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A Note About Isometry Groups of Chamfered Dodecahedron and Chamfered Icosahedron Spaces

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Abstract. Polyhedrons have been studied by mathematicians and geometers during many years, because of their symmetries. The theory of convex sets is a vibrant and classical field of modern mathematics with rich applications. The more geometric aspects of convex sets are developed introducing some notions, but primarily polyhedra. A polyhedra, when it is convex, is an extremely important special solid in $\mathbb{R}^n$. Some examples of convex subsets of Euclidean 3-dimensional space are Platonic Solids, Archimedean Solids and Archimedean Duals or Catalan Solids. There are some relations between metrics and polyhedra. For example, it has been shown that cube, octahedron, deltoidal icositetrahedron are maximum, taxicab, Chinese Checker's unit sphere, respectively. In this study, we introduce two new metrics, and show that the spheres of the 3-dimensional analytical space furnished by these metrics are chamfered dodecahedron and chamfered icosahedron. Also we give some properties about these metrics. We show that the group of isometries of the 3-dimesional space covered by $CD-$metric and $CI-$metric are the semi-direct product of I_h and $T(3)$, where icosahedral group I_h is the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the 3-dimensional space.

1. Introduction

A polyhedron is a solid in three dimensions with flat faces (two-dimensional), straight edges(one -dimensions) and vertices (zero-dimensional). The word polyhedron comes from the Classical Greek poly for "many" and hedron meaning "base". Polyhedra, like polygons, may be convex or non-convex. Polyhedra have very interesting symmetries. Therefore they have attracted the attention of scientists and artists from past to present. Thus mathematicians, geometers, physicists, chemists, artists have studied and continue to study on polyhedra. Consequently, polyhedra take place in many studies with respect to different fields.

Keywords and phrases: Polyhedron, Metric, Isometry group, Octahedral Symmetry, Chamfered Dodecahedron, Chamfered Icosahedron.

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A polyhedron is a three-dimensional figure made up of polygons. When discussing polyhedra one will use the terms faces, edges and vertices. Each polygonal part of the polyhedron is called a face. A line segment along which two faces come together is called an edge. A point where several edges and faces come together is called a vertex. That is, a polyhedron is a solid in three dimensions with flat faces, straight edges and vertices. In the early days of the study, the polyhedra involved to only convex polyhedra. If the line segment joining any two points in the set is also in the set, the set is called a convex set. There are many thinkers that have worked on convex polyhedra since the ancient Greeks. The Greek scientist defined two classes of convex equilateral polyhedron with polyhedral symmetry, the Platonic and the Archimedean. Johannes Kepler found a third class, the rhombic polyhedra and Eugène Catalan discovered a fourth class. The Archimedean solids and their duals the Catalan solids are less well known than the Platonic solids. Whereas the Platonic solids are composed of one shape, these forms that Archimedes wrote about are made of at least two different shapes, all forming identical vertices. They are thirteen polyhedra in this type. Since each solid has a dual there are also thirteen Catalan solids which is named after Belgian mathematician Eugène Catalan in 1865, these are made by placing a point in the middle of the faces of the Archimedean Solids and joining the points together with straight lines. The Catalan solids are all convex.

As it is stated in [17], Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not *uniform* in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set. The points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Some mathematicians studied and improved metric geometry in plane and space. (Some of these are [1, 2, 4, 6, 8, 9, 10, 11, 12, 13, 14]) According to studies on polyhedra, there are some Minkowski geometries in which unit spheres of these spaces furnished by some metrics are associated with convex solids. For example, unit spheres of maximum space and taxicab space are cubes and octahedrons, respectively, which are Platonic Solids. And unit sphere of CC-space is a deltoidal icositetrahedron which is a Catalan solid. Therefore, there are some metrics in which unit spheres of space furnished by them are convex polyhedra. That is, convex polyhedra are associated with some metrics. When a metric is given we can find its unit sphere. Naturally a question can be asked; "Is it possible to find the metric when a convex polyhedron is given?".

In this study, two new metrics are introduced, and showed that the spheres of the 3-dimensional analytical space furnished by these metrics are chamfered dodecahdron and chamfered icosahedron. Also some properties about these metrics are given. Morever, we show that the group of isometries of the 3-dimesional space covered by $CD-$metric and $CI-$metric are the semi-direct product of I_h and $T(3)$, where icosahedral group I_h are the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the 3-dimensional space.

