INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 8 (2019), No. 2, 5 - 13

THE SIX-POINT CIRCLE FOR THE QUADRANGLE

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Abstract. While seeking a pure-axiomatic proof for the Van Aubel's theorem [6], the author discovered a hitherto unknown, curious property.
The discovered property is that six points involved in the geometrical configuration of the forementioned theorem lie on a circle, as represented in figure 1.
In the following paper, the author, first calls to mind the Van Aubel's theorem, then, proposes a convenient nomenclature to describe the geometrical elements pertinent to the configuration, and finally, presents the original six-point circle theorem for the quadrangle.
The pure-axiomatic proofs of the various theorems are elaborated in section 3.

1. VAN AUBEL'S THEOREM

Theorem 1.1 (Van Aubel's theorem [6])**.** *Given a convex quadrangle, on each side construct a square external to the quadrangle. Connect the centers of the squares constructed over the opposite sides. The obtained segments are of equal length and orthogonal to each other.*

The theorem holds true also for re-entrant quadrangles [3], and when the squares are constructed internal to the given quadrangle.

Looking at figure 2, the equal and orthogonal segments are represented by O_1O_3 and O_2O_4, and they intersect at V. For the internal constructed squares, the equal and orthogonal segments are represented by $O'_1O'_3$ and $O'_2O'_4$, and they intersect at V' on the extension of segment $O'_1O'_3$.
The *external* and *internal* constructions for the squares are not definable for crossed quadrangles [3].
For crossed quadrangles, the theorem holds true when the constructions are carried out in the more general way:

Keywords and phrases: Van Aubel's theorem, Newton line, six-point circle
(2019)Mathematics Subject Classification: 51F02
Received: 30.04.2019. In revised form: 22.08.2019. Accepted: 19.08.2019.

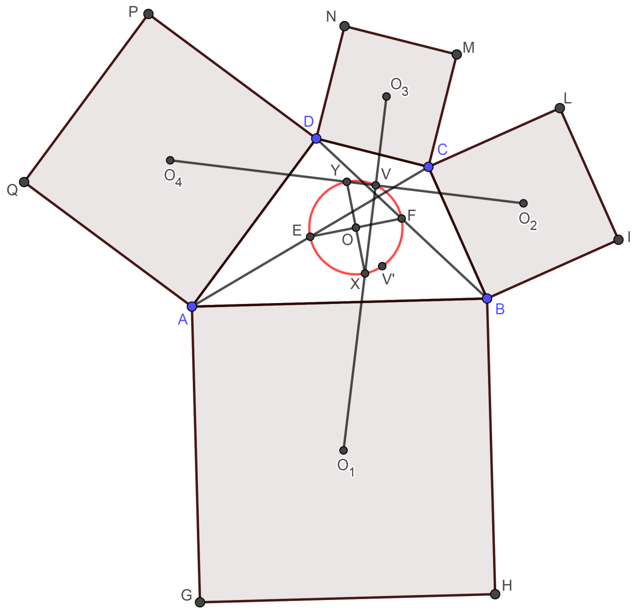

FIGURE 1. The six-point circle for the quadrangle is passing through the midpoints of the quadrangle's diagonals, E and F, the midpoints of the Van Aubel's segments, X and Y, and the Van Aubel's points, V and V'. The definitions for the Van Aubel's segments and the Van Aubel's points are given at the end of section 1.

- Follow the quadrangle's vertexes in a sequential direction (starting for example from the lower left vertex, point A, towards point B) and construct each square on the right hand side of each side of the given quadrangle.
- Follow the quadrangle's vertexes in the same sequential direction and construct each square on the left hand side of each side of the given quadrangle.

For the rest of the work, the author will focus on convex quadrangles, as the diagrams are easier to interpret than in the other cases.
The presented theorems remain true also for re-entrant and crossed quadrangles as the various proofs can be carried out essentially in the same way as for the convex case.
The following nomenclature is defined in order to simplify the exposition.

