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# QUADRATIC CONICS IN HYPERBOLIC GEOMETRY

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Abstract. We prove that no conic in any Cayley–Klein model of the hyperbolic plane can be quadratic.

## 1. Introduction

Let $(A, B; C, D)$ denote the *cross-ratio* of the (maybe ideal) points $A$, $B$, $C$, $D$ in $\mathbb{R}^n$ $(n = 1, 2, \dots)$, and let $\overline{CD}$ denote the open segment of the (maybe ideal) points $C, D \in \mathbb{R}^n$. If $\mathcal{M}$ is an open, strictly convex, proper subset of $\mathbb{R}^n$ $(n = 2, 3, \dots)$, then the function $d\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ defined by

$$(1.1)\quad d(A,B) = \begin{cases} 0, & \text{if } A = B, \\ \frac{1}{2}\big|\ln(A,B;C,D)\big|, & \text{if } A \neq B, \text{ where } \overline{CD} = \mathcal{M} \cap AB, \end{cases}$$

is a *metric* on $\mathcal{M}$ [1, page 297] which satisfies the strict triangle inequality, i.e. $d(A, B) + d(B, C) = d(A, C)$ if and only if $B \in \overline{AC}$. This function $d$ is called the *Hilbert metric on* $\mathcal{M}$, the pair $(\mathcal{M}, d)$ is a *Hilbert geometry*, and $\mathcal{M}$ is its *domain*.

A Hilbert geometry is a model of the hyperbolic geometry of Bolyai, Lobachevskii and Gauss, if and only if its domain is an ellipsoid [1, (29.3)]. These isomorphic models of the hyperbolic geometry are called Cayley–Klein models.

In a Hilbert geometry $(\mathcal{M}, d)$ a set

$(D_1)$ $\mathcal{C}^{\varepsilon}_{F,\mathcal{H}} := \{X \in \mathbb{R}^n : \varepsilon d(X, \mathcal{H}) = d(F, X)\}$ is called a *conic*,

where $\mathcal{H}$ is a hyperplane, the *leading hyperplane* or *directrix*, $F \notin \mathcal{H}$ is a point, the *focus*, and $\varepsilon > 0$ is a number, the *numeric eccentricity.* A conic is said to be *elliptic*, *parabolic* and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$ and $\varepsilon > 1$, respectively.

In [2, Theorem 5.1] Kurusa proved that if even one conic is quadratic in a Minkowski plane, then the Minkowski plane is Euclidean. He conjectures that no quadratic conic may exist in Hilbert geometry.

In this article we support Kurusa's conjecture by proving in Theorem 4.1 that no conic in the Cayley–Klein model of the hyperbolic plane can be quadratic.

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## 2. Preliminaries

Points of $\mathbb{R}^n$ are denoted as $A, B, \dots,$ vectors are $\overrightarrow{AB}$ or $\boldsymbol{a}, \boldsymbol{b}, \dots,$ but we use these latter notations also for points if the origin is fixed. The open segment with endpoints $A$ and $B$ is denoted by $\overline{AB} = (A, B)$, $\overline{A}B$ is the open ray starting from $A$ passing through $B$ and the line through $A$ and $B$ is denoted by $AB$.

We denote the *affine ratio* of the collinear points $A, B$ and $C$ by $(A, B; C)$ that satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$. The *cross ratio* of the collinear points $A, B$ and $C, D$ is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [1, page 243].

Notations $\boldsymbol{u}_\varphi = (\cos\varphi, \sin\varphi)$ and $\boldsymbol{u}_\varphi^\perp := (\cos(\varphi + \pi/2), \sin(\varphi + \pi/2))$ are frequently used.

