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MORE CHARACTERIZATIONS OF CYCLIC QUADRILATERALS

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Abstract. We continue the project of collecting a large number of characterizations of convex cyclic quadrilaterals with their proofs, which we started in [18]. This time we prove 15 more, focusing primarily on characterizations concerning trigonometry and the diagonals.

1. Introduction

In a convex quadrilateral $ABCD$, let the extensions of opposite sides AB and CD intersect at E. Suppose the angle bisector to angle AED intersects BC at G and AD at H in such a way that

$$AH \cdot BG = CG \cdot DH. \tag{1}$$

What can we conclude about quadrilateral $ABCD$?

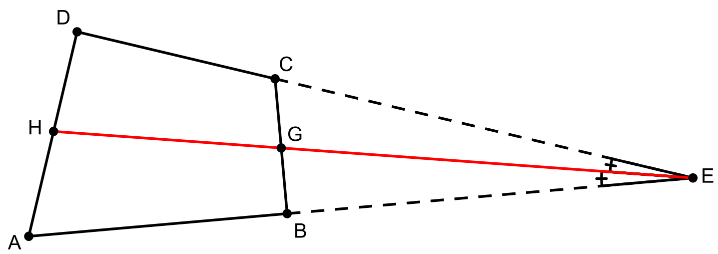

Figure 1. A quadrilateral in which $AH \cdot BG = CG \cdot DH$

Applying the angle bisector theorem in triangles ECB and EDA (see Figure 1), we get

$$\frac{CE}{BE} = \frac{CG}{BG}, \qquad \frac{DE}{AE} = \frac{DH}{AH};$$

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whence

$$\frac{CE}{BE}\cdot\frac{DE}{AE}=\frac{CG}{BG}\cdot\frac{DH}{AH}. \tag{2}$$

The right hand side is equal to 1 due to the assumption (1), implying that

$$AE\cdot BE = DE\cdot CE.$$

We recognize this equality as the external case of the intersecting chords theorem. According to its converse, it holds that $ABCD$ must be a cyclic quadrilateral. In fact we see in (2) that

$$AH\cdot BG = CG\cdot DH \quad \Leftrightarrow \quad AE\cdot BE = DE\cdot CE,$$

and since the intersecting chords theorem is a characterization of cyclic quadrilaterals (see Theorem A.5 in [18]), then so is equality (1). Thus we have proved:

Theorem 1.1. *If the extensions of opposite sides AB and CD intersect at E in a convex quadrilateral $ABCD$, and the bisector to angle AED intersects BC at G and AD at H, then*

$$AH\cdot BG = CG\cdot DH$$

if and only if it is a cyclic quadrilateral.

In this paper we shall prove 14 more characterizations of cyclic quadrilaterals. Almost all of the theorems we prove are known necessary conditions (properties) of cyclic quadrilaterals, but the fact that they are also sufficient conditions can hardly be considered well-known for the majority of them.

2. Trigonometric characterizations

Let us denote $\alpha=\angle BAC$, $\beta=\angle ABD$, $\gamma=\angle ACD$ and $\delta=\angle BDC$ (see Figure 2) in a quadrilateral $ABCD$. Then we have:

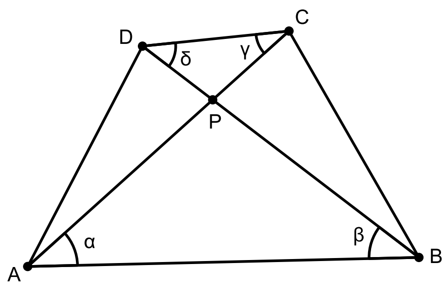

Figure 2. Angles between two opposite sides and the diagonals

Theorem 2.1. *In a convex quadrilateral $ABCD$, the equality*

$$\sin\alpha\sin\gamma=\sin\beta\sin\delta$$

holds if and only if it is a cyclic quadrilateral.

Proof. If the diagonals intersect at P, then applying the law of sines (see Figure 2) yields

$$\frac{AP}{\sin\beta} = \frac{BP}{\sin\alpha}, \qquad \frac{CP}{\sin\delta} = \frac{DP}{\sin\gamma}.$$

Thus

$$\frac{AP}{\sin\beta}\cdot\frac{CP}{\sin\delta} = \frac{BP}{\sin\alpha}\cdot\frac{DP}{\sin\gamma}.$$

Hence we have that

$$AP\cdot CP = BP\cdot DP \quad\Leftrightarrow\quad \sin\alpha\sin\gamma = \sin\beta\sin\delta$$

and since the left hand equality (the internal case of the intersecting chords theorem) is a characterization of cyclic quadrilaterals, then so is the right hand equality. □

We also give another, direct proof of the converse. (The direct theorem is trivial, since $\alpha = \delta$ and $\beta = \gamma$ in a cyclic quadrilateral.) Let $\theta = \angle BPC$. Then $\beta = \theta - \alpha$ and $\gamma = \theta - \delta$ (see Figure 2), so $\sin\alpha\sin\gamma = \sin\beta\sin\delta$ implies

$$\sin\alpha\sin(\theta-\delta) = \sin(\theta-\alpha)\sin\delta.$$

Applying a subtraction formula, we get after simplification

$$\sin\alpha\cos\delta = \cos\alpha\sin\delta$$

which is equivalent to

$$\sin(\alpha-\delta) = 0.$$

This equation only has one valid solution, $\alpha = \delta$ in a convex quadrilateral, proving that it is a cyclic quadrilateral (Theorem A.1 in [18]).

Next we have a trigonometric version of the famous supplementary angles characterization $\angle A + \angle C = \pi = \angle B + \angle D$ (Theorem A.3 in [18]). It is for instance stated in [27] and will be used in two subsequent proofs.