- Any segment that joins the centers of either internal or external squares that are constructed on opposite sides of the quadrangle will be referred to as a Van Aubel's segment.
- The pair of the resulting equal and orthogonal Van Aubel's segments, from the external construction, will be named as the first Van Aubel's segment pair.
- The intersection of the first Van Aubel's segment pair, point V, will be named as the first Van Aubel's point.
- The pair of the equal and orthogonal segments, from the internal construction, will be named as the second Van Aubel's segment pair.

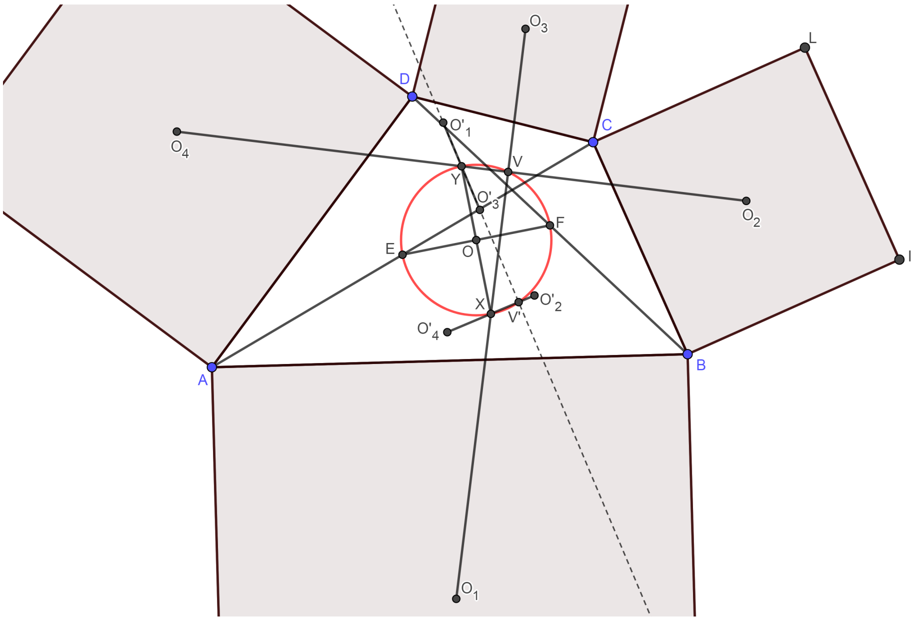

FIGURE 2. The six-point circle for the quadrangle - enlargement showing the construction for the second Van Aubel's point. The internal squares are not shown for the sake of clarity, their centers are represented by the points O'_i, with i=1,2,3,4.

- The intersection of the second Van Aubel's segment pair, Point V', will be named as the second Van Aubel's point.
- The segment joining the midpoints of the quadrangle's diagonals, represented as E and F in figure 2, will be named as the Newton's segment because the line through those points is known as the the Newton line [4].
- Any quadrangle with equal and orthogonal diagonals will be referred to as an iso-ortho-diagonal quadrangle. For example, quadrangle $O_1O_2O_3O_4$ is an an iso-ortho-diagonal quadrangle as a consequence of the Van Aubel's theorem.

For crossed quadrangles, an equivalent nomenclature can be defined that distinguishes the geometrical elements obtained with the *right hand side* construction from the ones obtained with the *left hand side* construction.

1.1. An interesting related theorem.

Theorem 1.2. *Each segment of a Van Aubel's segment pair is bisected by the Van Aubel's segment of the other pair that connects the centers of the squares constructed on the other couple of opposites side.*

For instance, the Van Aubel's segment O_1O_3 (obtained connecting the centers of the squares constructed external to the quadrangle over the opposites sides AB and DC) and the Van Aubel's segment $O'_2O'_4$ (obtained connecting the centers of the squares constructed internal to the quadrangle over the opposites sides AD and BC) bisect each other at point X, as shown

in figure 2.
The midpoints of the Van Aubel's segments are represented as X and Y in figures 1 and 2.
This theorem was found independently by the author while seeking the possible relations between the elements pertaining to both the external and internal constructions. After extensive research, the same theorem was found on the Alexander Bogomolny's website [2].