2. Chamfered Dodecahedron Metric and Some Properties

It has been stated in [16], there are many variations on the theme of the regular polyhedra. Firstly, one can meet the eleven solids which can be made by cutting off (truncating) the corners, and in some cases the edges, of the regular polyhedra so that all the faces of the faceted polyhedra obtained in this way are regular polygons. These polyhedra were first discovered by Archimedes (287-212 B.C.E.) and so they are often called Archimedean solids. Notice that vertices of the Archimedean polyhedra are all alike, but their faces, which are regular polygons, are of two or more different kinds. For this reason they are often called semiregular. Archimedes also showed that in addition to the eleven obtained by truncation, there are two more semiregular polyhedra: the snub cube and the snub dodecahedron.

The other operation about consctructing polyhedron from any polyhedra is chamfering. In geometry, chamfering or edge-truncation is a topological operator that modifies one polyhedron into another. It is similar to expansion, moving faces apart and outward, but also maintains the original vertices. For polyhedra, this operation adds a new hexagonal face in place of each original edge.

One of the solids which is obtained by chamfering from another solid is the chamfered dodecahedron. It has 12 regular pentagonal faces, 30 bi-mirror-symmetric hexagonal faces, 80 vertices and 120 edges. The chamfered dodecahedron can be obtained by chamfering operation from dodecahedron. This is the shape of the fullerene C_{80}; sometimes this shape is denoted $C_{80}(I_h)$ to describe its icosahedral symmetry and distinguish it from other less-symmetric 80-vertex fullerenes. It is one of only four fullerenes found by Deza, Deza and Grishukhin (1998) to have a skeleton that can be isometrically embeddable into an L1 space, and also it is the Goldberg polyhedron $G_V(2,0)$, containing pentagonal and hexagonal faces [19]. Figure 1 shows the chamfered dodecahdron and the C_{80} fullerene.

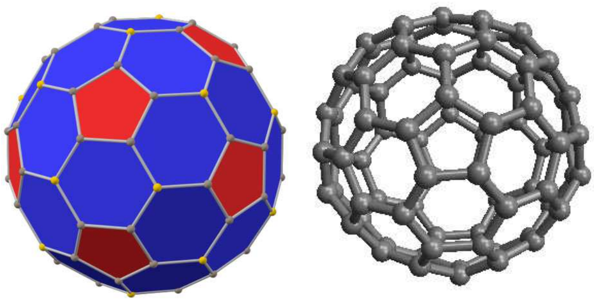

Figure 1: Chamfered Dodecahedron, C_{80} fullerene

Before we give a description of the chamfered dodecahedron distance function, we must agree on some impressions. Therefore U denote the maximum of quantities $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$. Also, $X - Y - Z - X$ orientation and $Z - Y - X - Z$ orientation are called positive $(+)$ direction and negative $(-)$ directions, respectively. Accordingly, U^+ and U^- will display the next term in the respective direction according to U. For example, if $U = |y_1 - y_2|$, then

$U^+ = |z_1 - z_2|$ and $U^- = |x_1 - x_2|$. The metric that unit sphere is chamfered dodecahedron is described as following:

Definition 2.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{CD} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ chamfered dodecahedron distance between P_1 and P_2 is defined by*

$$d_{CD}(P_1, P_2) = \max\left\{U, \beta U + \alpha U^+, \frac{\varphi}{2}U + \frac{1}{2}U^- + \frac{1}{2\varphi}U^+\right\},$$

where $\varphi = \frac{\sqrt{5}+1}{2}$ golden ratio, $\alpha = \frac{13\sqrt{5}-5}{82} + \frac{(7\sqrt{5}-9)\sqrt{25+10\sqrt{5}}}{205}$ and $\beta = \alpha\varphi$.

According to chamfered dodecahedron distance, there are three different paths from P_1 to P_2. These paths are

i) a line segment which is parallel to a coordinate axis,

ii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{1}{2}\right)$ angle with another coordinate axis,

iii) union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan(\frac{5\sqrt{5}-9}{8})$ and $\arctan\left(\frac{1}{2}\right)$ angles with one of the other coordinate axes .

Thus chamfered dodecahedron distance between P_1 and P_2 is for (*i*) Euclidean lengths of line segmentfor (*ii*) β times the sum of Euclidean lengths of mentioned two line segments, for (*iii*) φ times the sum of Euclidean lengths of mentioned three line segments. In case of $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$, Figure 2 illustrates some of chamfered dodecahedron way from P_1 to P_2.

