INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 8 (2019), No. 2, 54 - 59

REFACTORING EQUIAFFINITIES

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Abstract. In this article we show how to refactor an equiaffinity, which is a composition of two affine reflections. The refactoring replaces the two affine reflections with two other, in the same way an euclidean rotation is represented as a product by an infinity of pairs of appropriate euclidean reflections.

1 Introduction

This article was inspired by Coxeter's ingenious proof of the property of *"equiaffinities"* to be representable as a product (composition) of two affine reflections ([5, p.33]), in combination with an old unpublished result of mine on generating hyperbolas. The property rises immediately the question on the possible number of such decompositions of a given equiaffinity and I saw that I could apply a special case of my result to answer this question. There is here an analogy to euclidean rotations, which can be written in an infinity of ways as a composition of two euclidean reflections ([8, I,p.50]), whose mirrors $\{\alpha,\beta\}$ make an angle half the angle ω of rotation (See Figure 1). The corresponding problem

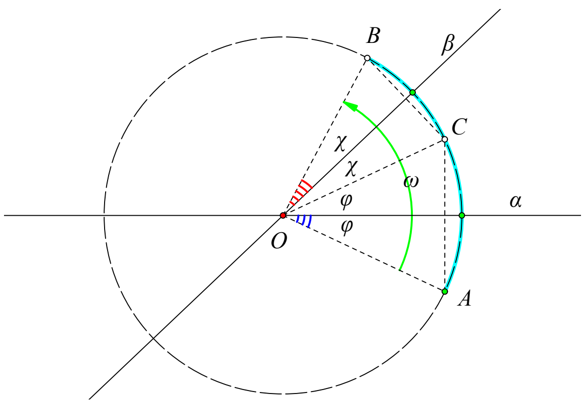

Figure 1: Parallelograms with sides parallel to arbitrary directions $\{\alpha,\beta\}$

for *"equiaffinities"*, i.e. compositions of two *"affine reflections"* ([5, p.17]) and the enumera-

Keywords and phrases: Affinity, Equiaffinity, Affine Reflection, Hyperbola
(2010)Mathematics Subject Classification: 51N10, 51N15, 51N20, 51N25
Received: 05.05.2019. In revised form: 26.09.2019. Accepted: 29.08.2019

tion of the possibilities of such a representation, as far as I know, seems to have not been handled yet. In section 2 we discuss the aforementioned generation of hyperbolas and its special case, corollary 2, which in section 3 is used to prove the main result, theorem 2.

2 Lines mapping to hyperbolas

Next theorem is useful in our context for its subsequent corollary 2. Probably, the proof of the corollary could be given directly, without the intervention of the theorem. I decided though to go through the more general theorem, since this reflects my personal path, and also adds, I hope, an interesting aspect, to the problem we are discussing.

The basic configuration here is a triangle ABC with $\{D,E,F\}$ the middles of corresponding opposite sides and a line ε on which varies a point P (See Figure 2). For each position of P on ε we consider the intersection point $M = PF \cap ED$ and the symmetric G of D w.r. to M.

Theorem 1. *The intersection point $Q = PE \cap AG$ describes a hyperbola as P varies on line ε. The hyperbola passes through the points $\{A,E\}$ and the two intersection points with the side-lines $\{A' = \varepsilon \cap BC, C' = \varepsilon \cap BA\}$. In addition, one of its asymptotes is parallel to the side BC.*