A curve in the plane is called *quadratical*, if it is part of a *quadric*

$$(D_q) \qquad \mathcal{Q}_{\mathfrak{s}}^\sigma := \left\{(x, y) : \begin{cases} 1 = x^2 + \sigma y^2, & \text{if } \sigma \in \{-1, 1\}, \\ x = y^2, & \text{if } \sigma = 0, \end{cases} \right\},$$

where $\mathfrak{s}$ is an affine coordinate system. A quadric is called *ellipse* (*affine circle*), *parabola* and *hyperbola*, if $\sigma = 1$, $\sigma = 0$ and $\sigma = -1$, respectively.

We use the Cayley–Klein model $(\mathcal{D}, \delta)$ (likewise called the Beltrami model) in the interior of the unit circular disc $\mathcal{D}$ of the plane $\mathbb{R}^2$ with the metric given by (1.1) as

$$(2.1) \qquad \delta(A, B) = \begin{cases} 0, & \text{if } A = B, \\ \frac{1}{2}\big|\ln(A, B; C, D)\big|, & \text{if } A \neq B, \text{ where } \overline{CD} = \mathcal{D} \cap AB. \end{cases}$$

Straight lines of $(\mathcal{D}, \delta)$, the *h-lines*, are the chords of $\mathcal{D}$, and segments of $(\mathcal{D}, \delta)$ are the segments of $\mathcal{D}$.

Isometries of $(\mathcal{D}, \delta)$, called *h-isometries*, are the restriction of those projectivities of the projective plane[4] $\mathbb{P}^2$ that leave $\mathcal{D}$ invariant. Any h-isometry is a product of at most three h-isometries which are restrictions of harmonic homologies, and any two non-degenerate triangles with pair-wisely equal side-lengths determine one and only one h-isometry that maps the first of these triangles onto the second one.

Let $\ell$ be an h-line in $(\mathcal{D}, \delta)$ and let $P \in \mathcal{H}$ be a point outside of $\ell$. The point $S \in \ell$ is the *$\ell$-foot of $P$*, if $\delta(P, X) \geq \delta(P, S)$ for every $X \in \ell$. An h-line $\ell'$ intersecting the h-line $\ell$ in a point $S$ is said to be *h-perpendicular to* $\ell$ if $S$ is an $\ell$-foot of $P$ for every $P \in \ell' \setminus \{S\}$. Notice that h-perpendicularity is invariant under isometries, because its definition is based on the metric. Further, the Euclidean line containing $\ell'$ is the one that connects $S$ and the intersection of those tangents of $\mathcal{D}$ that touch $\mathcal{D}$ at the points $\partial\mathcal{D} \cap \bar{\ell}$, where $\bar{\ell}$ is the Euclidean line containing $\ell$.

## 3. Conics in hyperbolic plane

As for any pair $(F, \ell)$ of a point $F$ in $\mathcal{D}$ and an h-line $\ell$ there exists an isometry $\imath$ such that $\imath(\ell)$ goes through the center $O$ of $\mathcal{D}$, and $O$ is the foot of $\imath(F)$ on $\imath(\ell)$, we can restrict without loss of generality the investigation of conics to those conics $\mathcal{C}_{F,\ell}^\varepsilon$ in $(\mathcal{D}, \delta)$ for which the directrix $\ell$ is the $y$-axis, and the focus $F$ is $(f, 0)$, where $f \in (0, 1)$.

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[4]For now, this is the affine plane $\mathbb{R}^2$ expanded with ideal points and straight line.

To calculate the points $P=(p,q)$ on $\mathcal{C}^{\varepsilon}_{F,\ell}$, we have to calculate $\delta(P,\ell)$ and $\delta(F,P)$, where $P=(p,q)\in\mathcal{C}^{\varepsilon}_{F,\ell}$.

