarrow{R_1P_x}$ is vertical

and one of the following holds:

1. $\overrightarrow{R_2P_x}$ is horizontal and $\overrightarrow{R_2P_c}$ is vertical
2. $\overrightarrow{R_2P_c}$ is horizontal and $\overrightarrow{R_2P_x}$ is vertical

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that $\angle P_xP_jR_1$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_xP_jR_1$ is not equal to either of $\hat{m}\angle R_2P_xP_c$ or $\hat{m}\angle R_2P_cP_x$. Note that we might have $R_1 - P_x - R_2$ and $P_j - P_x - P_c$, in which case angles $\angle R_1P_xP_j$ and $\angle R_2P_xP_c$ are vertical angles, and therefore congruent. However, this does not effect how we proceed in Case 9. As in Case 6, the case where only one of the triangles has a right angle with vertical and horizontal sides is similar, and we leave it to the reader to check the details.

Case 10: Assume that the triangles $\triangle P_jR_1P_x$ and $\triangle P_jR_2P_c$ have the vertex P_j in common, but do not have a common side. Assume that one of the following holds:

1. $\overrightarrow{R_1P_x}$ is horizontal and $\overrightarrow{R_1P_j}$ is vertical
2. $\overrightarrow{R_1P_j}$ is horizontal and $\overrightarrow{R_1P_x}$ is vertical

and one of the following holds:

1. $\overrightarrow{R_2P_j}$ is horizontal and $\overrightarrow{R_2P_c}$ is vertical
2. $\overrightarrow{R_2P_c}$ is horizontal and $\overrightarrow{R_2P_j}$ is vertical

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that $\angle P_xP_jR_1$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_xP_jR_1$ is not equal to $\hat{m}\angle R_2P_cP_j$. As above, we might have $R_1 - P_j - R_2$ and $P_x - P_j - P_c$, in which case angles $\angle R_1P_jP_x$ and $\angle R_2P_jP_c$ are vertical angles, and therefore congruent. Again, this does not effect how we proceed in Case 10. As in Cases 6 and 9, the case where only one of the triangles has a right angle with vertical and horizontal sides is similar, and we leave it to the reader to check the details.

Case 11: Assume that the triangles $\triangle P_xP_jR_1$ and $\triangle P_cP_jR_1$ have the side $\overline{P_jR_1}$ in common. Assume that one of the following holds:

1. $\overrightarrow{R_1P_x}$ and $\overrightarrow{R_1P_c}$ are horizontal, $\overrightarrow{R_1P_j}$ is vertical, and $P_c - R_1 - P_x$
2. $\overrightarrow{R_1P_x}$ and $\overrightarrow{R_1P_c}$ are vertical, $\overrightarrow{R_1P_j}$ is horizontal, and $P_c - R_1 - P_x$

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that $\angle P_xP_jR_1$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_xP_jR_1$ is not equal to either of $\hat{m}\angle R_1P_jP_c$ or $\hat{m}\angle R_1P_cP_j$. The case where only one of the triangles has a right angle with vertical and horizontal sides is similar, and we leave it to the reader to check the details.

Case 12: Assume that the triangles $\triangle P_jP_xR_1$ and $\triangle P_cP_xR_1$ have the side $\overline{P_xR_1}$ in common, where $x < j$. Assume that one of the following holds:

1. $\overrightarrow{R_1P_j}$ and $\overrightarrow{R_1P_c}$ are horizontal, $\overrightarrow{R_1P_x}$ is vertical, and $P_c - R_1 - P_j$
2. $\overrightarrow{R_1P_j}$ and $\overrightarrow{R_1P_c}$ are vertical, $\overrightarrow{R_1P_x}$ is horizontal, and $P_c - R_1 - P_j$

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that $\angle P_xP_jR_1$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_xP_jR_1$ is not equal to either of $\hat{m}\angle R_1P_xP_c$ or $\hat{m}\angle R_1P_cP_x$. Again, the case where only one of the triangles has a right angle with vertical and horizontal sides is similar, and we leave it to the reader to check the details.