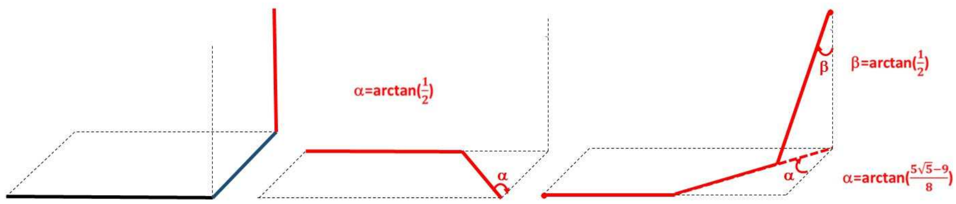

Figure 2: Some CD way from P_1 to P_2

In [5], the authour introduce a metric and show that spheres of 3-dimensional analytical space furnished by this metric is the dodecahedron. This metric for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ are defined as follows:

$$d_D(P_1, P_2) = U + (\varphi - 1)\, U^+.$$

Lemma 2.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. U_{12} denote the maximum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. Then*

$$d_{CD}(P_1, P_2) \geq U_{12}$$
$$d_{CD}(P_1, P_2) \geq \beta U_{12} + \alpha U_{12}^+,$$
$$d_{CD}(P_1, P_2) \geq \tfrac{\varphi}{2}U_{12} + \tfrac{1}{2}U_{12}^- + \tfrac{1}{2\varphi}U_{12}^+.$$

Proof. *Proof is trivial by the definition of maximum function.*

Theorem 2.1. *The distance function d_{CD} is a metric. Also according to d_{CD}, the unit sphere is a chamfered dodecahedron in $\mathbb{R}^3$.*

Proof. Let $d_{CD} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ be the chamfered dodecahedron distance function and $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ and $P_3 = (x_3, y_3, z_3)$

are distinct three points in $\mathbb{R}^3$. U_{12} denote the maximum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. To show that d_{CD} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

M1) $d_{CD}(P_1, P_2) \geq 0$ and $d_{CD}(P_1, P_2) = 0$ iff $P_1 = P_2$

M2) $d_{CD}(P_1, P_2) = d_{CD}(P_2, P_1)$

M3) $d_{CD}(P_1, P_3) \leq d_{CD}(P_1, P_2) + d_{CD}(P_2, P_3)$.

Since absolute values is always nonnegative value $d_{CD}(P_1, P_2) \geq 0$. If $d_{CD}(P_1, P_2) = 0$ then $d_{CD}(P_1, P_2) = \max\left\{U, \beta U + \alpha U^+, \frac{\varphi}{2}U + \frac{1}{2}U^- + \frac{1}{2\varphi}U^+\right\} = 0$, where U are the maximum of quantities $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. Therefore, U=0, $\beta U + \alpha U^+$=0, and $\frac{\varphi}{2}U + \frac{1}{2}U^- + \frac{1}{2\varphi}U^+ = 0$. Hence, it is clearly obtained by $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$. That is, $P_1 = P_2$. Thus it is obtained that $d_{CD}(P_1, P_2) = 0$ iff $P_1 = P_2$.

Since $|x_1 - x_2| = |x_2 - x_1|$, $|y_1 - y_2|$=$|y_2 - y_1|$ and $|z_1 - z_2| = |z_2 - z_1|$, obviously $d_{CD}(P_1, P_2) = d_{CC}(P_2, P_1)$. That is, d_{CD} is symmetric.

U_{13}, and U_{23} denote the maximum of quantities of $\{|x_1 - x_3|, |y_1 - y_3|, |z_1 - z_3|\}$ and $\{|x_2 - x_3|, |y_2 - y_3|, |z_2 - z_3|\}$, respectively.

$$\begin{aligned}
&d_{CD}(P_1, P_3) = \max\left\{U_{13}, \beta U_{13} + \alpha U_{13}^+, \tfrac{\varphi}{2}U_{13} + \tfrac{1}{2}U_{13}^- + \tfrac{1}{2\varphi}U_{13}^+\right\} \\
&\leq \max\left\{\begin{array}{c} U_{12} + U_{23}, \beta\left(U_{12} + U_{23}\right) + \alpha\left(U_{12}^+ + U_{23}^+\right), \\ \tfrac{\varphi}{2}\left(U_{12} + U_{23}\right) + \tfrac{1}{2}\left(U_{12}^- + U_{23}^-\right) + \tfrac{1}{2\varphi}\left(U_{12}^+ + U_{23}^+\right) \end{array}\right\} \\
&= I.
\end{aligned}$$

Therefore one can easily find that $I \leq d_{CD}(P_1, P_2) + d_{CD}(P_2, P_3)$ from Lemma 2.1. So $d_{CD}(P_1, P_3) \leq d_{CD}(P_1, P_2) + d_{CD}(P_2, P_3)$. Consequently, chamfered dodecahedron distance is a metric in 3-dimensional analytical space.