2. THE SIX-POINT CIRCLE THEOREM FOR THE QUADRANGLE

Theorem 2.1. *The midpoints of the quadrangle's diagonals, E and F, the first and second Van Aubel's points, V and V', and the midpoints of the Van Aubel's segments, X and Y, lie on a circle.*

Corollary 2.1. *The Newton's segment, EF, and the segment connecting the midpoints of the Van Aubel's segments, XY, are two mutually orthogonal diameters of the circle.*

Remark 2.1. *The center of the circle, O, coincides with the quadrangle's centroid* [1].

The circle is shown in figures 1, 2 and 7. The results expressed by this theorem and its corollary are original contributions by the author.

3. PROOFS

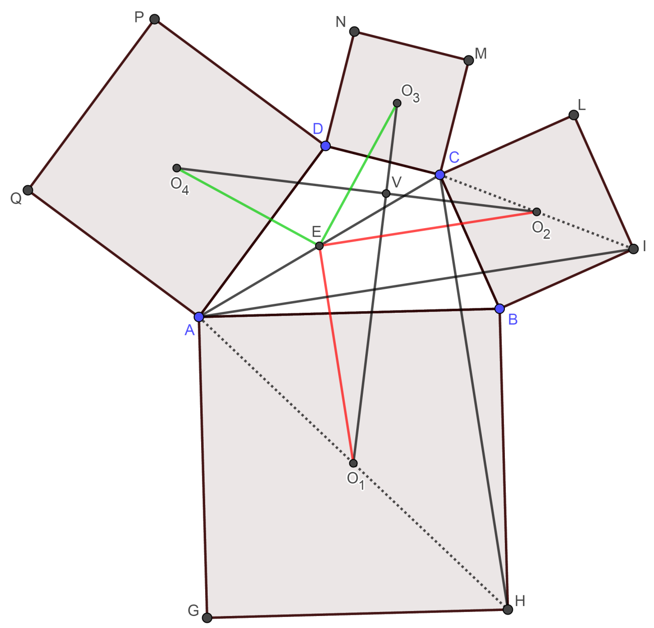

FIGURE 3. Diagram for the Van Aubel's theorem proof. The construction revolves around the midpoint E of the quadrangle's diagonal AC.

3.1. **Van Aubel's theorem. Proof.** Looking at figure 3, we deduce that segments AI and CH are of equal length and orthogonal to each other. This fact follows from the relation between the congruent triangles ABI and HBC: triangle HBC can be obtained from a right angle counterclockwise rotation around point B of triangle ABI.
If we apply the midsegment theorem [8] to triangles HAC and IAC respectively, we deduce that segments O_1E and O_2E are of equal length and meet at E orthogonally.
An analogous argument can be applied to show that segments O_3E and O_4E are of equal length and meet at E orthogonally.
Therefore, triangles O_1EO_3 and O_2EO_4 are congruent according to the SAS (Side-Angle-Side) rule: the included angles O_1EO_3 and O_2EO_4 are equal as they both measure the right angle plus the shared angle O_2EO_3.
Moreover, triangle O_2EO_4 can be obtained from a right angle counterclockwise rotation around point E of triangle O_1EO_3.
It follows that segments O_1O_3 and O_2O_4 are of equal length and orthogonal to each other.

