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A generalization of the theorem of Von Staudt-Hua-Buekenhout-Cojan in the real $\overline{\overline{\partial}} - \mathcal{F}\mathbb{R}^k_{td}$, 1≤k≤2n+1, space on real geometric projective P_k, 1≤k≤2n+1, finite dimensional space (II)

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Abstract. Quite often is possible to discover alternative way to define a geometric locus which is totally different from the original one. When this is possible we obtain new interesting insight on the geometric object analogous at the improvement achieved when different ways to prove a given theorem are discovered. The purpose of our article is to describe some well-known loci using an alternative approach.

1. Extending a well-known theorem of Von Staudt-Hua-Buekenhout-Cojan

We have reached the following result and obtain the **Remark**. If

$$\forall\ a, \psi, \mu, \omega, x, y \in (\mathcal{K} \cup \{\infty\}) \cap U_k^\beta := Not(\mathcal{K}^* \cap U_k^\beta) \neq \emptyset,$$

$\mathcal{K}$ is $\overline{\overline{\partial}}-$skew-field with characteristic different from 2, $Char\mathcal{K} \neq 2$, then $\forall\ x, y \in \mathcal{K}^*$, $\overline{\overline{\partial}} - f \in \overline{\overline{\partial}} - \mathcal{F}_{td}\mathbb{R}^{n+1}$ of the projective line $\mathcal{P}_{n+1}(\mathcal{K})$, $n \geq 0$, into itself, preserve the harmonic relations

$$\overline{\overline{\partial}} - f(x \oplus y) = \overline{\overline{\partial}} - f(x) \oplus \overline{\overline{\partial}} - f(y),\ \overline{\overline{\partial}} - f(x \bullet y) = \overline{\overline{\partial}} - f(x) \bullet f(y).$$

The aim is to show any $\overline{\overline{\partial}}-$skew-field, $\forall\ \mathcal{K}$, with $Char(\mathcal{K}) \neq 2$, that this result remain true if the harmonic cross ratio it replaced by a given cross ratio, $\forall\ a \notin \{0; 1; \infty\}$, of the homomorphism

$$H : \mathbb{Z} \ni x \to H(x) = x \bullet 1 = x \in \mathcal{K},\ H := Def(Char\mathcal{K}),$$

where $1 \in \mathcal{K}$ is unity element at $\mathcal{K}$. If $\mathcal{K}$ is commutative $\overline{\overline{\partial}}-$skew-field, then $\forall\ x \in \mathcal{K}$,

$$x \oplus \infty := Def(\infty \oplus x) := Def(\infty),\ \forall\ x \in \mathcal{K}^*,$$
$$0^{-1} := Def(\infty),\ \infty^{-1} = Def(0),$$

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then the symbols $\infty\oplus\infty$, $\infty\bullet 0$, $0\bullet\infty$ remain undefined. We shall identify the projective line

$$\mathcal{P}_{n+1}(\mathcal{K}\equiv\mathcal{K}^*,\ n\geq 0.$$

For $\forall\ x^1,x^2,x^3,x^4\in\mathcal{K}$ distinct elements, $x^1\neq x^2\neq x^3\neq x^4$, we define

$$[x^1,x^2,x^3,x^4]:=Def(x^3\ominus x^1)\bullet(x^4\ominus x^2)\bullet(x^3\ominus x^2)$$
$$\bullet(x^3\ominus x^2)^{-1}\bullet(x^4\ominus x^1)^{-1}=:Not(a).$$

Then

$$[x^1,x^3,x^2,x^4]=1\ominus a[x^1,x^2,x^4,x^3]=a^{-1}[x^2,x^1,x^4,x^3]$$
$$=[x^3,x^4,x^1,x^2]=a.$$

Next we define

$$[x^1,x^2,x^3,x^4]:=Def(1),$$

which are distinct elements of $\mathcal{K}$ and $x^4=\infty$, if $x^k\in\mathcal{K}^*$, $x^1\neq x^3$, $x^2\neq x^3$ and $x^3=x^4$. If among $x^k\in\mathcal{K}^*$, $1\leq k\leq 4$, then there are no three equal elements, the cross ratio a is defined. The symbols $[x,x,x,y]$, $[x,x,y,x]$, $[x,y,x,x]$, $[y,x,x,x]$ remain undefined. **A** $\overline{\overline{\partial}}-$**place** is a $\overline{\overline{\partial}}-$function

$$\overline{\overline{\partial}}-\mathcal{F}_{td}\mathbb{R}^{n+1}\ni\exists\ \overline{\overline{\partial}}-f:\mathcal{K}^*\ni x\to\overline{\overline{\partial}}-f(x)\in\mathcal{K}^*$$

i.e. $\forall\ x,y\in\mathcal{K}^*$,

$$\overline{\overline{\partial}}-f(x\oplus y)=-f(x)\oplus\overline{\overline{\partial}}-f(y),\ \overline{\overline{\partial}}-f(x\bullet y)=\overline{\overline{\partial}}-f(x)\bullet\overline{\overline{\partial}}-f(y)$$

i.e. the expressions in both formulae make sense.

Theorem 1.1. *If* $\mathcal{K}$ *is a* $\overline{\overline{\partial}}-$*skew-field with* $Char(\mathcal{K})\neq 2$,

$$\overline{\overline{\partial}}-\mathcal{F}_{td}\mathbb{R}^{n+1}\ni\exists\ \overline{\overline{\partial}}-\Psi:\mathcal{K}\setminus\{0;1\}\ni x\to\overline{\overline{\partial}}-\Psi(x)\in\mathcal{F}_{td}\mathbb{R}^{n+1}\cap\mathcal{P}_{n+1}\neq\emptyset$$

i.e.