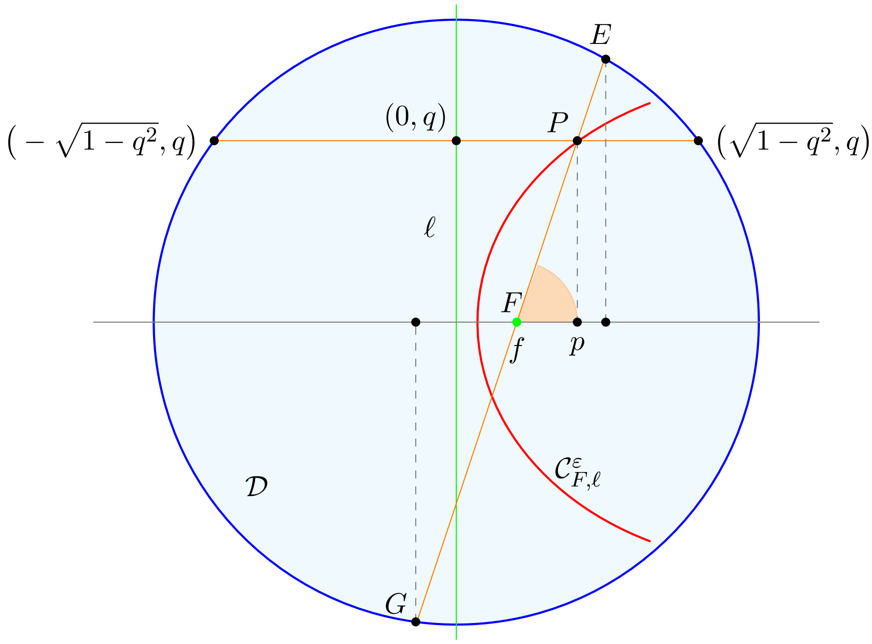


**Figure 1.** Directrix $\ell$ is through the center of the Cayley–Klein model, the focus $F$ is at $(f,0)$, where $f\in(0,1)$.

It is easy to get that

$$\delta(P,\ell)=\frac{1}{2}\Big|\log\Big\{\frac{p+\sqrt{1-q^2}}{p-\sqrt{1-q^2}}:\frac{0+\sqrt{1-q^2}}{0-\sqrt{1-q^2}}\Big\}\Big|. \tag{3.1}$$

To obtain $\delta(F,P)$, we firstly determine the points $\{E,G\}=\{(x_\pm,y_\pm)\}$, where line $FP$ intersects the unit circle, the border of $\mathcal{D}$. These points clearly satisfy the equations $x^2+y^2=1$ and $(x-f)q=y(p-f)$. So $(p-f)^2(1-x^2)=(x-f)^2q^2$, and we obtain $0=x^2((p-f)^2+q^2)-2fq^2x+(f^2q^2-(p-f)^2)$, hence

$$x_\pm=\frac{fq^2\pm(p-f)\sqrt{(p-f)^2+(1-f^2)q^2}}{(p-f)^2+q^2},$$
$$y_\pm=\frac{-qf(p-f)\pm q\sqrt{(p-f)^2+(1-f^2)q^2}}{(p-f)^2+q^2}.$$

Thus, we get

$$\begin{aligned}\delta(F,P)&=\frac{1}{2}\Big|\log\Big\{\frac{q^2+(p-f)^2+(f(p-f)+\sqrt{(p-f)^2+(1-f^2)q^2})}{q^2+(p-f)^2+(f(p-f)-\sqrt{(p-f)^2+(1-f^2)q^2})}\\&\qquad:\frac{f(p-f)+\sqrt{(p-f)^2+(1-f^2)q^2}}{f(p-f)-\sqrt{(p-f)^2+(1-f^2)q^2}}\Big\}\Big|\\&=\frac{1}{2}\Big|\log\Big\{\frac{(fp-1-\sqrt{(p-f)^2+(1-f^2)q^2})^2}{(1-f^2)(1-p^2-q^2)}\Big\}\Big|,\end{aligned} \tag{3.2}$$

where we have used the identities

$$\begin{aligned}(f(p-f)+\sqrt{(p-f)^2+(1-f^2)q^2})(f(p-f)-\sqrt{(p-f)^2+(1-f^2)q^2})\\ = f^2(p-f)^2-(p-f)^2-(1-f^2)q^2 = -(1-f^2)(q^2+(p-f)^2),\end{aligned}$$

and

$$\begin{aligned}(fp-1-\sqrt{(p-f)^2+(1-f^2)q^2})(fp-1+\sqrt{(p-f)^2+(1-f^2)q^2})\\ = (fp-1)^2-(p-f)^2-(1-f^2)q^2 = (1-f^2)(1-p^2-q^2).\end{aligned}$$