Finally, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that chamfered dodecahedron distance is 1 from $O = (0, 0, 0)$ is

$$S_{CD} = \left\{(x, y, z) : \max\left\{U, \beta U + \alpha U^+, \frac{\varphi}{2}U + \frac{1}{2}U^- + \frac{1}{2\varphi}U^+\right\} = 1\right\}.$$

Thus the graph of S_{CD} is as in the figure 3:

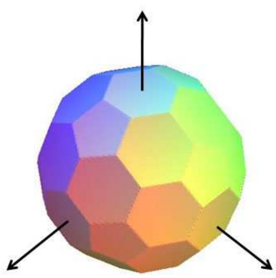

Figure 3 The unit sphere in terms of d_{CD}: Chamfered Dodecahedron

Corrolary 2.1. *The equation of the chamfered cube with center* (x_0, y_0, z_0) *and radius* r *is*

$$\max\left\{U_0, \beta U_0 + \alpha U_0^+, \frac{\varphi}{2}U_0 + \frac{1}{2}U_0^- + \frac{1}{2\varphi}U_0^+\right\} = r,$$

which is a polyhedron which has 42 faces and 80 vertices, where U_0 *are the maximum of quantities* $\{|x-x_0|,|y-y_0|,|z-z_0|\}$. *Coordinates of the vertices are translation to* (x_0,y_0,z_0) *all circular shift of the three axis components and all possible +/- sign changes of each axis component of* $(0,C_2r,r)$, (C_1r,C_7r,r), (C_3r,C_0r,C_6r), $(0,C_4r,C_5r)$ *and* (C_3r,C_3r,C_3r), *where* $C_0=\frac{2\sqrt{5}\sqrt{25+10\sqrt{5}}}{5}-\frac{5+3\sqrt{5}}{2}$, $C_1=2+2\sqrt{5}-\frac{2\sqrt{5}\sqrt{25+10\sqrt{5}}}{5}$, $C_2=\frac{3-\sqrt{5}}{2}$, $C_3=\frac{\sqrt{5}-1}{2}$, $C_4=\frac{7+3\sqrt{5}}{2}-\frac{2\sqrt{5}\sqrt{25+10\sqrt{5}}}{5}$, $C_5=\frac{(5-\sqrt{5})\sqrt{25+10\sqrt{5}}}{5}-3$, $C_6=\frac{7+\sqrt{5}}{2}+\frac{(\sqrt{5}-5)\sqrt{25+10\sqrt{5}}}{5}$, *and* $C_7=\frac{(5-\sqrt{5})\sqrt{25+10\sqrt{5}}}{5}-\frac{5+\sqrt{5}}{2}$.

Lemma 2.2. *Let* l *be the line through the points* $P_1=(x_1,y_1,z_1)$ *and* $P_2=(x_2,y_2,z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector* (p,q,r), *then*

$$d_{CD}(P_1,P_2)=\mu(P_1P_2)d_E(P_1,P_2)$$

where

$$\mu(P_1P_2)=\frac{\max\left\{U_d,\beta U_d+\alpha U_d^+,\frac{\varphi}{2}U_d+\frac{1}{2}U_d^-+\frac{1}{2\varphi}U_d^+\right\}}{\sqrt{p^2+q^2+r^2}},$$

U_d *are the maximum of quantities* $\{|p|,|q|,|r|\}$.

Proof. Equation of l gives us $x_1-x_2=\lambda p$, $y_1-y_2=\lambda q$, $z_1-z_2=\lambda r$, $\lambda\in\mathbb{R}$. Thus,

$$d_{CD}(P_1,P_2)=|\lambda|\left(\max\left\{U_d,\beta U_d+\alpha U_d^+,\frac{\varphi}{2}U_d+\frac{1}{2}U_d^-+\frac{1}{2\varphi}U_d^+\right\}\right),$$

where U_d are the maximum of quantities $\{|p|,|q|,|r|\}$, and $d_E(A,B)=|\lambda|\sqrt{p^2+q^2+r^2}$ which implies the required result.