1) $\overline{\overline{\partial}}-\Psi(0)=0$, $\overline{\overline{\partial}}-\Psi(1)=1$, $\overline{\overline{\partial}}-\Psi(\infty)=\infty$,
2) $\forall\ x^1,x^2,x^3,x^4\in\mathcal{K}\setminus\{0;1\}$,

$$[x^1,x^2,x^3,x^4]=:Not(a),$$

then $\overline{\overline{\partial}}-\Psi(x)$ *is a* $\overline{\overline{\partial}}-$*place of* $\mathcal{K}\setminus\{0;1\}$.

Proof. If $\forall\ \psi,\mu\in\mathcal{K}\setminus\{0;1\}$ i.e. $\psi\neq\mu$, then we may find

$$\forall\ x\in\mathcal{K}\setminus\{0;1\}\text{ i.e. }[\psi,\mu,\infty,\chi]=a$$

and since $(\chi\ominus\mu)\bullet(\chi\ominus\psi)=a$, or is undefined, that is one of the conditions holds

$$\forall\ a\notin\{0;1;\infty\},\ \overline{\overline{\partial}}-\Psi(x)=(a\bullet\overline{\overline{\partial}}-\Psi(\psi)\ominus\overline{\overline{\partial}}-\Psi(\mu))\bullet(a\ominus 1)^{-1},$$
$$\overline{\overline{\partial}}-\Psi(\psi)=\overline{\overline{\partial}}-\Psi(\mu)=\overline{\overline{\partial}}-\Psi(\chi)\neq\infty$$

and exactly one of $\overline{\overline{\partial}}-\Psi(\psi)$, $\overline{\overline{\partial}}-\Psi(\mu)$, $\overline{\overline{\partial}}-\Psi(\chi)$ equal to ∞, then

$$\overline{\overline{\partial}}-\Psi(\psi)=\overline{\overline{\partial}}-\Psi(\mu)=\overline{\overline{\partial}}-\Psi(\chi)=\infty.$$

We obtain

$$\overline{\overline{\partial}}-\Psi((a\bullet\psi\ominus\mu)\bullet(a\ominus 1)^{-1})=(a\bullet\overline{\overline{\partial}}-\Psi\psi)\ominus\overline{\overline{\partial}}-\Psi(\mu))\bullet(a\ominus 1)^{-1}$$

whenever the right-hand side is defined and if $\psi=0$ we get

$$\forall\ \mu\in\mathcal{K},\ \overline{\overline{\partial}}-(\ominus\mu\bullet(a\ominus 1)^{-1})=\ominus\overline{\overline{\partial}}-\Psi(\mu)\bullet(a\ominus 1)^{-1},$$

which together yield

$$\overline{\overline{\partial}} - \Psi(\ominus a \bullet \psi \oplus \mu) = \ominus a \bullet \overline{\overline{\partial}} - \Psi(\psi) \oplus \overline{\overline{\partial}} - \Psi(\mu)$$

or

$$\overline{\overline{\partial}} - \Psi(\psi) = \overline{\overline{\partial}} - \Psi(\mu) = \infty.$$

Then is clear

$$\forall\ x \in \mathcal{K},\ \overline{\overline{\partial}} - \Psi(\ominus a \bullet x) = \ominus a \bullet \overline{\overline{\partial}} - \Psi(x)$$

and hence

$$\overline{\overline{\partial}} - \Psi(\ominus x) = \ominus \overline{\overline{\partial}} - \Psi(x) \text{ and } \forall\ x, y \in \mathcal{K},\ \overline{\overline{\partial}} - (a^{-1}) = a^{-1}.$$

We shall identify the projective line

$$\mathcal{P}_{n+1}(\mathcal{K}) \equiv \mathcal{K}^* =: Not(\mathcal{K} \cup \{\infty\}),\ n \geq 0,$$

and we deduce

$$\forall\ x \in \mathcal{K}^*,\ \overline{\overline{\partial}} - \Psi((a \ominus 1) \bullet x) = (a \ominus 1) \bullet \overline{\overline{\partial}} - \Psi(x).$$

Consider now, $\forall\ \psi, \mu \in \mathcal{K}^*$ i.e. $\psi \neq \mu$ and select $\omega \in \mathcal{K}^*$ i.e.

$$[\psi, \mu, 0, \omega] = a,$$

then

$$\omega = (1 \ominus a) \bullet \psi \bullet \mu \bullet (\psi \ominus a)^{-1}.$$

Since

$$[\overline{\overline{\partial}} - \Psi(\psi), \overline{\overline{\partial}} - \Psi(\mu), 0, \overline{\overline{\partial}} - \Psi(\omega)] = a,$$

or is undefined and is easily seen that either

$$\overline{\overline{\partial}} - \Psi(\omega) = (1 \ominus a) \bullet \overline{\overline{\partial}} - \Psi(\psi) \bullet \overline{\overline{\partial}} - \Psi(\mu) \bullet (\overline{\overline{\partial}} - \Psi(\psi) \ominus a \bullet \overline{\overline{\partial}} - \Psi(\mu))^{-1}$$

or ins undefined. We have

$$(1 \ominus a) \bullet \overline{\overline{\partial}} - \Psi(\psi) \bullet \overline{\overline{\partial}} - \Psi(\psi \bullet \mu \bullet (\psi \ominus a \bullet \mu^{-1}))$$
$$= \overline{\overline{\partial}} - \Psi(\psi) \bullet \overline{\overline{\partial}} - \Psi(\mu) \bullet (\overline{\overline{\partial}} - \Psi(\psi) \ominus a \bullet \overline{\overline{\partial}} - \Psi(\mu))^{-1}$$

or it has no sense. The same is true if $\psi = 0$ or $\mu = 0$ or $\psi = \mu$. If

$$\psi := a \bullet x \in \mathcal{K},\ \mu := 1 \oplus x \in \mathcal{K},$$

then the relation, result

$$\overline{\overline{\partial}} - \Psi(x^2) = \overline{\overline{\partial}} - \Psi^2(x).$$