According to $(D_1)$ equations (3.1) and (3.2) give

$$\begin{aligned}&(1-q^2-p^2)\Big(1+\frac{2p}{\sqrt{1-q^2}-p}\Big)^{\epsilon}\\ &\qquad = \frac{\big(fp-1-\sqrt{q^2(1-f^2)+(p-f)^2}\big)^2}{1-f^2},\end{aligned} \tag{3.3}$$

where $\epsilon = \pm\varepsilon$. Figure 2 shows how these conics look like based on (3.3).

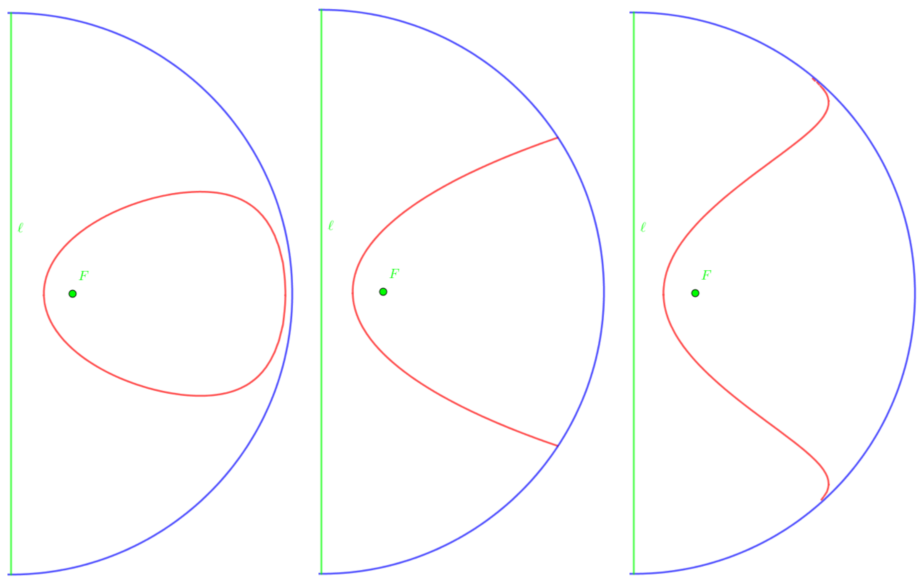

**Figure 2.** An elliptic ($\varepsilon = 0.9$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conic in the Cayley–Klein model of the hyperbolic geometry.

## 4. Quadratic conics in hyperbolic plane

We say that a conic is quadratic if it fits on a quadric $(D_q)$, hence satisfies an equation of the form $\bar{a}x^2+\bar{b}xy+\bar{c}y^2+\bar{d}x+\bar{e}y+\bar{f}=0$, where the coefficients are real and $\bar{a}\geq 0$.

As the conics $\mathcal{C}^{\varepsilon}_{F,\ell}$ are symmetric in the $x$-axis, the quadratic equation should be invariant under changing $y$ to $-y$, so $\bar{b}=\bar{e}=0$ follows. So the equation is of the form $\bar{a}x^2+\bar{c}y^2+\bar{d}x+\bar{g}=0$, hence $\bar{c}\neq 0$, because otherwise the curve will degenerate into straight lines. So the quadratic equation simplifies to

$$ax^2+y^2+bx+c=0, \quad a\geq 0. \tag{4.1}$$

For the sake of later contradiction in this section we assume from now on that

$$\text{conic } \mathcal{C}^{\varepsilon}_{F,\ell} \text{ is quadratic.}$$

As conic $\mathcal{C}^{\varepsilon}_{F,\ell}$ is quadratical, we have $q^2 = -ap^2 - bp - c, \quad a \geq 0$. Putting this into (3.3) gives the identity

$$(1 - p^2 + ap^2 + bp + c)\Big(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c} - p}\Big)^{\varepsilon} = \frac{\big(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}\big)^2}{1 - f^2}. \tag{4.2}$$