The above lemma says that d_{CD}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corrolary 2.2. *If* P_1, P_2 *and* X *are any three collinear points in* $\mathbb{R}^3$, *then* $d_E(P_1,X)=d_E(P_2,X)$ *if and only if* $d_{CD}(P_1,X)=d_{CD}(P_2,X)$.

Corrolary 2.3. *If* P_1, P_2 *and* X *are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{CD}(X,P_1)\ /\ d_{CD}(X,P_2)=d_E(X,P_1)\ /\ d_E(X,P_2)\ .$$

That is, the ratios of the Euclidean and $d_{CD}-$*distances along a line are the same.*

3. Chamfered Icosahedron Metric and Some Properties

The chamfered icosahedron can be obtained by using chamfering operation from icosahedron. The chamfered icosahedron has 20 equilateral triangular faces, 30 bi-mirror-symmetric hexagonal faces, 72 vertices and 120 edges. Figure 4 shows the chamfered icosahedron.

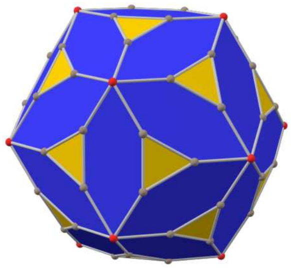

Figure 4: The chamfered icosahedron

The notations U, U^+, U^- shall be used as defined in the previous section. The metric that unit sphere is the chamfered icosahedron is described as following:

Definition 3.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{CI} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ chamfered icosahedron distance between P_1 and P_2 is defined by*

$$d_{CI}(P_1, P_2) = \max\left\{U, \varphi\alpha U + \frac{\alpha}{\varphi}U^-, \alpha\left(U + U^+ + U^-\right), \frac{\varphi}{2}U + \frac{1}{2}U^- + \frac{1}{2\varphi}U^+\right\},$$

where $\varphi = \frac{\sqrt{5}+1}{2}$ golden ratio and $\alpha = \frac{307+293\sqrt{5}}{1882} + \frac{\left(540-208\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{9410}$.

According to chamfered icosahedron distance, there are four different paths from P_1 to P_2. These paths are

$i)$ union of two line segments each of which is parallel to a coordinate axis,

$ii)$ union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{\sqrt{2}}{2}\right)$ angle with another coordinate axis,

$iii)$ union of three line segments each of which is parallel to a coordinate axis,

$iv)$ union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan(\frac{5\sqrt{5}-9}{8})$ and $\arctan\left(\frac{1}{2}\right)$ angles with one of the other coordinate axes.

Thus chamfered icosahedron distance between P_1 and P_2 is for (i) Euclidean lengths of line segment, for (ii) $\varphi\alpha$ times the sum of Euclidean lengths of mentioned two line segments, for (iii) α times the sum of Euclidean lengths of mentioned three line segments, and for (iv) φ times the sum of Euclidean lengths of mentioned three line segments. In case of $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$, Figure 5 illustrates some of chamfered dodecahedron way from P_1 to P_2.

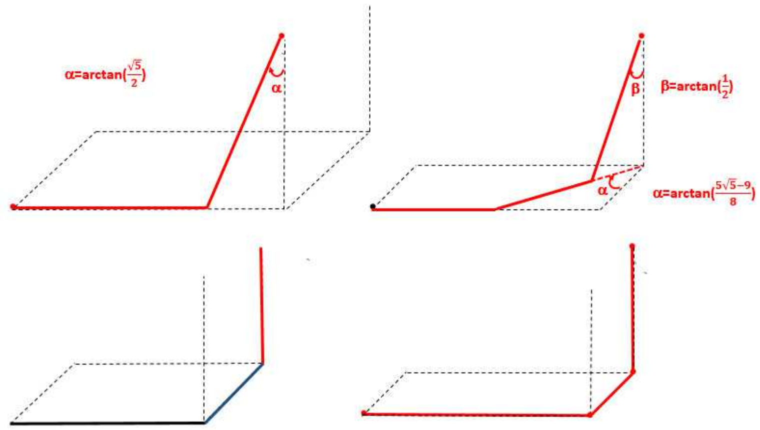

Figure 5: CI way from P_1 to P_2

In [5], the authour introduce a metric and show that spheres of 3-dimensional analytical space furnished by this metric is the icosahedron. This metric for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ are defined as follows:

$$d_I(P_1, P_2) = \max\left\{k_2\left(U + k_1 U^-\right), k_2\left(U + U^- + U^+\right)\right\}$$

where $k_1 = \dfrac{3-\sqrt{5}}{2}$, $k_2 = \dfrac{\sqrt{5}-1}{2}$.