Let $\forall\ x \in \mathcal{K}$ with

$$\overline{\overline{\partial}} - \Psi(x) \oplus \overline{\overline{\partial}} - \Psi(x^2) = \overline{\overline{\partial}} - \Psi(a \bullet x) \bullet (1 \oplus \overline{\overline{\partial}} - \Psi(a \bullet x)) \bullet a^{-1},$$

$$\psi := x \oplus 1,\ \mu := a^{-1},$$

we obtain

$$\forall\ x \in \mathcal{K},\ \overline{\overline{\partial}} - \Psi(x^{-1}) = (\overline{\overline{\partial}} - \Psi^{-1}(x).$$

If $\forall\ x, y \in \mathcal{K}$ i.e. $\overline{\overline{\partial}} - \Psi(x) \neq \infty,\ \overline{\overline{\partial}} - \Psi(y) \neq \infty$, then we have

$$\overline{\overline{\partial}} - \Psi(x^2) \oplus 2\overline{\overline{\partial}} - \Psi(x \bullet y) \oplus \Psi(y^2) = \overline{\overline{\partial}} - \Psi((x \oplus y)^2)$$
$$= (\overline{\overline{\partial}} - \Psi(x) \oplus \overline{\overline{\partial}} - \Psi(y))^2 = \overline{\overline{\partial}} - \Psi^2(x) \oplus 2\overline{\overline{\partial}} - \Psi(x) \bullet \overline{\overline{\partial}} - \Psi(y) \oplus \overline{\overline{\partial}} - \Psi^2(y)$$
$$= \overline{\overline{\partial}} - \Psi(x^2) \oplus 2\overline{\overline{\partial}} - \Psi(x) \bullet \overline{\overline{\partial}} - \Psi(y) \oplus \Psi(y^2)$$

and hence

$$\forall\ x,y \in \mathcal{K},\ \overline{\overline{\partial}} - \Psi(x \bullet y) = \overline{\overline{\partial}} - \Psi(x) \bullet \overline{\overline{\partial}} - \Psi(y).$$

It is also true if $\overline{\overline{\partial}} - \Psi(x) = \infty$ and $\overline{\overline{\partial}} - \Psi(y) \in \mathcal{K}^* \setminus \{0\}$ and if suppose to the contrary, that in this case,

$$\overline{\overline{\partial}} - \Psi(x \bullet y) \neq \infty,$$

$$\overline{\overline{\partial}} - \Psi(x) = \overline{\overline{\partial}} - \Psi((x \bullet y) \bullet y^{-1}) = \overline{\overline{\partial}} - \Psi(x \bullet y) \bullet \overline{\overline{\partial}} - \Psi(y^{-1})$$
$$= \overline{\overline{\partial}} - \Psi(x \bullet y) \bullet \overline{\overline{\partial}} - \Psi^{-1}(y) \neq \infty,$$

because $\overline{\overline{\partial}} - \Psi(y) \in \mathcal{K}^* \setminus \{0\}$. The $\overline{\overline{\partial}}-$function $\overline{\overline{\partial}} - \Psi^{-1}(y)$ is not inverse and $\overline{\overline{\partial}} - \Psi(x) \neq \infty$, a contradiction with the relation $\overline{\overline{\partial}} - \Psi(x) = \infty$. Indeed,

$$\overline{\overline{\partial}} - \Psi(x) = \overline{\overline{\partial}} - \Psi((x \bullet (y \bullet y^{-1}))) = \overline{\overline{\partial}} - \Psi(x) \bullet \overline{\overline{\partial}} - \Psi(y \bullet y^{-1})$$
$$= \overline{\overline{\partial}} - \Psi(x) \bullet \overline{\overline{\partial}} - \Psi(1) = \overline{\overline{\partial}} - \Psi(x) \bullet 1 = \overline{\overline{\partial}} - \Psi(x),$$

then $\forall\ x,y \in \mathcal{K}$ the relation

$$\overline{\overline{\partial}} - \Psi(x \bullet y) = \overline{\overline{\partial}} - \Psi(x) \bullet \overline{\overline{\partial}} - \Psi(y)$$

is valid i.e. the right-hand side of formulae is defined. Since $\overline{\overline{\partial}} - \Psi(\infty) = \infty$ by hypothesis, the formulae holds

$$\forall\ x,y \in \mathcal{K}^* = \mathcal{K} \cup \{\infty\}$$

whenever side make sense. Thus the proof is achieved. □

Corollary 1.2. *If the conditions of the above theorem are changed* $\forall\ x \in \mathcal{K}^*$, $Char(Im\overline{\overline{\partial}} - \psi) \geq 2$, *then there exists a* $\overline{\overline{\partial}}-$***place***,