Theorem 2.2. *In a convex quadrilateral $ABCD$, the equalities*

$$\cos A + \cos C = \cos B + \cos D = 0$$

are true if and only if it is a cyclic quadrilateral.

Proof. ($\Rightarrow$) If the quadrilateral is cyclic, then $\angle A + \angle C = \pi$. Hence

$$\cos A + \cos C = \cos A + \cos(\pi - A) = \cos A - \cos A = 0.$$

The second equality is proved in the same way.

($\Leftarrow$) We use the method of the contrapositive statement to prove the converse. Assume the quadrilateral is not cyclic and without loss of generality that $\angle A > \pi - \angle C$. Since $0 < \angle A < \pi$ and the cosine function is decreasing on that interval, we get $\cos A < \cos(\pi - C)$. Hence

$$\cos A + \cos C < \cos(\pi - C) + \cos C = 0.$$

In the same way

$$\angle A > \pi - \angle C \quad\Rightarrow\quad \angle B < \pi - \angle D \quad\Rightarrow\quad \cos B + \cos D > 0.$$

Thus, if the quadrilateral is not cyclic, then $\cos A + \cos C \neq \cos B + \cos D$ and neither side is equal to zero. □

A direct proof of the converse can be based on the trigonometric formula

$$\cos A + \cos C = 2\cos\frac{A+C}{2}\cos\frac{A-C}{2}.$$

We leave the details to be carried out by the reader.

The following characterization was proved in the same way in [9, p. 104] (but with vertex angles $2A$, $2B$, $2C$ and $2D$).

Theorem 2.3. *In a convex quadrilateral $ABCD$, the equalities*

$$\tan\frac{A}{2}\tan\frac{C}{2} = \tan\frac{B}{2}\tan\frac{D}{2} = 1$$

are true if and only if it is a cyclic quadrilateral.

Proof. ($\Rightarrow$) In a cyclic quadrilateral, $\angle A + \angle C = \angle B + \angle D = \pi$. Using these, the equalities in the theorem directly follow since $\tan\frac{C}{2} = \cot\frac{A}{2}$ and $\tan\frac{D}{2} = \cot\frac{B}{2}$.

($\Leftarrow$) Assume the quadrilateral is not cyclic and without loss of generality that $\angle A + \angle C > \pi$ and $\angle B + \angle D < \pi$. From the addition formula for tangent, we get

$$0 > \tan\left(\frac{A}{2} + \frac{C}{2}\right) = \frac{\tan\frac{A}{2} + \tan\frac{C}{2}}{1 - \tan\frac{A}{2}\tan\frac{C}{2}}.$$

The angles $\frac{A}{2}$ and $\frac{C}{2}$ are acute, so the numerator is positive. Then the denominator must be negative, so $\tan\frac{A}{2}\tan\frac{C}{2} > 1$. In the same way $\tan\frac{B}{2}\tan\frac{D}{2} < 1$. Hence

$$\tan\frac{A}{2}\tan\frac{C}{2} \neq \tan\frac{B}{2}\tan\frac{D}{2}$$

and neither side is equal to 1. □

The half angle formulas for cosine are also trigonometric characterizations of cyclic quadrilaterals. We use the notations $a = AB$, $b = BC$, $c = CD$ and $d = DA$ for the lengths of the sides of quadrilateral $ABCD$, and

$$s = \tfrac{1}{2}(a + b + c + d)$$

for the semiperimeter. The inequalities derived in the second half of the proof will be used in the proof of another converse later on.

Theorem 2.4. *In a convex quadrilateral with consecutive sides a, b, c and d, the half angle formulas for cosine are given by*

$$\cos\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{ad+bc}},$$

$$\cos\frac{B}{2} = \sqrt{\frac{(s-c)(s-d)}{ab+cd}},$$

$$\cos\frac{C}{2} = \sqrt{\frac{(s-d)(s-a)}{ad+bc}},$$

$$\cos\frac{D}{2} = \sqrt{\frac{(s-a)(s-b)}{ab+cd}}$$

if and only if it is a cyclic quadrilateral, where s is the semiperimeter.

Proof. (⇒) First we derive the third formula in a cyclic quadrilateral. Applying the law of cosines in triangles BCD and ABD yields

$$b^2 + c^2 - 2bc\cos C = a^2 + d^2 - 2ad\cos A$$

which implies

$$b^2 + c^2 - a^2 - d^2 = 2(ad + bc)\cos C$$

since $\cos A = \cos(\pi - C) = -\cos C$ in a cyclic quadrilateral. Using a trigonometric half angle formula, we get

$$\begin{aligned}\cos^2\left(\frac{C}{2}\right) &= \frac{1}{2}\left(1 + \frac{b^2 + c^2 - a^2 - d^2}{2(ad + bc)}\right)\\ &= \frac{(b+c)^2 - (a-d)^2}{4(ad+bc)}\\ &= \frac{(b+c+a-d)(b+c-a+d)}{4(ad+bc)}\\ &= \frac{(s-d)(s-a)}{ad+bc}\end{aligned}$$

and the third formula follows. The other proofs are similar, or more easily, these formulas follow from a symmetry argument.

