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A DIRECT TRIGONOMETRIC PROOF OF MORLEY'S THEOREM

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Abstract. We establish a direct short proof of Morley's theorem using trigonometry.

1. Introduction

Morley's theorem is one of the most beautiful theorems of plane geometry. We had seen a lot of proofs of Morley's theorem [1, 2, 3], some of these proofs used trigonometry. In this paper, we also use trigonometry to give a direct short proof for this theorem.

Theorem (Morley's theorem)**.** *In a triangle, the intersection points of the adjacent trisectors of angles make an equilateral triangle.*

2. Main proof

Given triangle ABC with $\angle A = 3\alpha$, $\angle B = 3\beta$, and $\angle C = 3\gamma$ then

$$\alpha + \beta + \gamma = 60^\circ. \tag{1}$$

Denote the intersection points of the adjacent trisectors of angles by X, Y, and Z. We shall prove that the first Morley triangle XYZ is equilateral. See Figure 1.

It is sufficient to prove only that $XY = XZ$ or since X is the incenter of triangle PBC, it is sufficient to prove that $PY = PZ$ where P is the intersection of line BZ and CY.

Let denote by $\angle PAY = x$ and $\angle PAZ = y$. Applying the law of sin for triangles PAY and PAZ, we obtain

$$\frac{PY}{PA} = \frac{\sin x}{\sin(\alpha+\gamma)}, \ \frac{PZ}{PA} = \frac{\sin y}{\sin(\alpha+\beta)}.$$

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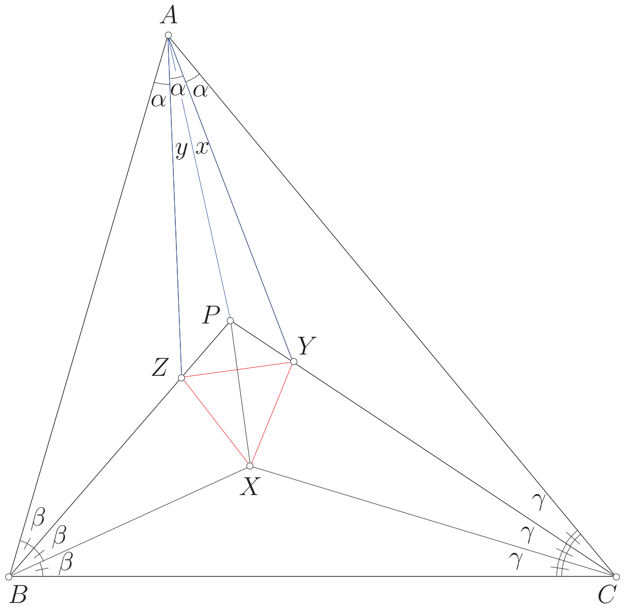

FIGURE 1. Proof of Morley's theorem

So the proof will be completed if we prove that

$$\frac{\sin x}{\sin y} = \frac{\sin(\alpha+\gamma)}{\sin(\alpha+\beta)}. \tag{2}$$

Applying the trigonometric form of Ceva's theorem [4] for triangle ABC and P, we obtain

$$\frac{\sin\angle PAC}{\sin\angle PAB} = \frac{\sin\angle PBC}{\sin\angle PBA}\cdot\frac{\sin\angle PCA}{\sin\angle PCB},$$

or

$$\frac{\sin(x+\alpha)}{\sin(y+\alpha)} = \frac{\sin 2\beta}{\sin\beta}\cdot\frac{\sin\gamma}{\sin 2\gamma} = \frac{\cos\beta}{\cos\gamma} = \frac{m}{n},$$

which means that

$$\begin{aligned}
\frac{2m\cdot\cos\alpha - n}{2n\cdot\cos\alpha - m} &= \frac{2\sin(x+\alpha)\cos\alpha - \sin(y+\alpha)}{2\sin(y+\alpha)\cos\alpha - \sin(x+\alpha)} \\
&= \frac{\sin(x+2\alpha)+\sin x - \sin(y+\alpha)}{\sin(y+2\alpha)+\sin y - \sin(x+\alpha)} \\
&= \frac{\sin x + 2\sin\left(\frac{x-y-\alpha}{2}\right)\cos\left(\frac{x+y+3\alpha}{2}\right)}{\sin y + 2\sin\left(\frac{y-x-\alpha}{2}\right)\cos\left(\frac{y+x+3\alpha}{2}\right)} \\
&= \frac{\sin x + 2\cos 2\alpha\sin x}{\sin y + 2\cos 2\alpha \sin y} \\
&= \frac{\sin x}{\sin y}. \qquad (3)
\end{aligned}$$

On the other hand, since from (1), we get $2\cos(\alpha+\beta+\gamma)=1$, therefore

$$\begin{aligned}
\frac{2m\cdot\cos\alpha-n}{2n\cdot\cos\alpha-m} &= \frac{2\cos\beta\cos\alpha-\cos\gamma}{2\cos\gamma\cos\alpha-\cos\beta}\\
&= \frac{2\cos\beta\cos\alpha-2\cos(\alpha+\beta+\gamma)\cos\gamma}{2\cos\gamma\cos\alpha-2\cos(\alpha+\beta+\gamma)\cos\beta}\\
&= \frac{\cos(\beta-\alpha)+\cos(\beta+\alpha)-\cos(\alpha+\beta)-\cos(\alpha+\beta+2\gamma)}{\cos(\gamma-\alpha)+\cos(\gamma+\alpha)-\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)}\\
&= \frac{\sin\frac{\alpha+\beta+2\gamma+\beta-\alpha}{2}\cdot\sin\frac{\alpha+\beta+2\gamma-\beta+\alpha}{2}}{\sin\frac{\alpha+2\beta+\gamma+\gamma-\alpha}{2}\cdot\sin\frac{\alpha+2\beta+\gamma-\gamma+\alpha}{2}}\\
&= \frac{\sin(\alpha+\gamma)}{\sin(\alpha+\beta)}. \qquad (4)
\end{aligned}$$

Thus from (3) and (4), we conclude (2) and that ends the proof.

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