$$\overline{\overline{\partial}} - \mathcal{F}_{td}\mathbb{R}^{n+1} \ni \exists\ \overline{\overline{\partial}} - f : \mathcal{K}^* \ni x \to \overline{\overline{\partial}} - f(x) \in \mathcal{K}^*$$

and a real $\overline{\overline{\partial}}-$*discontinuous* $\overline{\overline{\partial}}-$***projecivity***

$$\overline{\overline{\partial}} - \mathcal{F}_{td}\mathbb{R}^{n+1} \ni \exists\ \overline{\overline{\partial}} - \Phi : \mathbb{R}^{n+1} \setminus \{(0,\dots,0)\} \ni x \to \overline{\overline{\partial}} - \Phi(x) \in \mathcal{P}_{n+1},\ n \geq 0,$$

of $\mathcal{K}$ *such that*

$$\overline{\overline{\partial}} - \Psi = \overline{\overline{\partial}} - f \circ \overline{\overline{\partial}} - \Phi.$$

Theorem 1.3. ***(Lashkhi-Cojan).*** *If* U *is a non-commutative left principal ideal domain,* $\frac{1}{2} \in U$, $\mathcal{T}$ *is torsion-free over* U,

$$Dim_p\mathcal{T} = 1,\ \overline{\overline{\partial}} - \mathcal{F}_{td}\mathbb{R}^n \ni \exists\ \overline{\overline{\partial}} - f : \mathcal{P}_n(\mathcal{T}) \to \mathcal{P}_n(J_1,\mathcal{T}_1),\ n \geq 2,$$

is a harmonic $\overline{\overline{\partial}}-$*application, then there exist a* $\overline{\overline{\partial}}-$*isomorphism,*

$$\overline{\overline{\partial}} - \mathcal{F}_{td}\mathbb{R}^n \ni \exists\ \overline{\overline{\partial}} - \phi : \mathcal{K}_1 \to \mathcal{K}_2$$

a linear anti–$\overline{\overline{\partial}}-$*isomorphism,*

$$\overline{\overline{\partial}} - \mathcal{F}_{td}\mathbb{R}^n \ni \exists\ \overline{\overline{\partial}} - \chi : \mathcal{T} \to \mathcal{T}_1,$$

and a subring, $\exists\ \mathcal{K}_2 \subset \mathcal{K}_1$, $1 \in \mathcal{K}_2$ *such that*

$$\mathcal{K}_2\overline{\overline{\partial}} - \chi)\mathcal{T}) \subseteq \mathcal{T}_1\ \textit{i.e.}\ \overline{\overline{\partial}} - \phi(u) \to \mathcal{K}_2 \to \mathcal{K}_1 \leftarrow J_1 \leftarrow \overline{\overline{\partial}} - \phi(u)$$

and

$$\overline{\overline{\partial}} - f(Ux) = \mathcal{K}_2\overline{\overline{\partial}} - \chi(x).$$

Proof. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be arbitrary rings, the $\overline{\overline{\partial}}$−application

$$\overline{\overline{\partial}} - \mathcal{F}_{td}\mathbb{R}^n \ni \exists\, \overline{\overline{\partial}} - \phi : J_1 \to J_2$$

will be called **a linear** $\overline{\overline{\partial}}$**−isomorphism** with respect to $\overline{\overline{\partial}} - \phi$, if $\overline{\overline{\partial}} - \chi$ is defined on the $\overline{\overline{\partial}}$−basis B i.e. for e_1, e_2 the images $\overline{\overline{\partial}} - \chi(e_1)$, $\overline{\overline{\partial}} - \chi(e_2)$ are fixed, then we shall continue as follows

$$\forall\ a_0, a_1 \in J,\ a_0 \neq 0,\ a_1 \neq 0,$$

$$\overline{\overline{\partial}} - \chi(a_) \bullet e_k) = \overline{\overline{\partial}} - \phi(a_0) \bullet \overline{\overline{\partial}} - \chi(e_k),\ 1 \leq k \leq 2,$$

$$\overline{\overline{\partial}} - \chi(a_0 \bullet e_1 \oplus a_1 \bullet e_2) = (\overline{\overline{\partial}} - \phi^{-1}(a_1)) \bullet \overline{\overline{\partial}} - \chi(e_1) \oplus (\overline{\overline{\partial}} - \phi^{-1}(a_0)) \bullet \overline{\overline{\partial}} - \chi(e_k).$$

Now, let us turn back to our consideration and have the following alternatives:

1) $\overline{\overline{\partial}} - \phi$ is $\overline{\overline{\partial}}$−isomorphism,

$$\overline{\overline{\partial}} - g(a_0 \bullet x_0 \oplus a_1 \bullet x_1) = \overline{\overline{\partial}} - f(x_0 \oplus a_0^{-1} \bullet a_1 \bullet x_1)$$

$$= \mathcal{K}_1(\overline{\overline{\partial}} - \chi(x_0) \oplus \overline{\overline{\partial}} - \phi(a_0^{-1} \bullet a_1) \bullet \overline{\overline{\partial}} - \chi(x_1)) = \overline{\overline{\partial}} - g(\overline{\overline{\partial}} - \chi(x_0) \oplus \overline{\overline{\partial}} - \phi(a_0))^{-1},$$

$$\overline{\overline{\partial}} - \phi(a_1) \bullet \overline{\overline{\partial}} - \chi(x_0) = \overline{\overline{\partial}} - g(\overline{\overline{\partial}} - \phi(a_0)\overline{\overline{\partial}} - \chi(x_0) \oplus \overline{\overline{\partial}} - \phi(a_1) \bullet \overline{\overline{\partial}} - \chi(x_1)),\ ([6]),$$