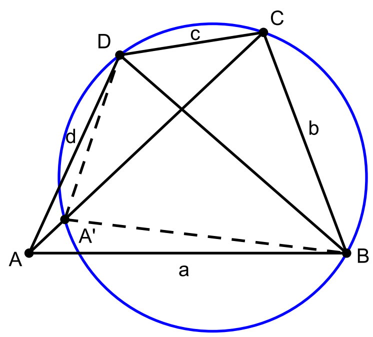

FIGURE 3. A non-cyclic quadrilateral $ABCD$

(⇐) If $ABCD$ is not cyclic, suppose vertex A lies outside of the circumcircle to triangle BCD. Let A' be the point where diagonal AC intersects this circumcircle. Then $\angle BA'D \equiv \angle A' > \angle A \equiv \angle BAD$ (see Figure 3), implying $\cos A' < \cos A$ since cosine is decreasing on the interval $[0, \pi]$. Thus $\cos A' = -\cos C$ since $A'BCD$ is cyclic, so $\cos A > -\cos C$. We have by the law of cosines

$$b^2 + c^2 - 2bc\cos C = a^2 + d^2 - 2ad\cos A < a^2 + d^2 + 2ad\cos C;$$

whence

$$\cos C > \frac{b^2 + c^2 - a^2 - d^2}{2(ad + bc)}.$$

Doing the same factorization as in the direct part of the proof, we get

$$\cos\frac{C}{2} > \sqrt{\frac{(s-a)(s-d)}{ad+bc}}.$$

For angle A, it holds that $\cos C > -\cos A$, so

$$a^2 + d^2 - 2ad\cos A < b^2 + c^2 + 2bc\cos A.$$

Thus

$$a^2 + d^2 - b^2 - c^2 < 2(ad + bc)\cos A$$

and we get

$$\cos\frac{A}{2} > \sqrt{\frac{(s-b)(s-c)}{ad+bc}}.$$

For the other two angles, we have $\angle B > \angle B' \equiv \angle CBA'$ and $\angle D > \angle D' \equiv \angle CDA'$ (see Figure 3). Then $\angle B + \angle D > \angle B' + \angle D' = \pi$. Thus $\cos D < \cos(\pi - B) = -\cos B$. We get

$$a^2 + b^2 - 2ab\cos B > c^2 + d^2 + 2cd\cos B$$

from which

$$a^2 + b^2 - c^2 - d^2 > 2(ab + cd)\cos B.$$

Hence

$$\cos B < \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} \quad \Rightarrow \quad \cos\frac{B}{2} < \sqrt{\frac{(s-c)(s-d)}{ab+cd}}.$$

In the same way we also have

$$\cos\frac{D}{2} < \sqrt{\frac{(s-a)(s-b)}{ab+cd}}.$$

When vertex A instead is inside the circumcircle to BCD, then all inequalities are reversed, completing the proof. □

3. Characterizations concerning the diagonals

In this section we will use the notations p and q for the lengths of the diagonals AC and BD respectively in a convex quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$ and $d = DA$.

A well-known necessary condition of cyclic quadrilaterals is *Ptolemy's theorem*, named after the Roman astronomer Claudius Ptolemy in the second century. That it is also a sufficient condition was proved by the Swiss mathematician Leonhard Euler in the eighteenth century. It is possible to find many different proofs of the direct part of this theorem in the mathematical literature, employing for instance similarity [24, pp. 11–12], area methods [6], trigonometry [21], complex numbers [24, pp. 12–13], vectors [19], and inversion [4, pp. 103–104]. Ptolemy's theorem is quite often stated as a characterization, but far from all such sources actually prove the converse. The most common way to prove the converse is with similarity, either using an internal or an external point to form similar triangles, as in [2, pp. 128–129]. Our proof is of this latter type. Other proofs of the converse use transformations (spiral similarity [3, pp. 40–41] or inversion [4, pp. 103–104]), but the key step is applying the triangle inequality.

Theorem 3.1 (Ptolemy)**.** *In a convex quadrilateral with consecutive sides a, b, c and d, the product of the diagonals p and q satisfies*

$$pq = ac + bd$$

if and only if it is a cyclic quadrilateral.

Proof. ($\Rightarrow$) In a cyclic quadrilateral $ABCD$, extend side AB and choose a point Q on this extension such that $\angle QDA = \angle BDC$ (see Figure 4). Then triangles QAD and BCD are similar since we also have $\angle QAD = \angle BCD$ (Theorem A.4 in [18]). Thus

$$\frac{BC}{QA} = \frac{DC}{DA} \quad \Rightarrow \quad BC \cdot DA = QA \cdot DC.$$

Triangles CDA and BDQ are also similar, so

$$\frac{CD}{BD} = \frac{AC}{QB} \quad \Rightarrow \quad AC \cdot BD = QB \cdot CD.$$

Hence

$$AB \cdot CD + BC \cdot DA = AB \cdot CD + QA \cdot CD = QB \cdot CD = AC \cdot BD.$$

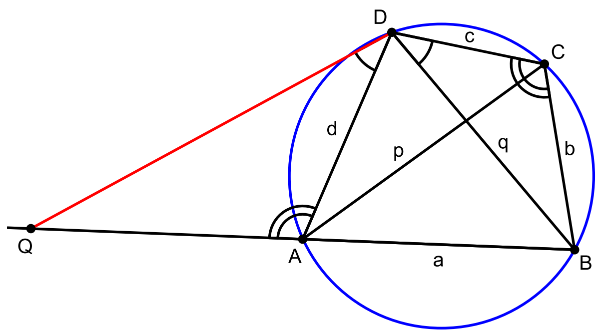

FIGURE 4. Point Q is chosen such that $\angle QDA = \angle BDC$

($\Leftarrow$) Outside of a convex quadrilateral where

$$AB \cdot CD + BC \cdot DA = AC \cdot BD, \tag{3}$$

we choose a point S such that $\angle SDA = \angle BDC$ and $\angle ASD = \angle CBD$. Note that we cannot yet be sure that SAB is a straight line segment. Triangles SAD and BCD are similar (see Figure 5), so

$$\frac{DC}{DA} = \frac{BC}{SA} = \frac{BD}{SD} \quad \Rightarrow \quad \frac{DA}{SD} = \frac{DC}{BD}. \tag{4}$$