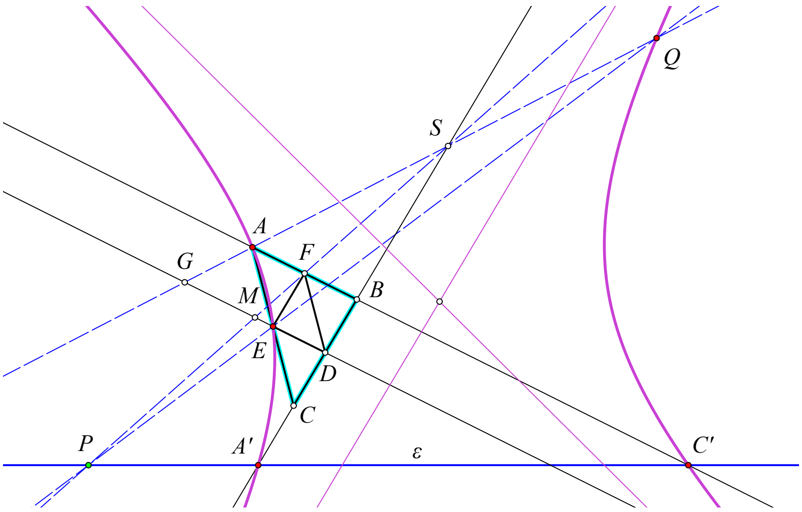

Figure 2: Hyperbola from a triangle and a line

Proof. We notice first that lines $\{PF, AG, BC\}$ are concurrent at a point S. This follows from the trapezium $ABDG$, since these lines are: the line joining the middles of the parallel sides and the two non parallel side-lines. Thus, we may consider the two lines $\{AG, EP\}$ intersecting at Q, as two rays of the pencils $\{A^*, E^*\}$, defined respectively by points $\{S,P\}$, which are related by a line (PS) turning about the fixed point F. But such a correspondence $P \leftrightarrow S$ of points on two lines $\{\varepsilon, BC\}$, through a line turning about a fixed point (F), defines a homographic relation ([2, p.67]) which, for coordinates $\{x,y\}$ on these lines, is represented by an invertible relation of the form

$$y = \frac{ax+b}{cx+d}, \quad \text{with constants satisfying:} \quad ad - bc \neq 0.$$

It follows ([2, p.66]) that the correspondence $EP \leftrightarrow AS$ between the rays of the pencils $\{E^*, A^*\}$ is also a homographic relation. Thus, by applying the Chasles-Steiner principle of generation of conics ([3, p.5], [1, p.72], [6, p. 259]), we conclude that the intersection points Q of the two homographically related rays of the pencils $\{A^*, E^*\}$ will generate a conic section passing through the centers $\{A, E\}$ of the two pencils.

The claim about the nature of the conic and its asymptote follows by taking $P \in \varepsilon$ at the position $\varepsilon \cap EF$. Then AG becomes parallel to BC and the two lines $\{EP, AG\}$ intersect at the point at infinity of line BC.

That the conic passes through $\{A', C'\}$ follows by taking into account that Q is always on line AS and realizing that for the positions of P at $\{A', C'\}$ point S obtains respectively the positions $\{A', B\}$, implying that Q coincides then respectively with $\{A', C'\}$. □

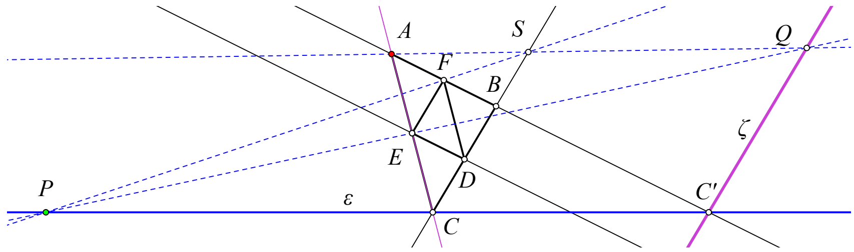

Figure 3: Line-locus from a triangle and a line

Corollary 1. *Given a triangle ABC with middles of sides $\{D, E, F\}$, we consider a line ε passing through C. For a variable point P on ε we consider the intersection points: the fixed: $C' = \varepsilon \cap AB$, and the variable: $\{S = BC \cap PF, Q = AS \cap PE\}$. Then, point Q describes a line ζ parallel to BC and passing through C'.*

Proof. This is a special case of theorem 2, resulting from it when line ε moves parallel to itself until to pass through point C. Then, the corresponding hyperbola-locus of Q, guaranteed by the theorem, passes through C. Thus, the hyperbola contains the three collinear points $\{A, E, C\}$, hence it degenerates to a product of lines, one of which is AC. The other line is the one parallel to BC passing through C'. This line is also the limit of the asymptote of the hyperbolas passing through the point at infinity of line BC. □

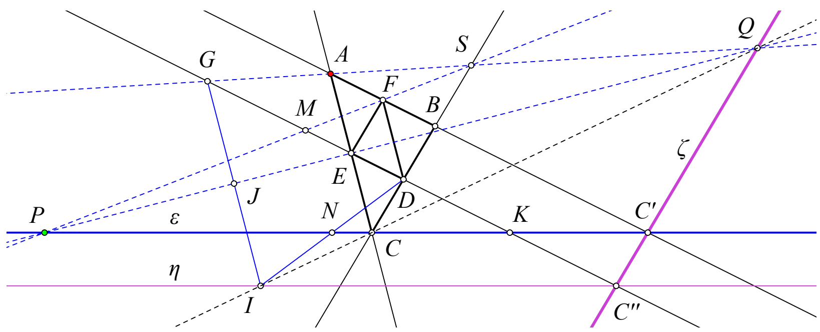

Figure 4: Locus of point I

Corollary 2. *With the definitions and notation of corollary 1, let $G = ED \cap AQ$ and GI be the segment parallel to AC, and such that its middle J is on line PE. Then, the middle N of ID is on the line ε.*