2) $\overline{\overline{\partial}} - \phi$ is anti−$\overline{\overline{\partial}}$−isomorphism, then

$$\overline{\overline{\partial}} - g(a_0 \bullet x_0 \oplus a_1 \bullet x_1) = \overline{\overline{\partial}} - g(\overline{\overline{\partial}} - \phi(a_0), \overline{\overline{\partial}} - \phi(a_1) \bullet \overline{\overline{\partial}} - \chi(x_1))$$

$$= \overline{\overline{\partial}} - g(\overline{\overline{\partial}} - \phi(a_1))^{-1}, \overline{\overline{\partial}} - \chi(x_1) \oplus (\overline{\overline{\partial}} - \phi(a_0))^{-1} \bullet \chi(x_1).$$

Thus for fixed point $x_0 \in B$ and $x_0 \in B$ and

$$\overline{\overline{\partial}} - \mathcal{F}_{td}\mathbb{R}^n \ni \overline{\overline{\partial}} - \chi : B \ni x \to \overline{\overline{\partial}} - \chi(x) \in \mathcal{T}$$

we have defined the $\overline{\overline{\partial}}$−isomorphism $\overline{\overline{\partial}} - \phi$ and the linear anti−$\overline{\overline{\partial}}$−isomorphism $\overline{\overline{\partial}} - \chi$, through for all, it is true that

$$\overline{\overline{\partial}} - g(x) = \mathcal{K}_1\overline{\overline{\partial}} - \chi(x).$$

Define the subring as follows,

$$\overline{\overline{\partial}} - f(Jx_0) := Def\mathcal{K}_2\overline{\overline{\partial}} - \chi(x_0)$$

and is clear that $\mathcal{K}_2$ is a J_1-submodule in $\mathcal{K}_1$. Let us show that

1.1) $\overline{\overline{\partial}} - f(Ux_1) = \mathcal{K}_2\overline{\overline{\partial}} - \chi(x_1)$ and

1.2) $\overline{\overline{\partial}} - f(U(a_0 \bullet x_0 \oplus a_1 \bullet x_1))$

$$= \mathcal{K}_2((\overline{\overline{\partial}}\phi(a_1)))^{-1} \bullet (\overline{\overline{\partial}} - \chi(x_0) \oplus (\overline{\overline{\partial}} - \phi(a_0)))^{-1} \bullet (\overline{\overline{\partial}} - \chi(x_1))$$

if $\overline{\overline{\partial}} - \phi$ is an anti$\overline{\overline{\partial}}$−isomorphism i.e

$$\overline{\overline{\partial}} - f(U(a_0 \bullet x_0 \oplus a_1 \bullet x_1)) = \mathcal{K}_2(\overline{\overline{\partial}} - \phi(a_0) \bullet \overline{\overline{\partial}} - \chi(x_0) \oplus \overline{\overline{\partial}} - \phi(a_1))$$

if $\overline{\overline{\partial}} - \phi$ is an $\overline{\overline{\partial}}$−isomorphism.

1.1) The U-points Ux_0, Ux_1, $U(x_0 \oplus x_1)$, $U(x_0 \ominus x_1)$ are harmonic and

$$Ux_0 \subset \{U(x_0 \oplus x_1), U(x_0 \ominus x_1)\} \oplus Ux_1.$$

From this in general we cannot conclude that

$$\overline{\overline{\partial}}-f(Ux_0)\subset\{\overline{\overline{\partial}}-f(U(x_0\oplus x_1),U(x_0\ominus x_1))\}\oplus\overline{\overline{\partial}}-f(\mathcal{K}_1(\overline{\overline{\partial}}-\chi(x_0)\oplus\chi(x_1)),$$
$$(\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1))\oplus\mathcal{K}_1(\overline{\overline{\partial}}-\chi(x_1)))$$

and the points

$$\mathcal{K}_1\overline{\overline{\partial}}-\chi(x_0),\ \mathcal{K}_1\overline{\overline{\partial}}-\chi(x_1),$$
$$\mathcal{K}(\overline{\overline{\partial}}-\chi(x_0)\oplus\overline{\overline{\partial}}-\chi(x_1),\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1))$$

are harmonic. By the definition and the images $\overline{\overline{\partial}}-f(Jx_0)$, $\overline{\overline{\partial}}-f(Jx_1)$, $\overline{\overline{\partial}}-f(J(x_0\oplus x_1))$, $\overline{\overline{\partial}}-f(J(x_0\ominus x_1))$ we can find harmonic J_1-points

$$J_1(\xi_1\bullet\overline{\overline{\partial}}-\chi(x_0)),$$
$$J_1(\xi_2\bullet\overline{\overline{\partial}}-\chi(x_1)),$$
$$J_1(\xi_3\bullet(\overline{\overline{\partial}}-\chi(x_0)\oplus\overline{\overline{\partial}}-\chi(x_1))),$$
$$J_1(\xi_4\bullet(\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1)))$$

and ξ_1,ξ_2,ξ_3,ξ_4 in such way that $\xi_k=\xi,\ 1\le k\le 4$,

$$J_1(\xi_1\bullet\overline{\overline{\partial}}-\chi(x_0))\subset J_1(\xi_2\bullet\overline{\overline{\partial}}-\chi(x_1))\oplus J_1(\xi_3 0\bullet(\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1)))$$
$$\Rightarrow\xi_1=\xi_2=\xi_3,\ J_1(\xi_3\bullet(\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1)))$$
$$\subset J_1(\xi_1\bullet\overline{\overline{\partial}}-\chi(x_0))\oplus J_1(\xi_2\bullet\overline{\overline{\partial}}-\chi(x_1))\Rightarrow\xi_1=\xi_2=\xi_3.$$