Case 13: Assume that the triangles $\triangle P_jP_xR_1$ and $\triangle P_jP_xR_2$ have the side $\overline{P_jP_x}$ in common, where $x < j$. Assume that one of the following holds:

1. $\overrightarrow{R_1P_j}$ is horizontal, $\overrightarrow{R_1P_x}$ is vertical, $\overrightarrow{R_2P_x}$ is horizontal, and $\overrightarrow{R_2P_j}$ is vertical
2. $\overrightarrow{R_1P_j}$ is vertical, $\overrightarrow{R_1P_x}$ is horizontal, $\overrightarrow{R_2P_x}$ is vertical, and $\overrightarrow{R_2P_j}$ is horizontal

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that $\angle P_x P_j R_1$ is not a right angle in $\mathbb{A}$ and such that $\hat{m}\angle P_x P_j R_1$ is not equal to either of $\hat{m}\angle R_2 P_x P_j$ or $\hat{m}\angle R_2 P_j P_x$. Again, the case where only one of the triangles has a right angle with vertical and horizontal sides is similar, and we leave it to the reader to check the details.

Note that in each of the above cases, there exists an angle in one of the two triangles which is not a right angle in $\mathbb{A}$, and which has a different angle measure in $\mathbb{A}$ than all three of the angles in the other triangle. We impose the following condition when choosing the values $b_1, b_2, \ldots, b_{k+h}$. We will need this condition in the next section when altering distance.

Condition:

Assume that there exist three distinct points $P_a, P_b, P_c \in (P_n)$ such that $a, b, c < j$, and there exists a point $R_1 \in \mathbb{Q}^2$ such that

1. Angle $\angle P_a P_c R_1$ is congruent to angle $\angle P_c P_a R_1$
2. $P_a - P_b - R_1$
3. Points P_j and P_c are on opposite sides of line $\overleftrightarrow{P_a P_b}$
4. The point R_1 comes later in the sequence (P_n) than the point P_j (by the construction above together with Condition (1), if R_1 comes before P_j in the sequence (P_n) then Condition (8) is unnecessary)

In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that angle $\angle P_b P_j R_1$ is not congruent to the angle $\angle P_j P_b R_1$. In fact, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that angle $\angle P_b P_j R_1$ is not congruent to any of the angles $\angle P_x P_y P_z$, where $x, y < j$ and $z \leq j$.

Note that after altering angle measure at P_j, there will still exist points $R_2, R_3 \in \mathbb{Q}^2$ such that angle $\angle P_a P_c R_2$ is congruent to angle $\angle P_c P_a R_2$ and angle $\angle P_j P_b R_3$ is congruent to angle $\angle P_b P_j R_3$. However, after altering angle measure at P_j, then for any possible point R_2, we have that angle $\angle P_j P_b R_2$ is not congruent to angle $\angle P_b P_j R_2$ and for any possible point R_3, we have that angle $\angle P_a P_c R_3$ is not congruent to angle $\angle P_c P_a R_3$.

The case where P_j and P_c are on the same side of line $\overleftrightarrow{P_a P_b}$ is similar. In this case, we again choose the values $b_1, b_2, \ldots, b_{k+h}$ so that angle $\angle P_b P_j R_1$ is not congruent to the angle $\angle P_j P_b R_1$.

Below are variations of the condition stated above. We leave it to the reader to check the details.

First assume that $\angle P_b P_c R_1$ is congruent to $\angle P_c P_b R_1$, that $P_a - P_b - R_1$, and that points P_j and P_c are on opposite sides of line $\overleftrightarrow{P_a P_b}$. In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that angle $\angle P_a P_j R_1$ is not congruent to the angle $\angle P_j P_a R_1$. Again, the case where P_j and P_c are on the same side of line $\overleftrightarrow{P_a P_b}$ is similar.

Next, assume that $\angle P_b P_c R_1$ is congruent to $\angle P_c P_b R_1$, that $P_j - P_b - R_1$, and that points P_a and P_c are on opposite sides of line $\overleftrightarrow{P_a P_b}$. In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that angle $\angle P_a P_j R_1$ is not congruent to the angle $\angle P_j P_a R_1$. The case where P_a and P_c are on the same side of line $\overleftrightarrow{P_a P_b}$ is similar.