Together with $\angle ADC = \angle SDB$, this implies that triangles ADC and SDB are also similar (SAS); whence

$$\frac{AD}{SD} = \frac{CD}{BD} = \frac{AC}{SB}. \tag{5}$$

From (4) and (5) we have $DA \cdot BC = CD \cdot SA$ and $AC \cdot BD = CD \cdot SB$. Inserting these into (3), we get

$$AB \cdot CD + CD \cdot SA = CD \cdot SB.$$

Thus $AB + SA = SB$ which proves that A is on SB according to the degenerate case of the triangle inequality (so SAB is a straight line segment). Then

$$\angle DAC = \angle DSB = \angle DSA = \angle DBC$$

which proves that $ABCD$ is a cyclic quadrilateral (Theorem A.1 in [18]). □

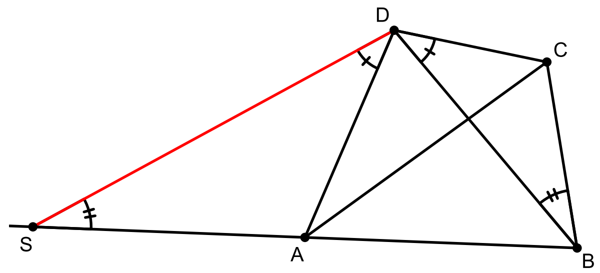

FIGURE 5. Point S is chosen so we get similar triangles

From the proof of the converse we also have that in all non-cyclic convex quadrilaterals, it holds

$$pq < ac + bd$$

due to the triangle inequality. This is usually called Ptolemy's inequality, but as far as we know, it was proved for the first time by Euler.

We note that an appealing way of writing Ptolemy's theorem is

$$AC \cdot BD = ac + bd.$$

A simple proof of both directions is by using a generalization of the theorem, which can be found in [22]. It was however first published in 1842 by the German mathematician Carl Anton Bretschneider [5]. This generalization of Ptolemy's theorem states that in a convex quadrilateral $ABCD$:

$$p^2q^2 = a^2c^2 + b^2d^2 - 2abcd\cos(A + C) \tag{6}$$

where a, b, c and d are the consecutive sides of the quadrilateral. We know that a convex quadrilateral is cyclic if and only if $\angle A + \angle C = \pi$. Inserting this into (6) and simplifying, we get

$$(pq)^2 = (ac + bd)^2$$

from which Ptolemy's theorem and its converse follow directly.

Another thing worth mentioning is that the equality in Ptolemy's theorem also holds true if the four points A, B, C and D are collinear, which was noted by Euler [11, p. 3]. This is one of the reasons why a straight line can be considered a special case of a circle with infinite radius.

There is a trigonometric version of Ptolemy's theorem that is also a characterization of cyclic quadrilaterals. This is not so well known, and we have only found it stated (with a misprint where the angles on the left hand side were interchanged) as a lemma in a collection of geometrical formulas [7, p. 6], without a proof.

Theorem 3.2. *In a convex quadrilateral $ABCD$, the relation*

$$AB \cdot \sin \angle CAD + AD \cdot \sin \angle CAB = AC \cdot \sin \angle BAD$$

holds if and only if it is a cyclic quadrilateral.

Proof. ($\Rightarrow$) In a cyclic quadrilateral $ABCD$ with circumradius R, we have $CD = 2R \sin \angle CAD$, $BC = 2R \sin \angle CAB$ and $BD = 2R \sin \angle BAD$ according to the extended law of sines. Inserting these into Ptolemy's theorem

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

directly yields the relation in the theorem after canceling the common factor $2R$.

($\Leftarrow$) For the converse, consider the circumcircle to triangle ABD. As we vary point C along the diagonal AC, the left hand side of the relation in the theorem stays constant, while the right hand side increases as C moves outside the circumcircle and decreases when C moves inside the circumcircle since this changes AC but angle BAD is constant (see Figure 6). Thus for equality to hold, point C must be on the circumcircle to triangle ABD, making $ABCD$ a cyclic quadrilateral. □

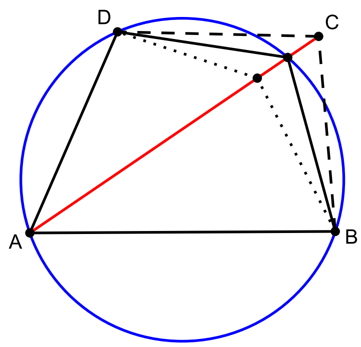

FIGURE 6. Point C moving along diagonal AC

How about if we substitute the remaining three lengths in this trigonometric version for sine expressions using the extended law of sines? Then we get a sine identity that is valid in cyclic quadrilaterals, but it is not a sufficient condition for a convex quadrilateral to be cyclic. This can be verified in a dynamic geometry computer program.

Applying Ptolemy's theorem, we get a necessary and sufficient condition on the angle between the diagonals for when a quadrilateral is cyclic.

Theorem 3.3. *In a convex quadrilateral with consecutive sides a, b, c and d, the acute angle between the diagonals satisfies*

$$\cos \theta = \frac{\left|a^2 - b^2 + c^2 - d^2\right|}{2(ac + bd)}$$

if and only if it is a cyclic quadrilateral.