The same version is also true for $\xi_4=\xi_k$, $1\le k\le 3$ and is obvious that we can choose ξ such that

$$J_1(\xi\bullet\overline{\overline{\partial}}-\chi(x_0))\subset J_1\overline{\overline{\partial}}-\chi(x_0)\subseteq\overline{\overline{\partial}}-f(Jx_0).$$

Thus we have

$$J_1(\xi\bullet\overline{\overline{\partial}}-\chi(x_0))\subset J_1(\xi\bullet((\overline{\overline{\partial}}-\chi(x_0)\oplus\overline{\overline{\partial}}-\chi(x_1))))(\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1))$$
$$J_1(\xi\bullet\overline{\overline{\partial}}-\chi(x_1))\subseteq J_1\overline{\overline{\partial}}-\chi(x_1)\subset\overline{\overline{\partial}}-f(Jx_1)$$
$$\subset\mathcal{K}_1\{\overline{\overline{\partial}}-\chi(x_0)\oplus\overline{\overline{\partial}}-\chi(x_1),\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1)\}\oplus\mathcal{K}_1\overline{\overline{\partial}}-\chi(x_1)$$

and

$$J_1(\xi\bullet\{\overline{\overline{\partial}}-\chi(x_0)\oplus\overline{\overline{\partial}}-\chi(x_1),\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1)\},$$
$$J_1(\xi\bullet\overline{\overline{\partial}}-\chi(x_0))\subseteq J_1\overline{\overline{\partial}}-\chi(x_0)\subset\overline{\overline{\partial}}-f(Ux_0)\subset\mathcal{K}_1\overline{\overline{\partial}}-\chi(x_0)$$
$$\subset\mathcal{K}_1\{\overline{\overline{\partial}}-\chi(x_0)\oplus\overline{\overline{\partial}}-\chi(x_1),\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1)\}\oplus\mathcal{K}_1\overline{\overline{\partial}}-\chi(x_1)$$

and

$$J_1(\xi\bullet\{\overline{\overline{\partial}}-\chi(x_0)\oplus\overline{\overline{\partial}}-\chi(x_1),\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1)\}\oplus\mathcal{K}_1\overline{\overline{\partial}}-\chi(x_1)).$$

For $\forall\ b\in\mathcal{K}_2$, we get

$$b\bullet(\xi\bullet\overline{\overline{\partial}}-\chi(x_0))=\xi_1\bullet(\xi\bullet(\overline{\overline{\partial}}-\chi(x_0)\ominus\overline{\overline{\partial}}-\chi(x_1)))\oplus\xi_2\bullet(\xi\bullet\overline{\overline{\partial}}-\chi(x_1))$$
$$\Rightarrow b=\xi_1=\xi_2\Rightarrow b\bullet\overline{\overline{\partial}}-\chi(x_1)\in\overline{\overline{\partial}}-f(Ux_1)\Rightarrow\mathcal{K}_2\subseteq\overline{\overline{\partial}}-f(Ux_1).$$

Suppose now that

$$c\bullet\overline{\overline{\partial}}-\chi(x_1)\in\overline{\overline{\partial}}-f(Ux_1).$$

Changing the roles of x_0 and x_1, we get

$$c\bullet\overline{\overline{\partial}}-\chi(x_0)\in\overline{\overline{\partial}}-f(Ux_0)=\mathcal{K}_2\overline{\overline{\partial}}-\chi(x_0)$$

$$\Rightarrow c\in\mathcal{K}_2\Rightarrow\overline{\overline{\partial}}-f(Ux_1)=\mathcal{K}_2\overline{\overline{\partial}}-\chi(x_1).$$

1.2) Suppose that $\overline{\overline{\partial}}-\phi$ is an anti$-\overline{\overline{\partial}}-$isomorphism, then the $U-$points $U(a_0\bullet x_0)$, $U(a_1\bullet x_1)$, $U(a_0\bullet x_0\oplus a_1\bullet x_1)$, $U(x_0\bullet x_0\ominus a_1\bullet x_1)$ are harmonic. So we have

$$\overline{\overline{\partial}}-f(U(a_k\bullet x_k))\to\mathcal{K}_1(a_k\bullet x_k),\ 1\le k\le 2,$$

$$\overline{\overline{\partial}}-f(U\{a_0\bullet x_0\oplus a_1\bullet x_1,a_0\bullet x_0\ominus a_1\bullet x_1\})$$
$$\to\mathcal{K}_1((\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet(\overline{\overline{\partial}}-\chi(x_0)\oplus(\overline{\overline{\partial}}-\phi(a_0)))).$$