Finally, assume that $\angle P_b P_c R_1$ is congruent to $\angle P_c P_b R_1$, that $P_b - P_j - R_1$, and that points P_a and P_c are on opposite sides of line $\overleftrightarrow{P_a P_b}$. In this case, we choose the values $b_1, b_2, \ldots, b_{k+h}$ so that angle $\angle P_a P_j R_1$ is not congruent to the angle $\angle P_jP_aR_1$. The case where P_a and P_c are on the same side of line $\overleftrightarrow{P_aP_b}$ is similar.

5. ALTERING DISTANCE BETWEEN TWO POINTS

In this section, we define distance between any two points in the model $\mathbb{A}$. Using the enumeration on the set of all points (P_n) in $\mathbb{Q}^2$, we can enumerate all (non-ordered) pairs of distinct points from $\mathbb{Q}^2$ in the following way: Given the pairs of points $(P_l, P_t), (P_h, P_k) \in \mathbb{Q}^2 \times \mathbb{Q}^2$, where $P_l \neq P_t$ and $P_h \neq P_k$, then assume that k is the largest of the four subscripts l, t, h, and k. If k is strictly larger than the other subscripts l, t, and h, then (P_l, P_t) comes before (P_h, P_k) in the enumeration. If $k = t$, then we compare l and h. In this case, (P_l, P_t) comes before (P_h, P_k) if and only if $l < h$. Similarly, if $k = l$, then (P_l, P_t) comes before (P_h, P_k) if and only if $t < h$. For example, the first ten pairs are enumerated in the following way:
(P_1, P_2), (P_1, P_3), (P_2, P_3), (P_1, P_4), (P_2, P_4), (P_3, P_4), (P_1, P_5), (P_2, P_5), (P_3, P_5), (P_4, P_5).

Note that the order in which the points in each pair are written is irrelevant when enumerating the pairs of points this way. In particular, (P_l, P_t) is considered the same as (P_t, P_l) for the purposes of enumeration. We use this enumeration when defining distance in $\mathbb{A}$. More specifically, if (P_l, P_t) comes before (P_h, P_k), then we define distance between P_l and P_t prior to defining distance between P_h and P_k.

Initially, we use the taxicab metric t to compute the distance $t(P_{i_1}, P_{i_2})$ between two points P_{i_1} and P_{i_2} in $\mathbb{Q}^2$. As we construct the model $\mathbb{A}$, there are several possible ways to get the distance between two points P_{i_1} and P_{i_2} in $\mathbb{A}$. The first of these is to use taxicab distance without any alteration. However, for most pairs of points P_{i_1} and P_{i_2}, we will have to alter the taxicab distance between P_{i_1} and P_{i_2} to get a new distance $\hat{t}(P_{i_1}, P_{i_2})$ between P_{i_1} and P_{i_2}. We define $\hat{t}(P_{i_1}, P_{i_2})$ in one of the following ways:

1. If there exists a triangle $\triangle P_{i_1}P_{i_2}P_{i_3}$ such that P_{i_3} comes after both P_{i_1} and P_{i_2} in the sequence (P_n) and such that the angles $\angle P_{i_1}P_{i_2}P_{i_3}$ and $\angle P_{i_2}P_{i_1}P_{i_3}$ are congruent, then sides $\overline{P_{i_1}P_{i_3}}$ and $\overline{P_{i_2}P_{i_3}}$ are congruent. In this case, we must define the distance between P_{i_1} and P_{i_3} to be equal to the distance between P_{i_2} and P_{i_3}.
2. If we have points $P_{i_1} - P_{i_3} - P_{i_4} - P_{i_2}$ such that each of P_{i_1} and P_{i_2} come before both P_{i_3} and P_{i_4} in the sequence (P_n), then $\overline{P_{i_3}P_{i_4}}$ inherits its length from $\overline{P_{i_1}P_{i_2}}$. More precisely, since each of P_{i_1} and P_{i_2} come before both P_{i_3} and P_{i_4} in the sequence (P_n), then we define the distance between P_{i_1} and P_{i_2} before we define the distance between P_{i_3} and P_{i_4}. If $r = \frac{t(P_{i_3}, P_{i_4})}{t(P_{i_1}, P_{i_2})}$, then $\hat{t}(P_{i_3}, P_{i_4}) = r\hat{t}(P_{i_1}, P_{i_2})$.
3. We use a similar method if $P_{i_1} = P_{i_3}$ or $P_{i_2} = P_{i_4}$.
4. If we have points $P_{i_3} - P_{i_1} - P_{i_2} - P_{i_4}$ such that each of P_{i_1} and P_{i_2} come before both P_{i_3} and P_{i_4} in the sequence (P_n), then $\overline{P_{i_3}P_{i_4}}$ is the sum of the lengths of the segments $\overline{P_{i_3}P_{i_1}}$, $\overline{P_{i_1}P_{i_2}}$, and $\overline{P_{i_2}P_{i_4}}$. That is, $\hat{t}(P_{i_3}, P_{i_4}) = \hat{t}(P_{i_3}, P_{i_1}) + \hat{t}(P_{i_1}, P_{i_2}) + \hat{t}(P_{i_2}, P_{i_4})$.