Remark 3.1. *The proof would have proceeded exactly in the same way if we would have considered the other diagonal midpoint (F, represented in figures 1 and 2).*

Remark 3.2. *The proof for the internal Van Aubel's configuration is exactly the same.*

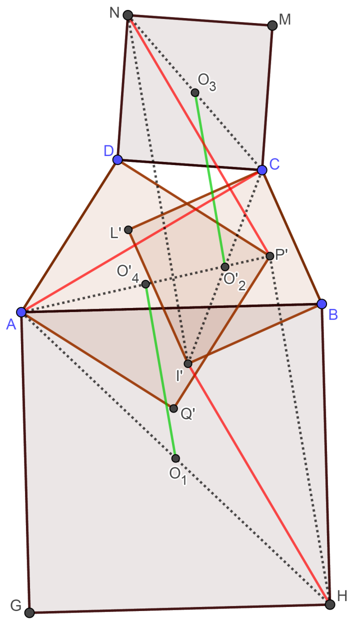

FIGURE 4. Diagram for the theorem 2 proof. The diagram shows the necessary geometrical elements to prove that quadrangle $O_1O'_2O_3O'_4$ is a parallelogram.

3.2. **Theorem 1.2. Proof.** Looking at figure 4, we deduce that segments HI' and AC (a quadrangle's diagonal) are of equal length and orthogonal

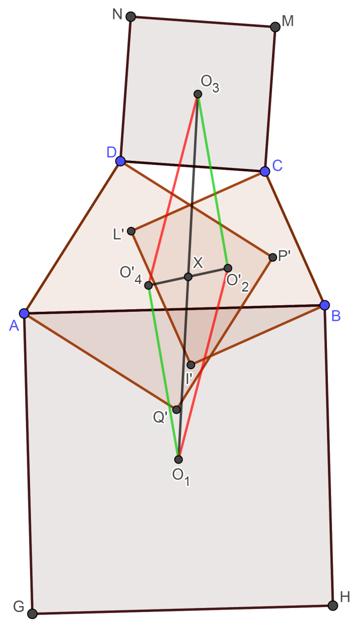

FIGURE 5. Diagram for the theorem 2 proof. The diagram shows the paralelogram O$_1$O'$_2$O$_3$O'$_4$ and its bisecting diagonals.

to each other. This fact follows from the relation between the congruent triangles HI'B and CAB: triangle HI'B can be obtained from a right angle counterclockwise rotation around point B of triangle CAB.
An analogous argument can be applied to show that segment NP' and the very same quadrangle's diagonal AC are of equal length and orthogonal to each other.
Therefore, segments HI' and NP' are of equal length and parallel, so that quadrangle HP'NI' is a parallelogram.
If we apply the midsegment theorem to triangles HAP' and NCI' respectively, we deduce that segments O$_1$O'$_4$ and O'$_2$O$_3$ are of equal length and parallel so that quadrangle O$_1$O'$_2$O$_3$O'$_4$ is a parallelogram.
It follows that the parallelogram O$_1$O'$_2$O$_3$O'$_4$ diagonals, O$_1$O$_3$ and O'$_2$O'$_4$, bisect each other at point X as shown in figure 5.

3.3. **The four concurring circles' lemma.**

Definition 3.1. *A circle will be defined by its center and radius. For instance, looking at figure 6, the circle, constructed with segment O_1O_2 as a diameter, will be referred to as the (C_1, C_1E) circle or equivalently as the (C_1, C_1V) circle, the (C_1, C_1O_1) circle and so on.*

The following lemma is central for the six-point circle theorem proof development.

Lemma 3.1. *The four circles, constructed with the sides of the obtained iso-ortho-diagonal quadrangle $O_1O_2O_3O_4$ (figure 6) as their diameters, concurr in the first Van Aubel's point, V.*