Proof. In fact, consider the intersection point $C'' = \zeta \cap ED$. Since PJ joins the middles of parallel sides of the trapezium $AGIC$, it passes through the intersection of the non parallel sides. Hence lines $\{PJ, GA, IC\}$ concur at Q and we have the equality of ratios:

$$\frac{CI}{CQ} = \frac{AG}{AQ} = \frac{C'C''}{C'Q}.$$

From this follows that lines $\eta = IC''$ and ε are parallel. Since the segments $\{C'C'', DC\}$ are parallel and equal, it follows that all segments DI with $I \in \zeta$ have their middle on line ε. □

3 Affine reflections

Affine reflections ([4, p.203]) are invertible transformations $f_{\mu,u}$ of the plane, defined by a line μ, called the *"axis"* and a *"direction"* called *"conjugate direction"* which is represented by a line u or a parallel to it. The transformation associates to each point X of the plane

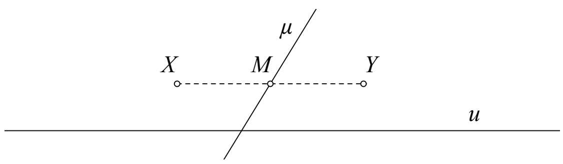

Figure 5: Affine reflection

the point Y (See Figure 5), such that XY : (i) is parallel to u, and (ii) has its middle M on line μ. Affine reflections generalize the usual euclidean reflections, which are defined in the same way with the only restriction that the lines $\{\mu, u\}$ are orthogonal.

Affine reflections, as do all affine transformations, preserve the parallelity of lines and the ratio BA/BC of three collinear points $\{A, B, C\}$. Obviously also they are *"involutive"* coinciding with their inverse transformation and satisfying $f^2 = e$, where e is the identity transformation of the plane. According to a theorem of Veblen ([7, II,p.110], [5, p.17]) the compositions of two reflections represent all *"equiaffinities"* i.e. all affine transformations preserving the area. Thus, our problem could be formulated as follows: *"In how many different ways can we represent an equiaffinity?"* An equivalent variation could be the following: *"starting from a pair of reflections (f_1, f_2), find all other pairs (g_1, g_2) satisfying $f_2 \circ f_1 = g_2 \circ g_1$."* This, given the involutive character of reflections, is equivalent with the problem of finding a reflection g_2 such that the product $g_1 \circ g_2 \circ f_2 \circ f_1 = e$ is the identity. Next theorem settles this question.

Theorem 2. *For a pair of affine reflections (f_1, f_2) and a given line μ through the intersection point P of their axes, there is another pair (f, f_3) of affine reflections, such that the axis of f is the line μ and the composition*

$$f \circ f_3 \circ f_2 \circ f_1 = e \quad \Leftrightarrow \quad f_3 \circ f \ = \ f_2 \circ f_1. \tag{1}$$

Proof. Assume first that the intersection point $P = \mu_1 \cap \mu_2$ of the axes of the reflections $\{f_1, f_2\}$ is a finite point and the corresponding conjugate directions $\{u_1, u_2\}$ are not parallel. Let $C \in \mu$ be a variable point on the given line μ through P. Applying the reflections $\{B = f_1(C),\ A = f_2(B)\}$ we obtain a triangle ABC, which is easily seen to remain similar and similarly placed to itself as C varies on μ (See Figure 6). Thus, the middles E of the sides AC of these triangles $\{ABC\}$ vary on a line μ_3, while the sides AC remain

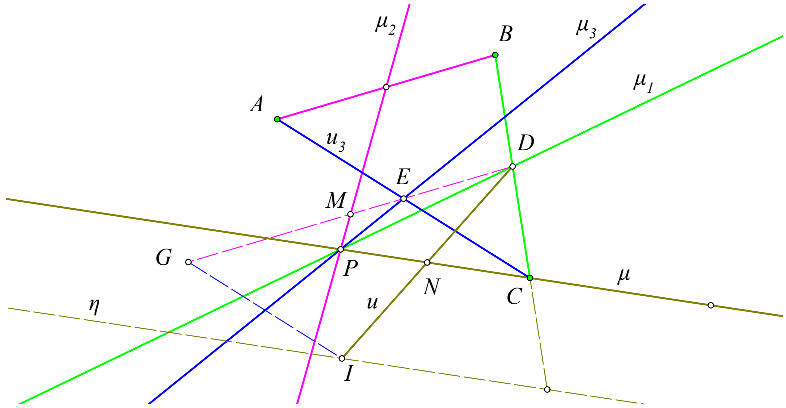

Figure 6: Refactoring the composition $f_2 \circ f_1$ (I)