Differentiating this with respect to $p$ gives

$$\begin{aligned}(-2p + 2ap + b)&\Big(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c} - p}\Big)^{\varepsilon} + \\ &+ \varepsilon(1 - p^2 + ap^2 + bp + c)\Big(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c} - p}\Big)^{\varepsilon - 1} \times \\ &\times \Big(\frac{2}{\sqrt{1 + ap^2 + bp + c} - p} - \frac{2p(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}} - 1)}{(\sqrt{1 + ap^2 + bp + c} - p)^2}\Big) \\ = &\frac{\big(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}\big)}{1 - f^2} \times \\ &\times \Big(2f - \frac{-(2ap + b)(1 - f^2) + 2(p - f)}{\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}}\Big).\end{aligned} \tag{4.3}$$

The exponential multiplier in (4.2) is

$$\Big(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c} - p}\Big)^{\varepsilon} = \frac{\big(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}\big)^2}{(1 - f^2)(1 - p^2 + ap^2 + bp + c)}.$$

Putting this into (4.3) leads to

$$\begin{aligned}(-2p + 2ap + b)&\frac{\big(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}\big)^2}{(1 - f^2)(1 - p^2 + ap^2 + bp + c)} + \\ &+ \varepsilon\frac{\big(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}\big)^2}{1 - f^2} \times \\ &\times \Big(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c} - p}\Big)^{-1} \times \\ &\times \Big(\frac{2}{\sqrt{1 + ap^2 + bp + c} - p} - \frac{2p(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}} - 1)}{(\sqrt{1 + ap^2 + bp + c} - p)^2}\Big) \\ = &\frac{\big(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}\big)}{1 - f^2} \times \\ &\times \Big(2f - \frac{-(2ap + b)(1 - f^2) + 2(p - f)}{\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}}\Big).\end{aligned}$$

This simplifies to

$$\begin{aligned}
(-2p+2ap+b)&\frac{\big(fp-1-\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}\big)^2}{(1-f^2)(1-p^2+ap^2+bp+c)}+\\
&+\varepsilon\frac{\big(fp-1-\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}\big)^2}{(1-f^2)(\sqrt{1+ap^2+bp+c}+p)}\times\\
&\times\Big(2-\frac{2p(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}}-1)}{\sqrt{1+ap^2+bp+c}-p}\Big)\\
=&\frac{\big(fp-1-\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}\big)}{1-f^2}\times\\
&\times\Big(2f-\frac{-(2ap+b)(1-f^2)+2(p-f)}{\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}}\Big).
\end{aligned}$$

Dividing by $\frac{\big(fp-1-\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}\big)}{(1-f^2)(1-p^2+ap^2+bp+c)}$ further simplifies the equation to

$$\begin{aligned}
&\Big((-2p+2ap+b)+\varepsilon(\sqrt{1+ap^2+bp+c}-p)\Big(2-\frac{2p(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}}-1)}{\sqrt{1+ap^2+bp+c}-p}\Big)\Big)\times\\
&\quad\times\big(fp-1-\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}\big)\\
&=(1-p^2+ap^2+bp+c)\Big(2f-\frac{-(2ap+b)(1-f^2)+2(p-f)}{\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}}\Big),
\end{aligned}$$

i.e.