Proof. Let e, f, g, h be the diagonal parts. Applying the law of cosines in the four subtriangles created by the diagonals (see Figure 7), we have

$$\begin{aligned} a^2 &= e^2 + f^2 - 2ef\cos(\pi - \theta), \\ b^2 &= f^2 + g^2 - 2fg\cos\theta, \\ c^2 &= g^2 + h^2 - 2gh\cos(\pi - \theta), \\ d^2 &= h^2 + e^2 - 2he\cos\theta. \end{aligned}$$

From these we get

$$a^2 - b^2 + c^2 - d^2 = 2(ef + fg + gh + he)\cos\theta = 2(e+g)(f+h)\cos\theta = 2pq\cos\theta$$

where p and q are the diagonal lengths. But we do not know which of the two angles between the diagonals that is the acute one, so in a convex quadrilateral the acute angle satisfies

$$\left|a^2 - b^2 + c^2 - d^2\right| = 2pq\cos\theta. \tag{7}$$

The quadrilateral is cyclic if and only if $pq = ac + bd$. Inserting this expression and solving for the cosine, we get the stated formula. □

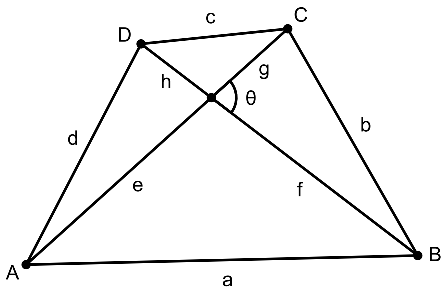

FIGURE 7. The diagonal parts

The next theorem concerns the lengths of the diagonals in terms of the four sides. This characterization was proved the same way in [27].

Theorem 3.4. *In a convex quadrilateral $ABCD$ with consecutive sides a, b, c and d, the lengths of the diagonals AC and BD are respectively*

$$p = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}},$$
$$q = \sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}}$$

if and only if $ABCD$ is a cyclic quadrilateral.

Proof. We begin by deriving a formula for the length of diagonal $BD = q$ in a convex quadrilateral. Applying the law of cosines in triangles ABD and BCD, we have

$$\begin{aligned} q^2 &= a^2 + d^2 - 2ad\cos A, \\ q^2 &= b^2 + c^2 - 2bc\cos C. \end{aligned}$$

Now we multiply the first of these equations by bc and the second by ad, so

$$bcq^2 = a^2bc + bcd^2 - 2abcd\cos A,$$
$$adq^2 = adb^2 + adc^2 - 2abcd\cos C.$$

Adding these two yields

$$(bc + ad)q^2 = ac(ab + cd) + bd(cd + ab) - 2abcd(\cos A + \cos C).$$

Hence

$$q^2 = \frac{(ab + cd)(ac + bd) - 2abcd(\cos A + \cos C)}{(ad + bc)}. \tag{8}$$

According to Theorem 2.2, the quadrilateral is cyclic if and only if $\cos A + \cos C = 0$, so it is cyclic if and only if

$$q^2 = \frac{(ab + cd)(ac + bd)}{ad + bc}. \tag{9}$$

In the same way we can derive a formula for the length of the diagonal AC in a convex quadrilateral. Then we get

$$p^2 = \frac{(ac + bd)(ad + bc) - 2abcd(\cos B + \cos D)}{(ab + cd)}. \tag{10}$$

Again using Theorem 2.2, the quadrilateral is cyclic if and only if $\cos B + \cos D = 0$, so it is cyclic if and only if

$$p^2 = \frac{(ac + bd)(ad + bc)}{ab + cd} \tag{11}$$

completing the proof. □

The direct part of the next theorem has been called Ptolemy's second theorem. We have however been unable to find any reference that it was actually known to Ptolemy. Instead we are quite sure that this metric relation first appeared in the book *Brāhmasphuṭasiddhānta* by Brahmagupta [29, pp. 198–199], a seventh century Indian mathematician. He was also the first to derive the formulas in Theorem 3.4. The converse in the following theorem is much newer. It is possible that it first appeared in 2003 [27]. However, the implication

$$\angle A + \angle C > \pi \quad \Rightarrow \quad \frac{p}{q} < \frac{ad + bc}{ab + cd},$$

which is half of the converse, was a problem by A. N. Danilov in 1969 according to [25, p. 401].

Theorem 3.5. *In a convex quadrilateral with consecutive sides a, b, c and d, the quotient of the the diagonals p and q satisfies*

$$\frac{p}{q} = \frac{ad + bc}{ab + cd}$$

if and only if it is a cyclic quadrilateral.

Proof. (⇒) We cite a short and clever proof of the necessary condition from [1]. If the quadrilateral $ABCD$ is cyclic, all the triangles ABD, BCA, CDB and DAC have the same circumcircle, with circumradius R. Using the well-known formula $T = \frac{abc}{4R}$ for the area of a triangle with sides a, b, c and circumradius R, we get

$$\frac{adq}{4R} + \frac{bcq}{4R} = \frac{abp}{4R} + \frac{cdp}{4R}$$

since each side is equal to the area of the quadrilateral. Hence

$$q(ad + bc) = p(ab + cd)$$

and the direct theorem follows. (Note that it also follows at once by simplifying the quotient of (11) and (9).)