parallel to a fixed direction u_3. By corollary 2, taking the middle D of BC and defining $\{G = f_2(D),\ I = f_3(G)\}$, where f_3 is the affine reflection with axis μ_3 and conjugate direction u_3, we obtain triangle GDI, whose side ID has its middle N on μ. As point C varies on μ, the direction u of DI remains constant and this defines the reflection f with axis μ and conjugate direction u.

The composition of transformations $g = f \circ f_3 \circ f_2 \circ f_1$, applied to points $C \in \mu$, gives obviously $g(C) = C$. The same is easily seen to be true for points $D \in \mu_1 : g(D) = D$. Thus, the affine transformation g coincides with the identity along the lines $\{\mu, \mu_1\}$, hence also at three non-collinear points, and consequently ([4, p.203]) coincides everywhere with the identity transformation, thereby proving the theorem in the case P is a finite point.

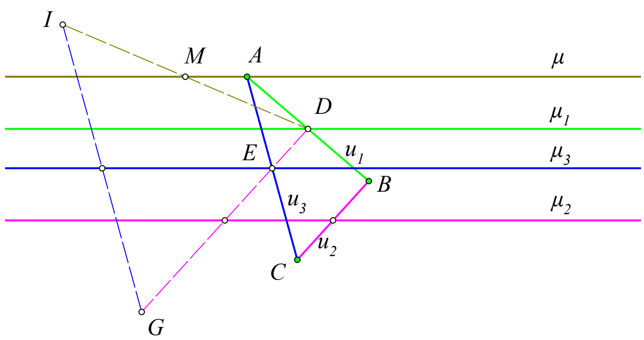

Figure 7: Refactoring the composition $f_2 \circ f_1$ (II)

The remaining three particular cases, that may occur, are much easier to handle and I describe them briefly, leaving some details as exercises. From these cases, the first occurs when the intersection point $P = \mu_1 \cap \mu_2$ is at infinity, i.e. the axes are parallel, and the conjugate directions $\{u_1, u_2\}$ are not parallel, then an arbitrary line μ through P and the corresponding line μ_3 are parallels to these two lines (See Figure 7). The proof in this case results with minor changes by the same arguments, and a simpler argument that point M is on line μ which can avoid the use of corollary 2.
The second particular case occurs when the axes $\{\mu_1, \mu_2\}$ intersect at P but the conjugate directions coincide with the direction of the line u through P (See Figure 8). Then the composition f_1, f_2 is a *"shear"* with axis u ([4, p.203]). It is then easily seen that every line μ intersecting u at a point P' defines another line μ_3 through P', such that the corre-

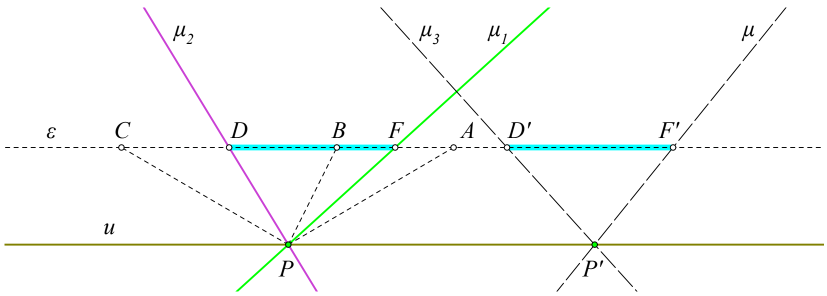

Figure 8: Refactoring the composition $f_2 \circ f_1$ (III)

sponding reflections with axes μ, μ_3 and conjugate directions coinciding with u, define the same shear: $f_3 \circ f = f_2 \circ f_1$.

Finally, the third case occurs when the reflections have parallel axes $\{\mu_1, \mu_2\}$ and the same conjugate direction u. Then, it is trivially seen that their product $f_2 \circ f_1$ is a *"translation"* by the vector $2EF$, where $\{E, F\}$ are the intersections of $\{\mu_1, \mu_2\}$ with a line parallel to u. Then, every parallel translates $\{\mu, \mu_3\}$ of these two lines and the same conjugate direction u will satisfy the theorem. □

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