INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 8 (2019), No. 1, 57 - 71

SPLIT QUATERNIONS AND TIME-LIKE CONSTANT SLOPE SURFACES IN MINKOWSKI 3-SPACE

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Abstract. In the present paper, we prove that time-like constant slope surfaces can be reparametrized by using rotation matrices corresponding to unit time-like split quaternions and also homothetic motions. Afterwards we give some examples to illustrate our main results by using Mathematica.

1 Introduction

Quaternions are very efficient tools for describing rotations about an arbitrary axis since the rotation matrices can be easily obtained by using them. Thus quaternions are usually used in computer/robot vision, computer graphics, robotics, animations, navigation systems, aerospace applications, flight simulators, visualizations, fractals and virtual reality, (see [1], [7], [10]).

Munteanu [8] showed that a constant slope surface can be constructed by using a unit speed curve on the Euclidean 2-sphere $\mathbb{S}^2$ and an equiangular spiral. Also in [1], we demonstrated that constant slope surfaces can be obtained by using quaternions. After that in [2], we found the relations between space-like constant slope surfaces [3] and split quaternions in Minkowski 3-space $\mathbb{R}^3_1$.

Keywords and phrases: Time-like constant slope surface, split quaternion, homothetic motion
(2010) Mathematics Subject Classification: 14Q10, 53A17, 53A35
Received: 22.02.2018. In revised form: 09.03.2019. Accepted: 24.03.2019.

Fu and Wang [4] studied time-like constant slope surfaces and classified these surfaces in $\mathbb{R}^3_1$. They showed that $S \subset \mathbb{R}^3_1$ is a time-like constant slope surface lying in the time-like cone if and only if it can be parametrized by

$$x(u,\, v) = u\cosh\theta\left(\cosh\xi_1(u)f(v) + \sinh\xi_1(u)f(v)\wedge f'(v)\right), \tag{1}$$

where, θ is a positive constant angle function, $\xi_1(u) = \tanh\theta\ln u$ and f is a unit speed space-like curve on pseudo-hyperbolic space $\mathbb{H}^2$.

$S \subset \mathbb{R}^3_1$ is a time-like constant slope surface lying in the space-like cone if and only if it can be parametrized by

$$x(u,\, v) = u\sin\theta\left(\cos\xi_2(u)g(v) + \sin\xi_2(u)g(v)\wedge g'(v)\right), \tag{2}$$

where, θ is a positive constant angle function satisfying $\theta\in(0,\pi/2]$, $\xi_2(u) = \cot\theta\ln u$ and g is a unit speed time-like curve on pseudo-sphere $\mathbb{S}^2_1$.

Also, $S \subset \mathbb{R}^3_1$ is a time-like constant slope surface lying in the space-like cone if and only if it can be parametrized by

$$x(u,\, v) = u\sinh\theta\left(\cosh\xi_3(u)h(v) + \sinh\xi_3(u)h(v)\wedge h'(v)\right), \tag{3}$$

where, θ is a positive constant angle function, $\xi_3(u) = \coth\theta\ln u$ and h is a unit speed space-like curve on pseudo-sphere $\mathbb{S}^2_1$.

In this paper, we will investigate the relations between split quaternions and time-like constant slope surfaces in $\mathbb{R}^3_1$. We will show that time-like constant slope surfaces can be reparametrized by using rotation matrices corresponding to unit time-like split quaternions and also homothetic motions. Afterwards we will give some examples to strengthen the both this paper and [4] by using Mathematica.

2 Preliminaries

In this section, we recall some important concepts and formulas regarding split quaternions and semi-Euclidean spaces.

A split quaternion p can be written as

$$p = p_1 1 + p_2 i + p_3 j + p_4 k,$$

where $p_1,\ p_2,\ p_3,\ p_4 \in \mathbb{R}$ and $i,\ j,\ k$ are split quaternion units which satisfy the following rules:

$i^2 = -1,\ j^2 = k^2 = 1,\ i\times j = -j\times i = k,\ j\times k = -k\times j = -i$ and $k\times i = -i\times k = j$.

The algebra of split quaternions is denoted by $\mathbb{H}'$ and its natural basis is given by $\{1,\ i,\ j,\ k\}$. Also an element of $\mathbb{H}'$ is called a split quaternion.