$$\begin{aligned}
&\Big((-2p+2ap+b)+\varepsilon(\sqrt{1+ap^2+bp+c}-p)\Big(2-\frac{2p(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}}-1)}{\sqrt{1+ap^2+bp+c}-p}\Big)\Big)\times\\
&\quad\times\big((fp-1)\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}+\\
&\qquad\qquad+(ap^2+bp+c)(1-f^2)-(p-f)^2\big)\\
&=(1-p^2+ap^2+bp+c)\times\\
&\quad\times\big(2f\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}+(2ap+b)(1-f^2)-2(p-f)\big).
\end{aligned}$$

With some rearrangement we obtain

$$\begin{aligned}
&2\varepsilon\Big(\sqrt{1+ap^2+bp+c}-\frac{p(2ap+b)}{2\sqrt{1+ap^2+bp+c}}\Big)\times\\
&\quad\times\big((fp-1)\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}+\\
&\qquad\qquad+(ap^2+bp+c)(1-f^2)-(p-f)^2\big)\\
&=(1-p^2+ap^2+bp+c)\times\\
&\quad\times\big(2f\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}+(2ap+b)(1-f^2)-2(p-f)\big)-\\
&\quad-(-2p+2ap+b)\times\\
&\quad\times\big((fp-1)\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}+\\
&\qquad\qquad+(ap^2+bp+c)(1-f^2)-(p-f)^2\big),
\end{aligned}$$

and again some rearrangement gives

$$
\begin{aligned}
2\varepsilon\Big(&\sqrt{1+ap^2+bp+c}-\frac{p(2ap+b)}{2\sqrt{1+ap^2+bp+c}}\Big)\times\\
&\times\big((fp-1)\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}+\\
&\qquad\qquad+(ap^2+bp+c)(1-f^2)-(p-f)^2\big)\\
=&\big(2f(1-p^2+ap^2+bp+c)-(-2p+2ap+b)(fp-1)\big)\times\\
&\times\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}+\\
&+\big((1-p^2+ap^2+bp+c)((2ap+b)(1-f^2)-2(p-f))-\\
&-(-2p+2ap+b)((ap^2+bp+c)(1-f^2)-(p-f)^2)\big),
\end{aligned}
$$

then

$$
\begin{aligned}
&\varepsilon(2(1+c)+pb)\\
&=\sqrt{1+ap^2+bp+c}\times\\
&\quad\times\Big(\big((fb+2a-2)p+2f+2fc+b\big)\sqrt{-(ap^2+bp+c)(1-f^2)+(p-f)^2}+\\
&\qquad+f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+b+2f(c+1)\Big),
\end{aligned}
$$

so we arrive at

$$
\begin{aligned}
&\varepsilon(2(1+c)+pb)\\
&\quad=\big((fb+2a-2)p+2f(c+1)+b\big)\times\\
&\qquad\times\sqrt{\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big)(1+ap^2+bp+c)}+\\
&\quad+(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b)\times\\
&\qquad\times\sqrt{1+ap^2+bp+c}.
\end{aligned}
$$

Squaring this gives

$$
\begin{aligned}
&\varepsilon^2(2(1+c)+pb)^2-\\
&\quad-\Big(\big((fb+2a-2)p+2f(c+1)+b\big)^2\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big)+\\
&\qquad+\big(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b\big)^2\Big)\times\\
&\quad\times(1+ap^2+bp+c)\\
&=2\big((fb+2a-2)p+2f(c+1)+b\big)(1+ap^2+bp+c)\times\\
&\quad\times\big(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b\big)\times\\
&\quad\times\sqrt{(p-f)^2-(ap^2+bp+c)(1-f^2)}.
\end{aligned}
$$