If $b\in\mathcal{K}_2$, then

$$b\bullet\overline{\overline{\partial}}-\chi(a_0\bullet x_0)=b\bullet\overline{\overline{\partial}}-\phi(a_0)\bullet\overline{\overline{\partial}}-\chi(x_0)$$
$$\in\mathcal{K}_1((\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_0)\oplus(\overline{\overline{\partial}}-\phi(a_0))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_1))$$
$$\oplus\mathcal{K}(\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_0)\ominus(\overline{\overline{\partial}}-\phi(a_0))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_1)$$
$$\Rightarrow a_0=a_1$$
$$\Rightarrow b\bullet\overline{\overline{\partial}}-\phi(a_0)\bullet\overline{\overline{\partial}}-\chi(x_0)=2a_0\bullet(\overline{\overline{\partial}}-\phi(a_0))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_0)$$
$$\Rightarrow 2a_0=b\bullet\overline{\overline{\partial}}-\phi(a_0)\bullet\overline{\overline{\partial}}-\phi(a_1)$$
$$\Rightarrow b\bullet\overline{\overline{\partial}}-\phi(a_0)\bullet\overline{\overline{\partial}}-\phi(a_1)\bullet(\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_0)\oplus(\overline{\overline{\partial}}-\phi(a_0))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_1)$$
$$\in\overline{\overline{\partial}}-f(U(a_1)\bullet a_0\bullet(a_0\bullet x_0\oplus a_1\bullet x_1)))$$
$$\to\mathcal{K}_1(\overline{\overline{\partial}}-\phi(a_0)\bullet(\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_0)\oplus(\overline{\overline{\partial}}-\phi(a_0))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_1))$$
$$\Rightarrow\mathcal{K}_2(((\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_0)\oplus(\overline{\overline{\partial}}-\phi(a_0))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_1))$$
$$\subseteq\overline{\overline{\partial}}-f(U(a_0\bullet x_0\oplus a_1\bullet x_1)).$$

On the other hand, if

$$c\bullet(\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet(\overline{\overline{\partial}}-\chi(x_0)\oplus(\overline{\overline{\partial}}-\phi(a_0))^{-1})\bullet\overline{\overline{\partial}}-\chi(x_1)$$
$$\in\overline{\overline{\partial}}-f(U(a_0\bullet x_0\oplus a_1\bullet x_1))$$

and we get

$$\overline{\overline{\partial}}-\chi(x_0)=c\bullet\overline{\overline{\partial}}-\chi(a_1^{-1}\bullet x_0)\in\overline{\overline{\partial}}-f(U\overline{\overline{\partial}}-\chi(a_1^{-1}\bullet x_0))\mathcal{K}_2\overline{\overline{\partial}}-\chi(a_1^{-1}\bullet x_0)$$
$$=\mathcal{K}_2(\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_0)$$

and

$$c\bullet(\overline{\overline{\partial}}-\phi(a_0))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_1)\in\mathcal{K}_2(\overline{\overline{\partial}}-\phi(a_0))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_1)$$
$$\Rightarrow c\in U\Rightarrow\overline{\overline{\partial}}-f(U(a_0\bullet x_0\oplus a_1\bullet x_1))$$
$$=\mathcal{K}_2((\overline{\overline{\partial}}-\phi(a_1))^{-1}\bullet\overline{\overline{\partial}}-\chi(x_0)\oplus(\overline{\overline{\partial}}-\phi(a_0)\bullet(\overline{\overline{\partial}}-\chi(x_1)).$$

The case when $\overline{\overline{\partial}}-\phi$ is an $\overline{\overline{\partial}}-$isomorphism is easily and can be proved with similar arguments. If $\xi\in U$, $x\in\mathcal{T}\setminus\{0\}$, then we have

$$\mathcal{K}_2\overline{\overline{\partial}}-\phi(U)\bullet\overline{\overline{\partial}}-\chi(x)=\mathcal{K}_2\overline{\overline{\partial}}-\chi(Ux)=\overline{\overline{\partial}}-f(U(\xi\bullet x))$$
$$\subseteq\overline{\overline{\partial}}-f(Ux)=\mathcal{K}_2\overline{\overline{\partial}}-\chi(x)$$
$$\Rightarrow\mathcal{K}_2\overline{\overline{\partial}}-\phi(Ux)\subseteq\mathcal{K}_2.$$

In general, the constructed subring, the $\overline{\overline{\partial}}-$applications $\overline{\overline{\partial}}-\chi$ and $\overline{\overline{\partial}}-\phi$ are not unique. If $0 \neq a \in \mathcal{K}_2$, then $\mathcal{K}_3 =: Not(\mathcal{K}_2 a^{-1})$ is a J_2-submodule and $\overline{\overline{\partial}}-\chi_1 =: Not(a \bullet \overline{\overline{\partial}}-\chi)$ is linear anti$-\overline{\overline{\partial}}-$isomorphism with respect to

$$\overline{\overline{\partial}}-\phi_1 =: Not(a \bullet \overline{\overline{\partial}}-\phi \bullet a^{-1})$$

and in fact

$$\mathcal{K}_3\overline{\overline{\partial}}-\phi_1(Ux) = \mathcal{K}_2 a^1 \bullet a \bullet \overline{\overline{\partial}}-\phi(Ux) = \mathcal{K}_2 a^{-1} = \mathcal{K}_3$$

$$\Rightarrow \mathcal{K}_3\overline{\overline{\partial}}-\chi_1(x) = \mathcal{K}_2 a^{-1} \bullet a \bullet \overline{\overline{\partial}}-\chi(x) = \mathcal{K}_2\overline{\overline{\partial}}-\chi(x).$$

Consequently, here exist a ring, $\exists\ \mathcal{K}_2 \ni 1$, and in fact $\mathcal{K}_2$ i.e. $\overline{\overline{\partial}}-\chi$ can be constructed up to a constant factor. Thus the following inclusions are true,

$$\overline{\overline{\partial}}-\phi(Ux) \hookrightarrow \mathcal{K}_2\overline{\overline{\partial}}-\phi(Ux) \hookrightarrow U_1 x \rightharpoonup \mathcal{K}_1$$

and by definition of

$$\overline{\overline{\partial}}-\mathcal{F}_{td}\mathbb{R}^n \ni \exists\ \overline{\overline{\partial}}-f : \mathcal{P}_n(\mathcal{T}) \to \mathcal{P}_n(\mathcal{T}_1),\ n \geq 2,$$

we have $\mathcal{K}_2\overline{\overline{\partial}}-\chi(\mathcal{T}) \subseteq \mathcal{T}_1$, thus we prove the theorem. □