(⇐) The idea we use for the proof of the converse comes from [23]. If the quadrilateral is not cyclic, assume first that $\angle A + \angle C > \pi$. Then $\angle B + \angle D < \pi$ and by the proof of Theorem 2.2 we have $\cos A + \cos C < 0$ and $\cos B + \cos D > 0$. From (10) and (8) we get

$$p^2 < \frac{(ac + bd)(ad + bc)}{ab + cd}$$

and

$$q^2 > \frac{(ab + cd)(ac + bd)}{ad + bc}.$$

Dividing these yields

$$\frac{p^2}{q^2} < \frac{(ac + bd)(ad + bc)}{ab + cd} \cdot \frac{ad + bc}{(ab + cd)(ac + bd)} = \frac{(ad + bc)^2}{(ab + cd)^2}.$$

Hence

$$\frac{p}{q} < \frac{ad + bc}{ab + cd}. \qquad (12)$$

If $\angle A + \angle C < \pi$, all inequalities are reversed, so we get

$$\frac{p}{q} > \frac{ad + bc}{ab + cd} \qquad (13)$$

completing the proof. □

Another characterization involving the diagonals and the sides is the following. The direct part of the theorem was discussed at [20].

Theorem 3.6. *The lengths of the diagonals p and q and the consecutive sides a, b, c and d of a convex quadrilateral satisfy*

$$\frac{|p - q|}{p + q} = \frac{|a - c|}{a + c} \cdot \frac{|b - d|}{b + d}$$

if and only if it is a cyclic quadrilateral.

Proof. Since the proof of the direct and the converse theorem are very similar, we only give the proof of the converse. A proof of the direct theorem is obtained simply by changing all inequalities to equalities.

If the quadrilateral is not cyclic, then

$$\frac{p}{q} < \frac{ad + bc}{ab + cd} \qquad \text{or} \qquad \frac{p}{q} > \frac{ad + bc}{ab + cd}$$

depending on if $\angle A + \angle C > \pi$ or $\angle A + \angle C < \pi$ respectively according to the proof of Theorem 3.5. The inequality in the first case is equivalent to

$$\frac{q}{p} > \frac{ab+cd}{ad+bc}.$$

Thus we get

$$\frac{|p-q|}{p+q} = \frac{\left|1-\frac{q}{p}\right|}{1+\frac{q}{p}} < \frac{\left|1-\frac{ab+cd}{ad+bc}\right|}{1+\frac{ab+cd}{ad+bc}} = \frac{|ad+bc-(ab+cd)|}{ad+bc+ab+cd} = \frac{|(a-c)(d-b)|}{(a+c)(d+b)}.$$

This proves that if $\angle A + \angle C > \pi$, then

$$\frac{|p-q|}{p+q} < \frac{|a-c||d-b|}{(a+c)(d+b)}.$$

In the same way we can prove that the second case $\angle A + \angle C < \pi$ implies

$$\frac{|p-q|}{p+q} > \frac{|a-c||d-b|}{(a+c)(d+b)}$$

which completes the proof. □

Let us now discuss the signs of the expressions $a-c$, $b-d$ and $p-q$ in the numerators in the previous theorem. From Theorem 3.4 we have that in a cyclic quadrilateral, $p^2 > q^2$ is equivalent to

$$\frac{(ac+bd)(ad+bc)}{ab+cd} - \frac{(ab+cd)(ac+bd)}{ad+bc} > 0.$$

Factoring this expression yields

$$-\frac{(a-c)(a+c)(b-d)(b+d)(ac+bd)}{(ab+cd)(ad+bc)} > 0.$$

Since $p > 0$ and $q > 0$, we get that

$$p-q > 0 \quad \Leftrightarrow \quad -(a-c)(b-d) > 0 \quad \Leftrightarrow \quad (a-c)(b-d) < 0.$$

This means that exactly one of the expressions $a-c$ and $b-d$ is negative.

In the same way we have

$$p-q < 0 \quad \Leftrightarrow \quad -(a-c)(b-d) < 0 \quad \Leftrightarrow \quad (a-c)(b-d) > 0.$$

Thus both of $a-c$ and $b-d$ are either positive or negative. In conclusion this proves that *in a cyclic quadrilateral, either one or all three of the expressions $a-c$, $b-d$ and $p-q$ are negative.*

Taking a closer look at Theorems 3.5 and 3.6 we might suspect there is another (shorter) way of deriving one of them from the other. This is true. A direct calculation confirms that

$$\begin{aligned}
&(p+q)(a-c)(b-d) - (q-p)(a+c)(b+d)\\
&= abp - bcp - adp + cdp + abq - bcq - adq + cdq\\
&\quad - (-abp - bcp - adp - cdp + abq + bcq + adq + cdq)\\
&= 2[p(ab+cd) - q(ad+bc)].
\end{aligned}$$

The last expression is equal to zero if and only if the first expression is equal to zero. Hence

$$\frac{p}{q} = \frac{ad+bc}{ab+cd} \quad \Leftrightarrow \quad \frac{q-p}{p+q} = \frac{(a-c)(b-d)}{(a+c)(b+d)}$$

which proves that the equality in Theorem 3.6 is a necessary and sufficient condition for a quadrilateral to be cyclic since Theorem 3.5 is so.

The next characterization is about the quotient of the diagonals expressed in terms of four subtriangle areas. To prove the direct theorem, that this equality holds in a cyclic quadrilateral, was given as a problem at [26].

Theorem 3.7. *If the diagonals of a convex quadrilateral $ABCD$ intersect at P and triangles ABP, BCP, CDP and DAP have areas S_1, S_2, S_3 and S_4 respectively, then the quotient of the diagonals p and q satisfies*

$$\frac{p}{q} = \frac{\sqrt{S_1S_4} + \sqrt{S_2S_3}}{\sqrt{S_1S_2} + \sqrt{S_3S_4}}$$

if and only if $ABCD$ is a cyclic quadrilateral.