Squaring again gives the identity of two polynomials

(4.4)

$$\begin{aligned}
&\varepsilon^4(2(1+c)+pb)^4+\\
&\quad+\Big(\big((fb+2a-2)p+2f(c+1)+b\big)^2\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big)+\\
&\qquad+\big(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b\big)^2\Big)^2\times\\
&\quad\times(1+ap^2+bp+c)^2+\\
&\quad+2\varepsilon^2(2(1+c)+pb)^2\times\\
&\quad\times\Big(\big((fb+2a-2)p+2f(c+1)+b\big)^2\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big)+\\
&\qquad+\big(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b\big)^2\Big)\times\\
&\quad\times(1+ap^2+bp+c)\\
&=4\big((fb+2a-2)p+2f(c+1)+b\big)^2(1+ap^2+bp+c)^2\times\\
&\quad\times(f(2-bf-2a)p^2+2(a-1+f^2(c+1))p+2f(c+1)+b)^2\times\\
&\quad\times\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big).
\end{aligned}$$

Two polynomials can only be equal on a segment, if their corresponding coefficients are pairwise equal. The coefficients of $p^{12}$ are equal, so

$$\begin{aligned}
&\Big((fb+2a-2)^2(1-a(1-f^2))+f^2(2-bf-2a)^2\Big)^2a^2\\
&\qquad=4(fb+2a-2)^2a^2f^2(2-bf-2a)^2(1-a(1-f^2)),
\end{aligned}\tag{4.5}$$

that is equivalent to

$$a^2(fb+2a-2)^4\Big(\big((1-a(1-f^2))+f^2\big)^2-4f^2(1-a(1-f^2))\Big)=0,$$

so either $a=0$, which is disclosed by $a>0$, or $1-a(1-f^2)=f^2$, i.e. $a=1$ that implies $b=0$ in (4.5), or $fb+2a-2=0$.

Assuming $fb+2a-2=0$ simplifies (4.4) into

(4.6)

$$\begin{aligned}
&\varepsilon^4(2(1+c)+pb)^4+\\
&\quad+\Big(\big(2f(c+1)+b\big)^2\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big)+\\
&\qquad+\big(2(a-1+f^2(c+1))p+2f(c+1)+b\big)^2\Big)^2(1+ap^2+bp+c)^2+\\
&\quad+2\varepsilon^2(2(1+c)+pb)^2\times\\
&\quad\times\Big(\big(2f(c+1)+b\big)^2\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big)+\\
&\qquad+\big(2(a-1+f^2(c+1))p+2f(c+1)+b\big)^2\Big)(1+ap^2+bp+c)\\
&=4\big(2f(c+1)+b\big)^2(1+ap^2+bp+c)^2\times\\
&\quad\times(2(a-1+f^2(c+1))p+2f(c+1)+b)^2\times\\
&\quad\times\big((p-f)^2-(ap^2+bp+c)(1-f^2)\big).
\end{aligned}$$

Two polynomials can only be equal on a segment, if their corresponding coefficients are pairwise equal. The coefficients of $p^8$ are equal, so

$$\begin{aligned}(4.7)\qquad &\Big(\big(2f(c+1)+b\big)^2\big(1-a(1-f^2)\big)+4(a-1+f^2(c+1))^2\Big)^2a^2\\ &\quad=4\big(2f(c+1)+b\big)^2a^24(a-1+f^2(c+1))^2\big(1-a(1-f^2)\big),\end{aligned}$$

that is equivalent to

$$a^2\Big(\big(2f(c+1)+b\big)^2\big(1-a(1-f^2)\big)-4(a-1+f^2(c+1))^2\Big)^2=0,$$

so either $a=0$, which is disclosed by $a>0$, or $\big(2f(c+1)+b\big)^2\big(1-a(1-f^2)\big)=4(a-1+f^2(c+1))^2$. Thus, $\big(2f^2(c+1)+fb\big)^2\big(1-a(1-f^2)\big)=4f^2(a-1+f^2(c+1))^2$, hence

$$\big(f^2(c+1)+1-a\big)^2\big(1-a(1-f^2)\big)=f^2(a-1+f^2(c+1))^2,$$

and therefore $1-a(1-f^2)\geq 0$, i.e. $a<1/(1-f^2)$. Assuming $f^2(c+1)+(1-a)\neq 0$, we can write this equation in the form