For a split quaternion $p = p_1 1 + p_2 i + p_3 j + p_4 k$, the conjugate $\bar{p}$ of p is

$$\bar{p} = p_1 1 - p_2 i - p_3 j - p_4 k$$

[5].

Scalar and vector parts of a split quaternion p are $S_p = p_1$ and $V_p = p_2 i + p_3 j + p_4 k$, respectively. The split quaternion product of two split quaternions $p = (p_1,\ p_2,\ p_3,\ p_4)$ and $q = (q_1,\ q_2,\ q_3,\ q_4)$ is given by

$$p \times q = p_1 q_1 + < V_p,\ V_q > + p_1 V_q + q_1 V_p + V_p \wedge V_q, \tag{4}$$

where

$$< V_p,\ V_q >= -p_2 q_2 + p_3 q_3 + p_4 q_4$$

and

$$V_p \wedge V_q = \begin{vmatrix} -i & j & k \\ p_2 & p_3 & p_4 \\ q_2 & q_3 & q_4 \end{vmatrix} = (p_4 q_3 - p_3 q_4) i + (p_4 q_2 - p_2 q_4)\ j + (p_2 q_3 - p_3 q_2)\ k.$$

If $S_p = 0$, then p is called as a pure split quaternion [10].

Definition 2.1. *If $\mathbb{R}^n$ is equipped with the metric tensor*

$$< u, v >= -\sum_{i=1}^{v} u_i v_i + \sum_{i=v+1}^{n} u_i v_i,$$

it is called semi-Euclidean space and given by $\mathbb{R}^n_v$, where v is called the index of the metric, $0 \leq v \leq n$ and $u, v \in \mathbb{R}^n$. If $v = 0$, semi-Euclidean space $\mathbb{R}^n_v$ is reduced to $\mathbb{R}^n$. For $n \geq 2$, $\mathbb{R}^n_1$ is called Minkowski n-space [9].

Definition 2.2. *A vector $w \in \mathbb{R}^n_v$ is called*
(i) space-like if $< w, w >> 0$ or $w = 0$,
(ii) time-like if $< w, w >< 0$,
(iii) light-like (null) if $< w, w >= 0$ and $w \neq 0$.

The norm of a vector $w \in \mathbb{R}^n_v$ is $\sqrt{|< w, w >|}$. Two vectors w_1 and w_2 in $\mathbb{R}^n_v$ are said to be orthogonal if $< w_1, w_2 >= 0$ [9].

In $\mathbb{R}^n_v$, pseudo-sphere and pseudo-hyperbolic space are given by

$$\mathbb{S}^{n-1}_v = \left\{ (v_1, ..., v_n) \in \mathbb{R}^n_v | -\sum_{i=1}^{v} v_i^2 + \sum_{i=v+1}^{n} v_i^2 = 1 \right\},$$

and

$$\mathbb{H}^{n-1}_{v-1} = \left\{ (v_1, ..., v_n) \in \mathbb{R}^n_v | -\sum_{i=1}^{v} v_i^2 + \sum_{i=v+1}^{n} v_i^2 = -1 \right\},$$

respectively. If we take $v = 1$ and $v_1 > 0$, then $\mathbb{H}^{n-1} = \mathbb{H}^{n-1}_0$ is called a pseudo-hyperbolic space.

Since $-p \times \bar{p} = -p_1^2 - p_2^2 + p_3^2 + p_4^2$, we identify $\mathbb{H}'$ with semi-Euclidean space $\mathbb{R}^4_2$ [5]. As a result, we can define time-like, space-like and light-like split quaternions as follows:

Definition 2.3. *A split quaternion p is space-like, time-like or light-like, if $I_p < 0$, $I_p > 0$ or $I_p = 0$, respectively, where $I_p = p_1^2 + p_2^2 - p_3^2 - p_4^2$,* [10].

Definition 2.4. *The norm of $p = (p_1, p_2, p_3, p_4)$ is defined as*

$$N_p = \sqrt{\left| p_1^2 + p_2^2 - p_3^2 - p_4^2 \right|}.$$

If $N_p = 1$, then p is called as a unit split quaternion and $p_0 = p/N_p$ is a unit split quaternion for $N_p \neq 0$. Also space-like and time-like split quaternions have multiplicative inverses having the property $p \times p^{-1} = p^{-1} \times p = 1$ and they are constructed by $p^{-1} = \bar{p}/I_p$. Light-like split quaternions have no inverses [10].