Proof. Let θ be one of the angles between the diagonals. The four subtriangles have areas $S_1 = \frac{1}{2}ef\sin\theta$, $S_2 = \frac{1}{2}fg\sin\theta$, $S_3 = \frac{1}{2}gh\sin\theta$ and $S_4 = \frac{1}{2}he\sin\theta$ (see Figure 7), since $\sin(\pi - \theta) = \sin\theta$. Thus

$$\sqrt{S_1S_4} = \tfrac{1}{2}e\sqrt{fh}\sin\theta, \qquad \sqrt{S_2S_3} = \tfrac{1}{2}g\sqrt{fh}\sin\theta,$$
$$\sqrt{S_1S_2} = \tfrac{1}{2}f\sqrt{eg}\sin\theta, \qquad \sqrt{S_3S_4} = \tfrac{1}{2}h\sqrt{eg}\sin\theta.$$

Hence we get that

$$\frac{\sqrt{S_1S_4} + \sqrt{S_2S_3}}{\sqrt{S_1S_2} + \sqrt{S_3S_4}} = \frac{\sqrt{fh}(e+g)}{\sqrt{eg}(f+h)} = \sqrt{\frac{fh}{eg}} \cdot \frac{p}{q}.$$

According to the intersecting chords theorem and its converse (Theorem A.5 in [18]), $fh = eg$ if and only if $ABCD$ is a cyclic quadrilateral. □

We have no known reference for the following formula, which is yet another characterization regarding the quotient of the diagonals, this time expressed in terms of the four vertex angles.

Theorem 3.8. *In a convex quadrilateral $ABCD$, the quotient of the diagonals p and q satisfies*

$$\frac{p}{q} = \frac{\cos\frac{B}{2}\cos\frac{D}{2}}{\cos\frac{A}{2}\cos\frac{C}{2}}$$

if and only if $ABCD$ is a cyclic quadrilateral.

Proof. (⇒) In a cyclic quadrilateral, using formulas from Theorem 2.4 we have

$$\frac{\cos\frac{B}{2}\cos\frac{D}{2}}{\cos\frac{A}{2}\cos\frac{C}{2}} = \frac{\sqrt{\frac{(s-c)(s-d)}{ab+cd}}\sqrt{\frac{(s-a)(s-b)}{ab+cd}}}{\sqrt{\frac{(s-b)(s-c)}{ad+bc}}\sqrt{\frac{(s-a)(s-d)}{ad+bc}}} = \frac{ad+bc}{ab+cd} = \frac{p}{q}$$

where we applied Theorem 3.5 in the last equality.

(⇐) If the quadrilateral is not cyclic, assume first that vertex A lies outside the circumcircle to triangle BCD. Then, from inequalities derived in the proof of Theorems 2.4, we get

$$\frac{\cos\frac{B}{2}\cos\frac{D}{2}}{\cos\frac{A}{2}\cos\frac{C}{2}} < \frac{\sqrt{\frac{(s-c)(s-d)}{ab+cd}}\sqrt{\frac{(s-a)(s-b)}{ab+cd}}}{\sqrt{\frac{(s-b)(s-c)}{ad+bc}}\sqrt{\frac{(s-a)(s-d)}{ad+bc}}} = \frac{ad+bc}{ab+cd} < \frac{p}{q}.$$

The last inequality is due to (13), since $\angle A + \angle C < \pi$.

In the other case, when A lies inside the circumcircle to triangle BCD, all inequalities are reversed. Thus

$$\frac{\cos\frac{B}{2}\cos\frac{D}{2}}{\cos\frac{A}{2}\cos\frac{C}{2}} > \frac{ad+bc}{ab+cd} > \frac{p}{q}$$

where the last inequality is due to (12), completing the proof. □

Note that the formula in Theorem 3.8 can also be written in the symmetric form

$$AC\cos\frac{A}{2}\cos\frac{C}{2} = BD\cos\frac{B}{2}\cos\frac{D}{2}.$$

4. A characterization concerning an incircle

A quadrilateral that can have an incircle is often called a tangential quadrilateral. To prove the following characterization was given as a problem by Juan-Bosco Romero Márquez in the Canadian problem solving journal *Crux Mathematicorum*, with a solution published in [28]. We give a much shorter proof of the converse.

Theorem 4.1. *In a convex quadrilateral $ABCD$ where the diagonals intersect at P, let E, F, G and H be the feet of the perpendiculars to the sides through P. Then $EFGH$ is a tangential quadrilateral if and only if $ABCD$ is a cyclic quadrilateral.*

Proof. Let E, F, G and H be on AB, BC, CD and DA respectively. Quadrilaterals $AEPH$ and $BFPE$ are always cyclic since they each have a pair of opposite right angles.

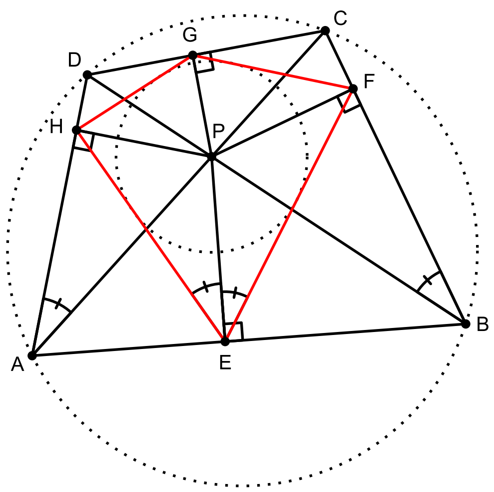

Figure 8. $EFGH$ is tangential iff $ABCD$ is cyclic

(⇒) When $ABCD$ is cyclic, we have that (see Figure 8)

$$\angle HEP = \angle HAP = \angle DAC = \angle CBD = \angle FBP = \angle FEP$$

so EP is the angle bisector to angle HEF. In the same way FP, GP and HP are angle bisectors to the other vertex angles EFG, FGH and GHE respectively. This makes $EFGH$ a tangential quadrilateral since a quadrilateral has an incircle if and only if the angle bisectors to the vertex angles are concurrent according to a well-known characterization.