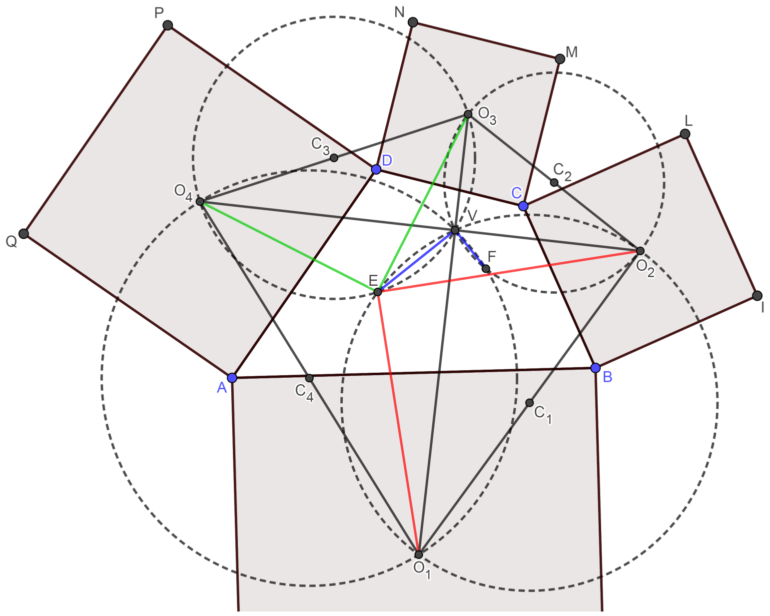

FIGURE 6. The four concurring circles' lemma.

Proof. The circumscribed circles of the right-angled triangles O_1VO_2, O_2VO_3, O_3VO_4 and O_1VO_4 concurr at point V (see figure 6). If we apply the converse of Thales's theorem [7] to any of these triangles, i.e. triangle O_1VO_2, we deduce that its hypotenuse, i.e. O_1O_2, coincides with a diameter of its circumscribed circle, i.e. (C_1, C_1V).
Incidentally, we notice that point E (the midpoint of the ABCD quadrangle's diagonal AC) is the other point of intersection of circles (C_1, C_1V) and (C_3, C_3V). This statement is a direct consequence of the converse of Thales's theorem applied to the right-angled triangles O_1EO_2 and O_3EO_4, represented in figures 3 and 6: quadrangles O_1O_2VE and O_3O_4EV are cyclic quadrangles inscribed in circles (C_1, C_1V) and (C_3, C_3V), respectively. An analogous argument can be applied to show that point F (the midpoint of the ABCD quadrangle's diagonal BD) is the other point of intersection of circles (C_2, C_2V) and (C_4, C_4V).

Remark 3.3. *The centers of the four circles, represented in figures 6 and 7 as C_1, C_2, C_3 and C_4, coincide with the midpoints of the sides of the iso-ortho-diagonal quadrangle $O_1O_2O_3O_4$.*

Remark 3.4. *Segments EV and VF are the common chords of the circles constructed with the opposite sides of the iso-ortho-diagonal quadrangle $O_1O_2O_3O_4$ as diameters (see figures 6 and 7).*

Remark 3.5. *The analogous result holds for the second Van Aubel's point.*

3.4. **The six-point circle theorem for the quadrangle. Proof.** According to the Van Aubel's theorem and the converse of Thales's theorem, it follows that quadrangle XV'VY (vertexes represented in figure 2) is cyclic, and XY is a diameter of its circumscribed circle. The center of the circle, O, is the midpoint of segment XY. By definition, point O is the centroid of