$$\frac{(f^2(c+1)-(1-a))^2}{\big(f^2(c+1)+(1-a)\big)^2}=\frac{af^2+1-a}{f^2},$$

which, if $c+1\neq 0$, leads to

$$\begin{aligned}1&>\frac{(f^2(c+1)-(1-a))^2}{\big(f^2(c+1)+(1-a)\big)^2}=\frac{af^2+1-a}{f^2}>1\qquad\text{if } a\in(0,1),\\ 1&<\frac{(f^2(c+1)-(1-a))^2}{\big(f^2(c+1)+(1-a)\big)^2}=\frac{af^2+1-a}{f^2}<1\qquad\text{if } a\in(1,1/(1-f^2)),\end{aligned}$$

a clear contradiction. Thus, $c=-1$, and consequently $a=1$. If $f^2(c+1)+(1-a)=0$, then also $f^2(c+1)-(1-a)=0$, so $a=1$.

Thus, in any case, we obtained $a=1$. This simplifies (4.4) into

(4.8)

$$\begin{aligned}&\varepsilon^4(2(1+c)+pb)^4+\\ &\quad+\Big(\big(fbp+2f(c+1)+b\big)^2\big(f^2-p(2f+b(1-f^2))-c(1-f^2)\big)+\\ &\qquad+\big(-bf^2p^2+2f^2(c+1)p+2f(c+1)+b\big)^2\Big)^2\times\\ &\qquad\times(1+p^2+bp+c)^2+\\ &\quad+2\varepsilon^2(2(1+c)+pb)^2\times\\ &\qquad\times\Big(\big(fbp+2f(c+1)+b\big)^2\big(f^2-p(2f+b(1-f^2))-c(1-f^2)\big)+\\ &\qquad+\big(-bf^2p^2+2f^2(c+1)p+2f(c+1)+b\big)^2\Big)\times\\ &\qquad\times(1+p^2+bp+c)\\ &=4\big(fbp+2f(c+1)+b\big)^2(1+p^2+bp+c)^2\times\\ &\qquad\times(-bf^2p^2+2f^2(c+1)p+2f(c+1)+b)^2\times\\ &\qquad\times\big(f^2-p(2f+b(1-f^2))-c(1-f^2)\big).\end{aligned}$$

Two polynomials can only be equal on a segment, if their corresponding coefficients are pairwise equal. The coefficients of $p^{12}$ are equal, so $b^4 f^8 = 0$, hence $b = 0$.

Thus, in any case, we obtained $a = 1$ and $b = 0$. This simplifies (4.8) into

$$
\begin{aligned}
(4.9)\quad & \varepsilon^4 16(1+c)^4+ \\
& + \big(2f(c+1)\big)^4\Big(\big(f^2-2fp-c(1-f^2)\big)+\big(fp+1\big)^2\Big)^2(1+p^2+c)^2+ \\
& + 2\varepsilon^2 4(1+c)^2\big(2f(c+1)\big)^2\times \\
& \quad \times\Big(\big(f^2-2fp-c(1-f^2)\big)+\big(f(c+1)p+1\big)^2\Big)(1+p^2+c) \\
& = 4\big(2f(c+1)\big)^4(1+p^2+c)^2(fp+1)^2\big(f^2-2fp-c(1-f^2)\big).
\end{aligned}
$$

Two polynomials can only be equal on a segment, if their corresponding coefficients are pairwise equal. The coefficients of $p^8$ are equal, so $\big(2f(c+1)\big)^4(5f^2)^2 = 0$, hence $c = -1$.

Thus, by (4.1), if the conic $\mathcal{C}^{\varepsilon}_{F,\ell}$ is quadratic, then it is of the form $x^2+y^2 = 1$, a clear contradiction that proves the following:

**Theorem 4.1.** *No conic of the hyperbolic plane can be quadratic in Cayley–Klein models.*

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