In case (4) above, we first find the lengths of $\overline{P_{i_3}P_{i_1}}$, $\overline{P_{i_1}P_{i_2}}$, and $\overline{P_{i_2}P_{i_4}}$ separately, and then add those lengths up to find the length of $\overline{P_{i_3}P_{i_4}}$.

Theorem 5.1. *There does not exist a finite sequence of* $k+1$ *points* $C, P_{i_1}, \ldots, P_{i_k}$ *such that*

(1) $C - P_{i_k} - P_{i_1}$
(2) *For each* $j = 1, \ldots, k-1$, $\angle CP_{i_j}P_{i_{j+1}} \cong \angle CP_{i_{j+1}}P_{i_j}$

Proof. If such a sequence existed, then there would exist five points $C, P_{i_1}, P_{i_2}, P_{i_{k-1}}, P_{i_k}$ such that $C - P_{i_k} - P_{i_1}$, $\angle CP_{i_1}P_{i_2} \cong \angle CP_{i_2}P_{i_1}$, and $\angle CP_{i_k}P_{i_{k-1}} \cong \angle CP_{i_{k-1}}P_{i_k}$. However, by the way that angle measure is defined in the model $\mathbb{A}$, five such points can not exist.

Theorem 5.1 assures us that distance in $\mathbb{A}$ is well-defined in the sense that we can use exactly one of (1)-(4) to define $\hat{t}(P_{i_1}, P_{i_2})$.

6. PRECONDITIONS AND POSTCONDITIONS FOR DISTANCE

In this section, we give a case where we need to alter distance between two points P_{i_t} and P_{i_k} prematurely while going through the sequence (P_n) due to the relationship between P_{i_t} and P_{i_k} and other distances and angle measures already defined. We also examine the situation where we might need to alter the distance between two points P_{i_t} and P_{i_k} multiple times, again due to the relationship between P_{i_t} and P_{i_k} and other distances and angle measures already defined.

Assume that we have points $P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}$, and P_{i_5} such that $P_{i_1} - P_{i_5} - P_{i_2}$ and $P_{i_3} - P_{i_5} - P_{i_4}$, and such that P_{i_4} comes after P_{i_1}, P_{i_2}, and P_{i_3}, and such that P_{i_5} comes after P_{i_4} in the sequence (P_n). Furthermore assume that P_{i_1}, P_{i_3}, and P_{i_5} are not collinear, and that angles $\angle P_{i_5}P_{i_1}P_{i_3}$ and $\angle P_{i_5}P_{i_3}P_{i_1}$ are congruent. In this case, since $\hat{t}(P_{i_1}, P_{i_2})$ is already defined, then $\hat{t}(P_{i_1}, P_{i_5})$ is already defined by inheriting distance from $\overline{P_{i_1}P_{i_2}}$. Thus, when defining $\hat{t}(P_{i_3}, P_{i_4})$, we need to first define $\hat{t}(P_{i_3}, P_{i_5})$ such that $\hat{t}(P_{i_1}, P_{i_5}) = \hat{t}(P_{i_3}, P_{i_5})$. By the way that angle measure is defined in $\mathbb{A}$, we have that angles $\angle P_{i_3}P_{i_4}P_{i_2}$ and $\angle P_{i_1}P_{i_2}P_{i_4}$ are not congruent. Since $P_{i_1} - P_{i_5} - P_{i_2}$ and $P_{i_3} - P_{i_5} - P_{i_4}$, then it follows that angles $\angle P_{i_5}P_{i_4}P_{i_2}$ and $\angle P_{i_5}P_{i_2}P_{i_4}$ are not congruent. We define $\hat{t}(P_{i_4}, P_{i_5})$ by $\hat{t}(P_{i_4}, P_{i_5}) = t(P_{i_3}, P_{i_4})$ - $\hat{t}(P_{i_3}, P_{i_5})$ Finally, we define $\hat{t}(P_{i_3}, P_{i_4})$ by means of segment addition. In particular, $\hat{t}(P_{i_3}, P_{i_4}) = \hat{t}(P_{i_3}, P_{i_5}) + \hat{t}(P_{i_4}, P_{i_5}) = t(P_{i_3}, P_{i_4})$. Note that we have defined $\hat{t}(P_{i_1}, P_{i_5})$ while working with points that come before the point P_{i_5} in the sequence (P_n).