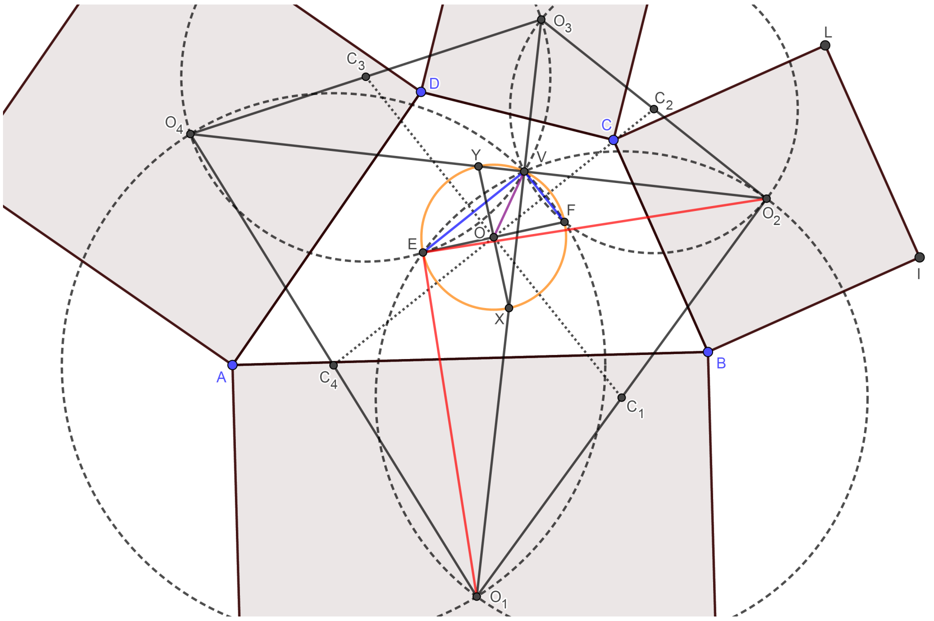

FIGURE 7. Diagram for the six-point circle theorem proof development.

the iso-ortho-diagonal quadrangle $O_1O_2O_3O_4$ [1]. Segment OV, represented in figure 7, is a radius of the circle.
With lemma 1, we noted that segments EV and VF are the common chords of the circles constructed with the opposite sides of the iso-ortho-diagonal quadrangle as diameters (see figure 7). Therefore, segment EV is bisected orthogonally by segment C_1C_3, and segment VF is bisected orthogonally by segment C_2C_4. Segments C_1C_3 and C_2C_4 are the segments which connect the midpoints of the opposite sides of the iso-ortho-diagonal quadrangle $O_1O_2O_3O_4$.
They bisect each other at point O forming a right angle.
The fact that they bisect each other forming a right angle is a very well known property of any iso-ortho-diagonal quadrangle: applying the mid-segment theorem to triangles $O_1O_3O_4$, $O_1O_3O_2$, $O_2O_4O_3$ and $O_2O_4O_1$, we deduce that quadrangle $C_1C_2C_3C_4$ is indeed a square.
The fact that the segments which connect the midpoints of the opposite sides of any quadrangle bisect each other at the quadrangle's centroid (point O for the $O_1O_2O_3O_4$ quadrangle) is another quite known property [1].
With segments C_1C_3 and C_2C_4 the segment bisectors of the EV and FV segments, respectively, we deduce that:
– EVF is a right angle. Again, according to the converse of Thales's theorem, segment EF is a diameter of the circumscribed circle of triangle EVF, also shown in figure 7.
– The center of the circle coincides with point O. Trivially, OV is a radius of the circle.
Thus, all six points E, X, V', F, V and Y lie on the same circle (O, OV) and segments EF and XY are two diameters of the circle.

Remark 3.6. *Point O (the centroid of the obtained iso-ortho-diagonal quadrangle) coincides with the centroid of the generic quadrangle ABCD as it bisects the Newton's segment EF.*

The corollary proof follows hereafter.

3.5. **Corollary 2.1. Proof.** This proof can be carried out via angles inspection. Looking at figure 7 we notice that angle EVX measures half of a right angle.
Indeed, angle EVX is equal to angle O_1O_2E which measures half of a right angle (see the Van Aubel's theorem proof) as both angles subtend arc O_1E of circle (C_1, C_1E).
As a consequence of the the six-point circle theorem for the quadrangle, we know that angle EVX subtend the EX arc of circle (O, OV) and thus angle EOX is a right angle according to the inscribed angle theorem [5].

4. ACKNOWLEDGMENT

The author wishes to thank his friend Jackie Clayton, whose precious help was crucial to improve dramatically the exposition of the topic.

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