($\Leftarrow$) Conversely, if $EFGH$ is a tangential quadrilateral (see Figure 8), then we directly get

$$\angle DAC = \angle HAP = \angle HEP = \angle FEP = \angle FBP = \angle CBD$$

which proves that $ABCD$ is cyclic (according to Theorem A.1 in [18]). $\square$

It is an easy exercise to prove that when $EFGH$ is a tangential quadrilateral, then P is the center of its incircle.

In [13, p. 16] we proved a related result: Quadrilateral $ABCD$ has perpendicular diagonals if and only if $EFGH$ is a cyclic quadrilateral.

5. An area characterization

We can model a convex quadrilateral as being built by four very thin rods connected with hinges at their endpoints. If we push on it, its area will be the largest possible if and only if the quadrilateral can be inscribed in a circle according to our last characterization.

Theorem 5.1. *A convex quadrilateral with given sides has maximal area if and only if it is cyclic.*

Proof. Suppose we have a convex quadrilateral with consecutive sides a, b, c and d, which all have lengths that cannot be changed, but the angles between them can vary. Then we are to prove that the area of this quadrilateral is maximal if and only if it is cyclic. The area K of a convex quadrilateral with diagonals p and q is given by the formula

$$K = \tfrac{1}{2}pq\sin\theta$$

where θ is one of the angles between the diagonals. This formula is easy to derive and quite well-known (one proof was given in [10]). Rewriting it, we have that the quadrilateral area satisfies

$$16K^2 = 4p^2q^2\left(1 - \cos^2\theta\right)$$

and inserting (6) and (7) into this equality, we get

$$16K^2 = 4\left(a^2c^2 + b^2d^2 - 2abcd\cos(A+C)\right) - \left(a^2 - b^2 + c^2 - d^2\right)^2.$$

Thus the area is maximal if and only if the angle term $-2abcd\cos(A+C)$ is maximal. This is equivalent to that the cosine factor is minimal, that is $\cos(A+C) = -1$, which in turn is equivalent to $\angle A + \angle C = \pi$, that is, if and only if the quadrilateral is cyclic. $\square$

Defining the semiperimeter $s = \frac{1}{2}(a+b+c+d)$ and performing some factorizations of the area expression in the last theorem, we have as a consequence:

Corollary 5.1. *The area of a convex quadrilateral with consecutive sides a, b, c and d satisfies*

$$K \leq \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where s is the semiperimeter. Equality holds if and only if the quadrilateral is cyclic.

The equality case is the well-known Brahmagupta's formula. We leave the details of these skipped steps to be carried out by the reader.

6. A remark on proving converses

This paper has been a continuation to [18], where we collected 19 characterizations of cyclic quadrilaterals with proofs. One thing that has been a considerable difference between that paper and this is how the converses have been proved. In the present paper, half of the converses were proved using the contrapositive statement, a method that was applied only once in [18]. In that paper on the other hand, half of the converses were proved by showing that a theorem is equivalent to a previously proved characterization, thus proving both the necessary and sufficient condition at the same time – what we like to call an equivalence proof (used for instance to prove Theorem 1.1 in this paper). The second most applied method in [18] was using a direct proof of the converse, which was applied only twice in this paper. The reason for these differences is certainly since there were more basic theorems in [18].

Anyway, there are four different techniques for proving a converse (direct proof, contrapositive proof, proof by contradiction, and equivalence proof) and all four methods have been applied in our two papers on characterizations of cyclic quadrilaterals. Therefor this collection may be of interest not only for its geometrical contents, but also when studying different proof techniques in some university course or even at the secondary school level.

7. Concluding remarks

Properties of cyclic quadrilaterals receive much attention in Olympiad problem solving. In our two part exploration of cyclic quadrilaterals, we have seen that lots of their properties are in fact not only necessary conditions but sufficient conditions as well. In quadrilateral geometry it is quite common that the converse to a property is also true, and this especially holds for the top classes in a classification of convex quadrilaterals (see [15, p. 81]). This has been demonstrated in our previous papers about characterizations of orthodiagonal quadrilaterals [13], tangential quadrilaterals [12], extangential quadrilaterals [16], trapezoids [14], and cyclic quadrilaterals [18].

But not all properties even for those classes can be reversed. In the case of cyclic quadrilaterals, when the converse is not true, then instead of $\angle A + \angle C = \pi$ the quadrilateral often has the property $\angle A = \angle C$ (a tilted kite, see [17]). This is likely to occur if the problem can be solved using the law of sines or other formulas with sine functions, such as the well-known trigonometric formula $T = \frac{1}{2}ab\sin C$ for the area of a triangle.

As an example, consider the following theorem, which was Problem 332 at [8]: If the diagonals intersect at P in a cyclic quadrilateral $ABCD$, then

$$\frac{AP}{CP} = \frac{DA \cdot AB}{BC \cdot CD}.$$

Here the converse does not hold. It is easy to see that kites also satisfy the relation, and in fact, a calculation shows that it is true in all quadrilaterals where $\angle A = \angle C$ (see [17, pp. 94–95]).

There certainly exists other characterizations of cyclic quadrilaterals than the 34 we have collected in our two papers on this subject. Several more of both known och yet unknown properties of cyclic quadrilaterals can surely be proved to be sufficient conditions as well. These are just waiting to be discovered and presented to the world in mathematics competitions, in problem solving journals, on mathematical websites, or in new intriguing papers. We hope to see more of this fascinating subject in the future.

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