Lemma 3.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. U_{12} denote the maximum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. Then*

$$\begin{aligned} &d_{CI}(P_1, P_2) \geq U_{12}, \\ &d_{CI}(P_1, P_2) \geq \varphi\alpha U_{12} + \tfrac{\alpha}{\varphi} U_{12}^-, \\ &d_{CI}(P_1, P_2) \geq \alpha\left(U_{12} + U_{12}^+ + U_{12}^-\right), \\ &d_{CI}(P_1, P_2) \geq \tfrac{\varphi}{2} U_{12} + \tfrac{1}{2} U_{12}^- + \tfrac{1}{2\varphi} U_{12}^+. \end{aligned}$$

Proof. Proof is trivial by the definition of maximum function.

Theorem 3.1. *The distance function d_{CI} is a metric. Also according to d_{CI}, unit sphere is a chamfered icosahedron in $\mathbb{R}^3$.*

Proof. One can easily show that the chamfered icosahedron distance function satisfies the metric axioms by similar way in Theorem 2.1.

Consequently, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that chamfered icosahedron distance is 1 from $O = (0, 0, 0)$ is

$$S_{CI} = \left\{(x, y, z) : \max\left\{U, \varphi\alpha U + \frac{\alpha}{\varphi} U^-, \alpha\left(U + U^+ + U^-\right), \frac{\varphi}{2} U + \frac{1}{2} U^- + \frac{1}{2\varphi} U^+\right\} = 1\right\},$$

where U are the maximum of quantities $\{|x|, |y|, |z|\}$. Thus the graph of S_{CI} is as in the figure 6:

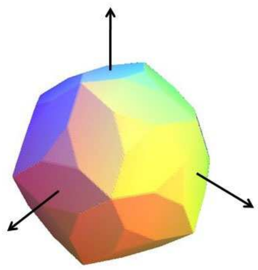

Figure 6 The unit sphere in terms of d_{CO}: Chamfered Icosahedron

Corrolary 3.1. *The equation of the chamfered icosahedron with center (x_0, y_0, z_0) and radius r is*

$$\max\left\{U_0, \varphi\alpha U_0 + \frac{\alpha}{\varphi}U_0^-, \alpha\left(U_0 + U_0^+ + U_0^-\right), \frac{\varphi}{2}U_0 + \frac{1}{2}U_0^- + \frac{1}{2\varphi}U_0^+\right\} = r,$$

which is a polyhedron which has 50 faces and 72 vertices, where U_0 are the maximum of quantities $\{|x - x_0|, |y - y_0|, |z - z_0|\}$. Coordinates of the vertices are translation to (x_0, y_0, z_0) all circular shift of the three axis components and all possible +/- sign changes of each axis component of $(C_3 r, 0, r)$, $(r, C_6 r, C_0 r)$, $(0, C_2 r, C_5 r)$, and $(C_3 r, C_1 r, C_4 r)$, where $t = \frac{(11\sqrt{5}-25)\sqrt{25+10\sqrt{5}}}{5}$, $C_0 = 14 - 6\sqrt{5} + \frac{1}{\varphi}t$, $C_1 = 4\sqrt{5} - 8 + t$, $C_2 = 9 - 4\sqrt{5} - t$, $C_3 = \frac{1}{\varphi}$, $C_4 = 6\sqrt{5} - 13 - \frac{1}{\varphi}t$, $C_5 = \frac{27-11\sqrt{5}}{2} + \frac{1}{\varphi}t$, and $C_6 = \frac{15-7\sqrt{5}}{2} - t$.

Lemma 3.2. *Let l be the line through the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p, q, r), then*

$$d_{CI}(P_1, P_2) = \mu(P_1P_2)d_E(P_1, P_2)$$

where

$$\mu(P_1P_2) = \frac{\max\left\{U_d, \varphi\alpha U_d + \frac{\alpha}{\varphi}U_d^-, \alpha\left(U_d + U_d^+ + U_d^-\right), \frac{\varphi}{2}U_d + \frac{1}{2}U_d^- + \frac{1}{2\varphi}U_d^+\right\}}{\sqrt{p^2 + q^2 + r^2}},$$

U_d are the maximum of quantities $\{|p|, |q|, |r|\}$.

