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ON THE EIGHT CIRCLES THEOREM AND ITS DUAL

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Abstract. Problem 3845 in Crux Mathematicorum is a nice configuration about eight circles with many special cases. We sketch a proof of this problem and rename it as *eight circles theorem*. Using Miquel's six circles theorem, the bundle theorem and the eight circles theorem, we give a proof of the dual of the eight circles theorem and its converse.

1. Introduction

Consider six points A_1, A_2, $\cdots$, A_6 on a circle (O_A) and a point B_1 on another circle (O_B). Let the circle $(A_iA_{i+1}B_i)$ meet (O_B) again at B_{i+1} for $i = 1, \cdots, 5$. *Then the four points A_6, A_1, B_1, B_6 lie on a circle.* Denoting by (O_i) the circle $(A_iA_{i+1}B_{i+1}B_i)$ for $i = 1, \cdots, 6$, taking subscripts modulo 6, *it can be shown that the three lines O_1O_4, O_2O_5, O_3O_6 are concurrent*; we denote the point of concurrence by O. The first author proposed this result which appears as Problem 3845 in Crux Mathematicorum [1]. We sketch a proof of the problem following J. Chris Fisher [2]. That A_6, A_1, B_1, B_6 are concyclic is easily seen by applying Miquel's six circles theorem (cf. the first part of the proof of Theorem 1.1 below). To prove that the three lines O_1O_4, O_2O_5, O_3O_6 are concurrent, by Brianchon's theorem it suffices to show that there is a conic tangent to the sides of the hexagon $O_1O_2O_3O_4O_5O_6$. By the lemma of J. Chris Fisher [2], this conic is the unique conic with foci O_A and O_B such that it is tangent to O_1O_2 (resp. O_iO_{i+1} for $i = 2, \cdots, 6$, taking subscripts modulo 6). A proof of Fisher's lemma was given by M. Bataille [3]. We now call this problem *eight circles theorem.* There are some other proofs of the theorem by Ákos G.Horváth [4] and Gábor Gévay [5].

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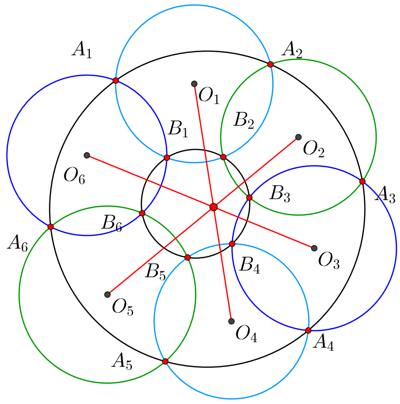

FIGURE 1. Three lines O_1O_4, O_2O_5, O_3O_6 are concurrent

The dual of the eight circles theorem is as follows:

Theorem 1.1. *[6] We keep the notations as above. If (O_1) meets (O_4) at two points C_3, C_6, (O_2) meets (O_5) at two points C_4, C_1 and (O_3) meets (O_6) at two points C_2, C_5, then the six points C_1, C_2, C_3, C_4, C_5, C_6 lie on a circle with center O.*

The eight circles theorem is a generalization of a special case of Brianchon's theorem and its dual is a generalization of a special case of Pascal's theorem [6]. The dual is also a generalization of the Dao-symmedial circle. The reader may find the Dao-symmedial circle in [7].

We reformulate and prove the converse of Theorem 1.1 in the following form.

Theorem 1.2. *[8] Let (O_1), (O_2), $\cdots$, (O_6) be six circles on the plane, and (O_i) meet (O_{i+1}) at A_{i+1}, B_{i+1} for $i = 1, \cdots, 6$ (taking subscripts modulo 6) such that A_1, A_2, $\cdots$, A_6 lie on a circle (in particular (O_i) contains the points $A_i, B_i, A_{i+1}, B_{i+1}$). Let (O_1) meet (O_4) at two points C_3, C_6, (O_2) meet (O_5) at two points C_4, C_1 and (O_3) meet (O_6) at two points C_2, C_5 such that C_1, C_2, C_3, C_4, C_5 lie on a circle (C). Then C_6 lies on the circle (C) and the six points B_1, B_2, B_3, B_4, B_5, B_6 lie on a circle.*