Next assume that we have points $P_{i_1}, P_{i_2}, P_{i_3}$, and P_{i_4} such that P_{i_1} comes before P_{i_2} and P_{i_3} in the sequence (P_n), and such that P_{i_4} comes after P_{i_2} and P_{i_3} in the sequence(P_n).

First assume that $P_{i_2} - P_{i_4} - P_{i_3}$, and that $\angle P_{i_2}P_{i_1}P_{i_4} \cong \angle P_{i_2}P_{i_4}P_{i_1}$ in $\mathbb{A}$. In this case, we define the distance from P_{i_2} to P_{i_4} to be the same as the distance from P_{i_1} to P_{i_2}. Since $P_{i_2} - P_{i_4} - P_{i_3}$, then we might possibly need to alter the distance from P_{i_2} to P_{i_3} a second time. However, since there are only a finite number of points in the sequence (P_n) that are prior to P_{i_2} and P_{i_3}, then we need only alter the distance from P_{i_2} to P_{i_3} at most a finite number of times, and consequently, this altering will eventually stop. We do a similar alteration if $\angle P_{i_3}P_{i_1}P_{i_4} \cong \angle P_{i_3}P_{i_4}P_{i_1}$.

Similarly, assume that $P_{i_1} - P_{i_4} - P_{i_3}$, and that $\angle P_{i_1}P_{i_2}P_{i_4} \cong \angle P_{i_1}P_{i_4}P_{i_2}$ in $\mathbb{A}$. In this case, we define the distance from P_{i_1} to P_{i_4} to be the same as the distance from P_{i_1} to P_{i_2}. As in the previous case, since $P_{i_1} - P_{i_4} - P_{i_3}$, then we might possibly need to alter the distance from P_{i_1} to P_{i_3} a second time. However, since there are only a finite number of points in the sequence (P_n) that are prior to P_{i_1} and P_{i_3}, then we need only alter the distance from P_{i_1} to P_{i_3} at most a finite number of times, and consequently, this altering will eventually stop.

Next, assume that $P_{i_1} - P_{i_4} - P_{i_2}$, and that $\angle P_{i_2}P_{i_3}P_{i_4} \cong \angle P_{i_2}P_{i_4}P_{i_3}$ in $\mathbb{A}$. Since $P_{i_1} - P_{i_4} - P_{i_2}$, then the distance $\hat{t}(P_{i_2}, P_{i_4})$ is inherited from to the distance $\hat{t}(P_{i_1}, P_{i_2})$. Since $\angle P_{i_2}P_{i_3}P_{i_4} \cong \angle P_{i_2}P_{i_4}P_{i_3}$, then we define the distance $\hat{t}(P_{i_2}, P_{i_3})$ to be the same as the distance $\hat{t}(P_{i_2}, P_{i_4})$. As above, we might possibly need to alter the distance from P_{i_2} to P_{i_3} multiple times. However, since there are only a finite number of points in the sequence (P_n) that are prior to P_{i_2} and P_{i_3}, then we need only alter the distance from P_{i_2} to P_{i_3} at most a finite number of times, and consequently, this process will eventually stop. We do a similar alteration if $\angle P_{i_1}P_{i_3}P_{i_4} \cong \angle P_{i_1}P_{i_4}P_{i_3}$.