Corollary 1.4. *If U is a noncommutative left principal ideal domain, $\frac{1}{2} \in U$, $\mathcal{T}$ is a torsion-free module over U, $Dim_p\mathcal{T} = 1$, then the $\overline{\overline{\partial}}-$bijection*

$$\overline{\overline{\partial}}-\mathcal{F}_{td}\mathbb{R}^n \ni \exists\ \overline{\overline{\partial}}-f : \mathcal{P}_n(\mathcal{T}) \to \mathcal{P}_n(\mathcal{T}_1),\ n \geq 2,$$

is harmonic if and only if there exist an $\overline{\overline{\partial}}-$isomorphism or an anti$-\overline{\overline{\partial}}-$isomorphism,

$$\overline{\overline{\partial}}-\mathcal{F}_{td}\mathbb{R}^n \ni \exists\ \overline{\overline{\partial}}-\chi : \mathcal{T} \to \mathcal{T}_1$$

such that

$$\forall\ x \in \mathcal{T},\ \overline{\overline{\partial}}-f(Ux) = U_1\overline{\overline{\partial}}-\chi(x).$$

Theorem 1.5. ***(S.P. Cojan).*** *Let $\overline{\overline{\partial}}-\chi$ and $\overline{\overline{\partial}}-\chi_1$ be the linear $\overline{\overline{\partial}}-$isomorphism with respect to*

$$\overline{\overline{\partial}}-\mathcal{F}_{td}\mathbb{R}^n \ni \exists\ \overline{\overline{\partial}}-\Omega : \mathcal{P}_n \ni x \to \overline{\overline{\partial}}-\Omega(x) \in \mathcal{P}_n^1,\ n \geq 2,$$

i.e. $Dim\mathcal{T} \geq 2$, where $\mathcal{T}$ is a torsion-free module over noncommutative left principal ideal domain U. If $\mathcal{K}$ and $\mathcal{K}_1$ i.e. $1 \in \mathcal{K}$, $1 \in \mathcal{K}_1$ are subrings of $\mathcal{P}_n$ i.e.

$$\forall\ x \in \mathcal{T},\ \mathcal{K}_1\overline{\overline{\partial}}-\chi_1(x) = \mathcal{K}\overline{\overline{\partial}}-\chi(x),$$

$$\mathcal{K} \leftarrow \overline{\overline{\partial}}-\Omega(Ux) \to \mathcal{K}_1 \to \mathcal{P}_n^1 \leftarrow U_1 x \to \mathcal{P}_n^1 \leftarrow \mathcal{K},$$

where U is a noncommutative left principal ideal domain i.e. $\frac{1}{2} \in U$, then there exists an element,

$$\exists\ a \in \mathcal{K}\ i.e.\ \mathcal{K}_1 = \mathcal{K}a_1\overline{\overline{\partial}}-\chi_1(*) = a \bullet \overline{\overline{\partial}}-\chi(*).$$

Proof. For this we suppose that the points,

$$\forall\ x, y \in \mathcal{T},\ \text{i.e. } x \neq y \Rightarrow \mathcal{P}_n x \neq \mathcal{P}_n y,$$

then there exist,

$$\exists\ a, b, c \in \mathcal{P}_n,\ \text{i.e. } \overline{\overline{\partial}} - \chi_1(x) = a \bullet \overline{\overline{\partial}} - \chi(x),$$

$$\overline{\overline{\partial}} - \chi_1(y) = b \bullet \overline{\overline{\partial}} - \chi(y),$$

$$\overline{\overline{\partial}} - \chi(x \oplus y) = c \bullet \overline{\overline{\partial}} - \chi(x \oplus y)$$

$$\Rightarrow a \bullet \overline{\overline{\partial}} - \chi(x) \oplus b \bullet \overline{\overline{\partial}} - \chi(y) \overline{\overline{\partial}} - \chi_1(x \oplus y) = c \bullet \overline{\overline{\partial}} - \chi(x \oplus y)$$

$$\Rightarrow a = b = c \Rightarrow \forall\ m \in \mathcal{T},\ \overline{\overline{\partial}} - \chi_1(m) = a \bullet \overline{\overline{\partial}} - \chi(m).$$

Since $\mathcal{P}_n \mathcal{T} = \mathcal{T}_1$, $n \geq 2$, we get for all,

$$\forall\ x \in \mathcal{T},\ \overline{\overline{\partial}} - \chi_1(x) = a \bullet \overline{\overline{\partial}} - \chi(x),$$

but if $\forall\ x \in \mathcal{T} \setminus \{0\}$, then

$$\mathcal{K}\overline{\overline{\partial}} - \chi(x) = \mathcal{K}_1 \overline{\overline{\partial}} - \chi_1(x) = \mathcal{K}_1 a \bullet \overline{\overline{\partial}} - \chi(x)$$

$$\Rightarrow \mathcal{K} = \mathcal{K}_1 a,\ \mathcal{K}_1 = \mathcal{K} a^{-1}$$

and as $1 \in \mathcal{K}_1$ it is clear that $a \in \mathcal{K}$. □

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