In order to prove Theorem 1.1 and Theorem 1.2 we need to recall some important theorems:

Theorem 1.3 (Miquel). *Let (O_1), (O_2), (O_3), (O_4) be four circles on the plane, (O_i) meet (O_{i+1}) at A_{i+1}, B_{i+1} for $i = 1, \cdots, 4$, where we take subscripts modulo 4. If A_1, A_2, A_3, A_4 lie on a circle, then B_1, B_2, B_3, B_4 lie on a circle.*

Theorem 1.4 (Bundle theorem). *Let (O_1), (O_2), (O_3), (O_4) be four circles on the plane, (O_i) meet (O_{i+1}) at A_{i+1}, B_{i+1} for $i = 1, \cdots, 4$, where we take subscripts modulo 4. If A_2, B_2, A_4, B_4 lie on a circle, then A_1, B_1, A_3, B_3 lie on a circle.*

Using Miquel's six circles theorem, the bundle theorem and the eight circles theorem, we give a proof of Theorem 1.1 and Theorem 1.2. Another proof of Theorem 1.2 is given by Futurologist in the Stackexchange forum [8].

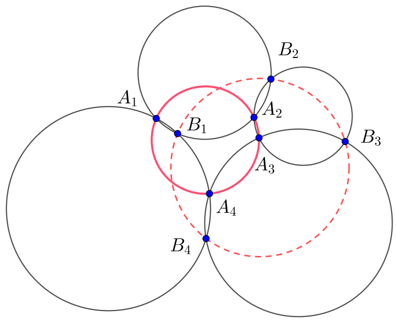

Figure 2. Miquel's six circles theorem

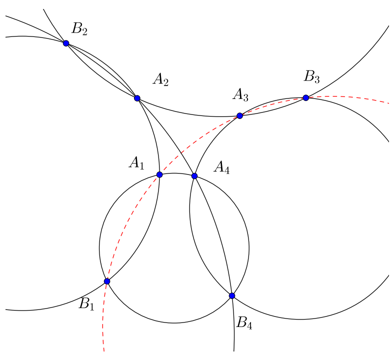

Figure 3. The bundle theorem

2. A Proof of Theorem 1.1

Proof. Applying Miquel's six circles theorem for (See Figure 4a):

- The circle (O_2) and four circles (O_3), (O_B), (O_1), (O_A), we have four points A_1, B_1, B_4, A_4 lying on a circle.
- The circle (O_3) and four circles (O_4), (O_B), (O_2), (O_A), we have four points A_2, B_2, B_5, A_5 lying on a circle.
- The circle (O_4) and four circles (O_5), (O_B), (O_3), (O_A), we have four points A_3, B_3, B_6, A_6 lying on a circle.
- The circle $(A_2B_2B_5A_5)$ and four circles (O_5), (O_B), (O_2), (O_A), we have four points A_1, B_1, B_6, A_6 lying on a circle.

Applying the bundle theorem for (See Figure 4b):

- The eight points $A_2, B_2, C_3, C_6, A_5, B_5, C_1, C_4$ in (O_1), (O_4), (O_5), (O_2) with four points A_2, B_2, B_5, A_5 lying on a circle as above, we have that C_1, C_4, C_3, C_6 lie on a circle. Similarly, C_1, C_4, C_2, C_5 lie on a circle and C_2, C_5, C_3, C_6 lie on a circle.

By the eight circles theorem, the lines O_1O_4, O_2O_5, O_3O_6 are concurrent, so the perpendicular bisectors of C_3C_6, C_1C_4, C_2C_5 are concurrent, where the point of concurrence is denoted by O. Then the three circles $(C_1C_4C_3C_6)$, $(C_2C_5C_3C_6)$, $(C_1C_4C_2C_5)$ have the same center. It follows that the six points C_1, C_2, C_3, C_4, C_5, C_6 lie on a circle with center O.