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R}$. Thus,

$$d_{CI}(P_1, P_2) = |\lambda|\left(\max\left\{U_d, \varphi\alpha U_d + \frac{\alpha}{\varphi}U_d^-, \alpha\left(U_d + U_d^+ + U_d^-\right), \frac{\varphi}{2}U_d + \frac{1}{2}U_d^- + \frac{1}{2\varphi}U_d^+\right\}\right)$$

where U_d are the maximum of quantities $\{|p|, |q|, |r|\}$, and $d_E(A, B) = |\lambda|\sqrt{p^2 + q^2 + r^2}$ which implies the required result.

The above lemma says that d_{CI}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corrolary 3.2. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{CI}(P_1, X) = d_{CI}(P_2, X)$.*

Corrolary 3.3. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{CI}(X,P_1)\ /\ d_{CI}(X,P_2) = d_E(X,P_1)\ /\ d_E(X,P_2)\ .$$

That is, the ratios of the Euclidean and $d_{CI}-$distances along a line are the same.

4. Isometry Group of Chamfered Dodecahedron and Chamfered Icosahedron Spaces

Three essential methods geometric investigations; synthetic, metric and group approach. The group approach takes isometry groups of a geometry and convex sets plays an substantial role in indication of the group of isometries of geometries. Those properties are invariant under the group of motions and geometry studies those properties. There are a lot of studies about group of isometries of a space (See [9, 10, 6])

It is mentioned in introduction section that in a Minkowski geometry the linear structure is the same as the Euclidean one but distance is not uniform in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a certain symmetric closed convex set. In [15] the author give the following thereom:

Theorem 4.1. *If the unit ball C of $(V, \|\|)$ does not intersect a two-plane in an ellipse, then the group $I\,(3)$ of isometries of $(V, \|\|)$ is isomorphic to the semi-direct product of the translation group $T(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant ± 1.*

After this theorem remains a single question. This question is that what is the relevant subgroup?

Now we show that the group of isometries of the 3-dimesional space covered by $CD-$metric and $CI-$metric are the semi-direct product of I_h and $T(3)$, where icosahedral group I_h are the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the 3-dimensional space. In the rest of article we take $\triangle = CD$ or $\triangle = CI$. That is, $\triangle \in \{CD, CI\}$.

Definition 4.1. *Let P, Q be two points in $\mathbb{R}^3_\triangle$. The minimum distance set of P, Q is defined by*

$$\{X \mid d_\triangle(P,X) + d_\triangle(Q,X) = d_\triangle(P,Q)\}$$

and denoted by $[PQ]$.

In general, $[PQ]$ stand for a hexagonal dipyramid which is not necessary uniform in $\mathbb{R}^3_{CC}$ and $\mathbb{R}^3_{CO}$ as shown in Figure 7.

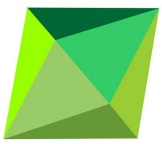

Figure 7

Proposition 4.1. *Let $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ be an isometry and let $[PQ]$ be the minimum distance set of P, Q. Then $\phi([PQ]) = [\phi(P)\phi(Q)]$.*

Proof. Let $Y \in \phi([PQ])$. Then,

$$\begin{aligned} Y \in \phi([PQ]) &\Leftrightarrow \exists X \in [PQ] \ni Y = \phi(X) \\ &\Leftrightarrow d_\triangle(P,X) + d_\triangle(Q,X) = d_\triangle(P,Q) \\ &\Leftrightarrow d_\triangle(\phi(P),\phi(X)) + d_\triangle(\phi(Q),\phi(X)) = d_\triangle(\phi(P),\phi(Q)) \\ &\Leftrightarrow Y = \phi(X) \in [\phi(P)\phi(Q)]\,. \end{aligned}$$

Corrolary 4.1. *Let $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ be an isometry and $[PQ]$ be the minimum distance set. Then ϕ maps vertices to vertices and preserves the lengths of the edges of $[PQ]$.*

Proposition 4.2. *Let $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_{\triangle T}$ be an isometry such that $\phi(O) = O$. Then $\phi \in I_h$.*