Finally, assume that $P_{i_1} - P_{i_4} - P_{i_3}$, and that $\angle P_{i_3}P_{i_2}P_{i_4} \cong \angle P_{i_3}P_{i_4}P_{i_2}$ in $\mathbb{A}$. Since $P_{i_1} - P_{i_4} - P_{i_3}$, then the distance $\hat{t}(P_{i_3}, P_{i_4})$ is inherited from to the distance $\hat{t}(P_{i_1}, P_{i_3})$. Since $\angle P_{i_3}P_{i_2}P_{i_4} \cong \angle P_{i_3}P_{i_4}P_{i_2}$, then we define the distance $\hat{t}(P_{i_2}, P_{i_3})$ to be the same as the distance $\hat{t}(P_{i_3}, P_{i_4})$.

7. Congruence in the Model $\mathbb{A}$

We define two angles in $\mathbb{A}$ to be congruent if they have the same angle measure in $\mathbb{A}$. We define two segments in $\mathbb{A}$ to be congruent if they have the same length in $\mathbb{A}$. It follows by the way that angle measure is defined in $\mathbb{A}$, together with the fact that the functions $f_{j,m}$ are increasing bijections that are continuous on their domains, that congruence axioms (C4) and (C5) hold in $\mathbb{A}$.

Moreover, it follows by the way that distance is defined in $\mathbb{A}$, together with the extra condition in the previous section, that given a point P_j, given a ray $\overrightarrow{r}$ emanating from P_j, and given $n \in \mathbb{Z}^+$, then there exists a point $P_k \in \mathbb{Q}^2$ such that P_k is on $\overrightarrow{r}$, such that P_k comes after P_j in the sequence (P_n), and such that $\hat{t}(P_j, P_k) > n$. One can now apply arguments similar to those given in [5] to show that Congruence Axioms (C1)-(C3) hold in $\mathbb{A}$.

We have two cases when applying Angle-Angle-Angle. The first case is when we apply Angle-Angle-Angle to a single triangle two of whose angle are congruent to each other. In this case, the sides opposite to these angles must be congruent to each other. The second case is when we have two distinct triangles whose angles can be paired up into pairs of congruent angles, with one of the angles in each pair coming from one of the triangles, and the other angle coming from the other triangle.

If we are given a triangle $\triangle P_{i_1}P_{i_2}P_{i_3}$ in $\mathbb{A}$ such that $\angle P_{i_1}P_{i_2}P_{i_3} \cong \angle P_{i_1}P_{i_3}P_{i_2}$, then it follows by the way that distance (and consequently congruence of segments) is defined in the model $\mathbb{A}$ that $\hat{t}(P_{i_1}, P_{i_2}) = \hat{t}(P_{i_1}, P_{i_3})$. Thus, Angle-Angle-Angle holds when it is applied to a single triangle $\triangle P_{i_1}P_{i_2}P_{i_3}$ whose base angles are congruent to each other.

Assume that we are given two distinct triangles $\triangle P_{i_1}P_{i_2}P_{i_3}$ and $\triangle P_{i_4}P_{i_5}P_{i_6}$. It might be the case that two of the six possible angles in these triangles are vertical angles or that two of these angles are right angles. However, at least one the angles in these triangles is not a right angle and is not congruent with any of the angles from the other triangle. We may assume without loss of generality that angle $\angle P_{i_1}P_{i_2}P_{i_3}$ is such an angle. By the way that angle measure is defined in the model $\mathbb{A}$, it follows that $\angle P_{i_1}P_{i_2}P_{i_3}$ is not congruent to any of the angles in $\triangle P_{i_4}P_{i_5}P_{i_6}$. Therefore, triangles $\triangle P_{i_1}P_{i_2}P_{i_3}$ and $\triangle P_{i_4}P_{i_5}P_{i_6}$ do not satisfy the hypotheses of Angle-Angle-Angle. In either case, we have that Angle-Angle-Angle holds.

Finally, we note that since Angle-Angle-Angle and Playfair's parallel postulate hold in $\mathbb{A}$ (as opposed to the hyperbolic parallel postulate), then it can not be the case that Side-Angle-Side holds in $\mathbb{A}$ [9].

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