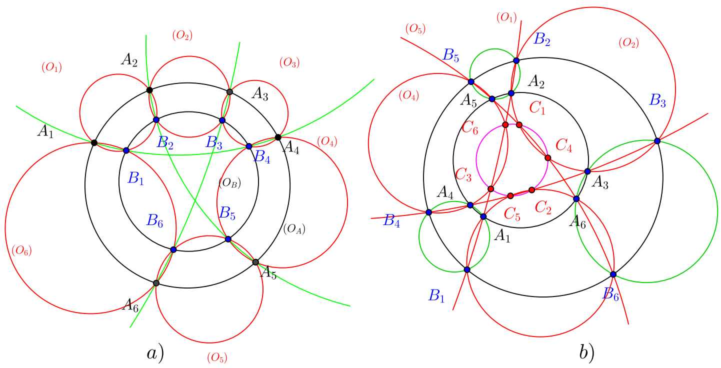

FIGURE 4. Six points C_1, C_2, C_3, C_4, C_5, C_6 lie on a circle

3. A PROOF OF THEOREM 1.2

Proof.

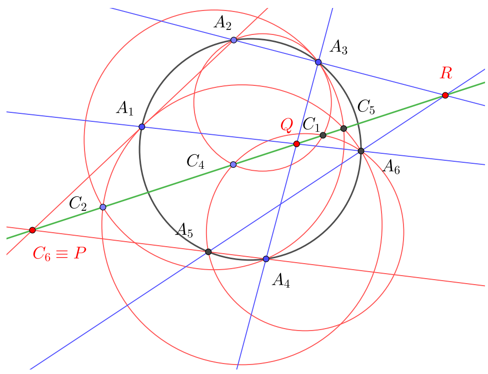

FIGURE 5.

To prove that C_6 lies on circle (C), map C_3 to ∞ by a Möbius transformation. Then it results in a configuration as in Figure 5, where line A_1A_2 (resp. A_4A_5) corresponds to circle $(A_1A_2C_3)$ (resp. $(A_4A_5C_3)$) and circle (C) becomes line C_1C_2. That C_6 lies on (C) is equivalent to the fact that the lines A_1A_2, A_4A_5 and C_1C_2 are concurrent (denote the intersection of A_1A_2 and A_4A_5 by P). To see this, note that by a special case of the bundle theorem (see Figure 6 which is obtained from Figure 3 by mapping B_3 to ∞), we have that lines A_1A_6, A_3A_4 (resp. A_2A_3, A_5A_6) and C_1C_2 are concurrent at a point Q (resp. R), where line QR is the same as C_1C_2. By Pascal's theorem for circles, it follows that the three points P, Q and R are collinear, which shows that P must lie on C_1C_2. This proves the assertion that C_6 lies on (C).

It remains to prove that $B_1, B_2, \cdots, B_6$ lie on a circle. Applying the bundle theorem for the four circles (O_2), (O_3), (O_5), (O_6) and the four concyclic points $\{C_1, C_4, C_2, C_5\}$, one sees that A_3, B_3, B_6, A_6 are concyclic. Then applying Miquel's six circles theorem for the four circles (O_4), (O_5), $(A_3B_3B_6A_6)$, (O_3) and the four concyclic points $\{A_3, A_4, A_5, A_6\}$, it follows

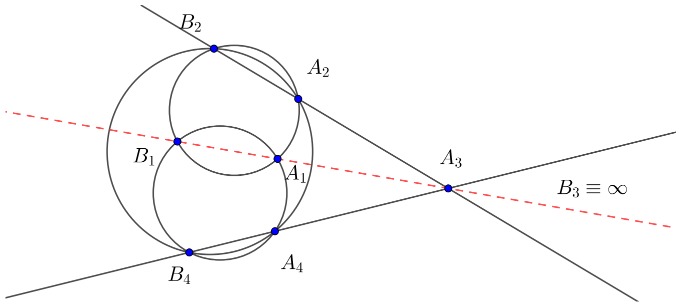

Figure 6.

that B_3, B_4, B_5, B_6 are concyclic. Repeating similar arguments by shifting the indices, it is easy to see that B_4, B_5, B_6, B_1 (resp. B_2, B_3, B_4, B_5) are concyclic. Consequently the six points $B_1, B_2, \cdots, B_6$ lie on a circle. This finishes the proof.

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