Proof. Since $\triangle \in \{CD, CI\}$, there are two possibility for $\triangle$. Let $\triangle = CD$, $C_0 = \frac{2\sqrt{5}\sqrt{25+10\sqrt{5}}}{5} - \frac{5+3\sqrt{5}}{2}$, $C_1 = 2+2\sqrt{5} - \frac{2\sqrt{5}\sqrt{25+10\sqrt{5}}}{5}$, $C_2 = \frac{3-\sqrt{5}}{2}$, $C_3 = \frac{\sqrt{5}-1}{2}$, $C_4 = \frac{7+3\sqrt{5}}{2} - \frac{2\sqrt{5}\sqrt{25+10\sqrt{5}}}{5}$, $C_5 = \frac{(5-\sqrt{5})\sqrt{25+10\sqrt{5}}}{5} - 3$, $C_6 = \frac{7+\sqrt{5}}{2} + \frac{(\sqrt{5}-5)\sqrt{25+10\sqrt{5}}}{5}$, $C_7 = 1$ ve $C_8 = \frac{(5-\sqrt{5})\sqrt{25+10\sqrt{5}}}{5} - \frac{5+\sqrt{5}}{2}$, and let $P_1 = (0, C_2, C_7)$, $P_2 = (C_1, C_8, C_7)$, $P_3 = (C_3, C_0, C_6)$, $P_4 = (C_0, C_6, C_3)$, $P_5 = (0, C_4, C_5)$, $P_6 = (C_3, C_3, C_3)$ and $R = \left(\frac{\sqrt{5}-1}{2}, 1, \frac{\sqrt{5}+1}{2}\right)$ be seven points in $\mathbb{R}^3_{CD}$. Consider $[OR]$ which is the hexagonal dipyramid (Figure 8(a)).

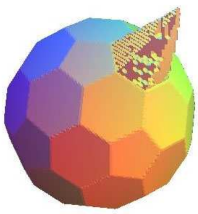
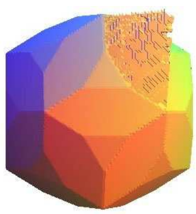

Figure 8(a) Figure 8(b)

Also points P_1, P_2, P_3, P_4, P_5, P_6 lie on minimum distance set $[OR]$ and unit sphere with center at origin. Moreover these six points are the corner points of a chamfered dodecahedron's hexagonal face. ϕ maps points P_i to the vertices of a chamfered dodecahedron by Corollary 4.1. Since ϕ preserve the lengths of the edges and chamfered dodecahedron have 30 hexagonal faces, there are 30 possibility to points which they can map, and also there are four possibility to points which they can map on the hexagonal face of chamfered dodecahedron. Therefore total number of possibility are one hundred and twenty. If these possibilities are handled one by one, it is seen that the elements of the desired subgroup are obtained.

Let $\triangle{=}CI$, $t = \frac{(11\sqrt{5}-25)\sqrt{25+10\sqrt{5}}}{5}$, $C_0 = 14 - 6\sqrt{5} + \frac{1}{\varphi}t$, $C_1 = 4\sqrt{5} - 8 + t$, $C_2 = 9 - 4\sqrt{5} - t$, $C_3 = \frac{1}{\varphi}$, $C_4 = 6\sqrt{5} - 13 - \frac{1}{\varphi}t$, $C_5 = \frac{27-11\sqrt{5}}{2} + \frac{1}{\varphi}t$, $C_6 = 1$, $C_7 = \frac{15-7\sqrt{5}}{2} - t$ and let $P_1 = (C_3, 0, C_6)$, $P_2 = (C_6, C_3, 0)$, $P_3 = (C_6, C_7, C_0)$, $P_4 = (C_5, 0, C_2)$, $P_5 = (C_3, C_1, C_4)$, $P_6 = (C_4, C_3, C_1)\,\mathbb{R}^3_{CI}$. Consider $[OR]$ such that $R = \left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}-1}{2}, 1\right)$. that is the hexagonal dipyramid with diagonal OR. (Figure 8(b)) Also points P_i lie on minimum distance set $[OR]$ and unit sphere with center at origin. Moreover these six points are the corner points of a chamfered icosahedron's hexagonal face. ϕ maps points P_i to the vertices of a chamfered icosahedron by Corollary 4.1. Since ϕ preserve the lengths of the edges, and chamfered icosahedron have 30 hexagonal faces, there are 30 possibility to points which they can map, and also there are four possibility to points which they can map on the octagonal face of chamfered icosahedron. Therefore total number of possibility are one hundred and twenty. Similar way, If these possibilities are handled one by one, it is seen that the elements of the desired subgroup are obtained.

Theorem 4.2. *Let* $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ *be an isometry. Then there exists a unique* $T_A \in T(3)$ *and* $\psi \in I_h$ *where* $\phi = T_A \circ \psi$

Proof. Let $\phi(O) = A$ such that $A = (a_1, a_2, a_3)$. Define $\psi = T_{-A} \circ \phi$. We know that $\psi(O) = O$ and ψ is an isometry. Thereby, $\psi \in I_h$ and $\phi = T_A \circ \psi$ by Proposition 4.2. The proof of uniqueness is trivial.

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