The set of space-like split quaternions is not a group because it is not closed under multiplication. Therefore we are only interested in time-like split quaternions. The vector part of any time-like split quaternion can be space-like or time-like. Trigonometric forms of them are as below:

(i) Every time-like split quaternion with the space-like vector part can be written as

$$p = N_p(\cosh\theta + w \sinh\theta),$$

where w is a unit space-like vector in $\mathbb{R}^3_1$.

(ii) Every time-like split quaternion with the time-like vector part can be written as

$$p = N_p(\cos\theta + w \sin\theta),$$

where w is a unit time-like vector in $\mathbb{R}^3_1$ ([6], [10]).

Unit time-like split quaternions are used to construct rotations in $\mathbb{R}^3_1$:
If $p = (p_1, p_2, p_3, p_4)$ is a unit time-like quaternion, using the transformation law

$$(p \times V \times p^{-1})_i = \sum_{j=1}^{3} \mathbf{R}_{ij} V_j,$$

one can find the follwing rotation matrix

$$R_p = \begin{bmatrix} p_1^2 + p_2^2 + p_3^2 + p_4^2 & 2p_1p_4 - 2p_2p_3 & -2p_1p_3 - 2p_2p_4 \\ 2p_2p_3 + 2p_4p_1 & p_1^2 - p_2^2 - p_3^2 + p_4^2 & -2p_3p_4 - 2p_2p_1 \\ 2p_2p_4 - 2p_3p_1 & 2p_2p_1 - 2p_3p_4 & p_1^2 - p_2^2 + p_3^2 - p_4^2 \\ & & \end{bmatrix}, \tag{5}$$

where V is a pure split quaternion. The causal character of vector part of the unit time-like split quaternion p is important. If the vector part of p is time-like or space-like then the rotation angle is spherical or hyperbolic, respectively [10].

In $\mathbb{R}^3_1$, one-parameter homothetic motion of a body is generated by the transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix},$$

where X and C are real matrices of 3×1 type and h is a homothetic scale and $A \in SO_1(3)$. A, h and C are differentiable functions of C^∞ class of a parameter t, (see [11]).

3 Split Quaternions and Time-like Constant Slope Surfaces Lying in the Time-like Cone

A unit time-like split quaternion with the space-like vector part

$$Q(u, v) = \cosh(\xi_1(u)/2) - \sinh(\xi_1(u)/2) f'(v)$$

defines a 2-dimensional surface on pseudo-hyperbolic space $\mathbb{H}^3_1$, where $\xi_1(u) = \tanh\theta \ln u$, θ is a positive constant angle function, $f' = (f'_1, f'_2, f'_3)$ and f is a unit speed space-like curve on $\mathbb{H}^2$. Thus, using the equation (5), the corresponding rotation matrix can be found as

$$R_Q = \left[\begin{matrix} \cosh^2\frac{\xi_1}{2} + \sinh^2\frac{\xi_1}{2}(f_1'^2 + f_2'^2 + f_3'^2) & -2\sinh^2\frac{\xi_1}{2} f'_1 f'_2 - \sinh\xi_1 f'_3 \\ 2\sinh^2\frac{\xi_1}{2} f'_1 f'_2 - \sinh\xi_1 f'_3 & \cosh^2\frac{\xi_1}{2} + \sinh^2\frac{\xi_1}{2}(-f_1'^2 - f_2'^2 + f_3'^2) \\ 2\sinh^2\frac{\xi_1}{2} f'_1 f'_3 + \sinh\xi_1 f'_2 & -2\sinh^2\frac{\xi_1}{2} f'_2 f'_3 - \sinh\xi_1 f'_1 \end{matrix} \right.$$

$$\left.\begin{array}{c} -2\sinh^2\frac{\xi_1}{2}f_1'f_3'+\sinh\xi_1 f_2' \\ -2\sinh^2\frac{\xi_1}{2}f_2'f_3'+\sinh\xi_1 f_1' \\ \cosh^2\frac{\xi_1}{2}+\sinh^2\frac{\xi_1}{2}(-f_1'^2+f_2'^2-f_3'^2) \end{array}\right]. \tag{6}$$

Now we give the relation between unit time-like split quaternions with the space-like vector parts and time-like constant slope surfaces lying in the time-like cone.

Theorem 3.1. *Let $x : S \to \mathbb{R}_1^3$ be a time-like constant slope surface immersed in Minkowski 3-space $\mathbb{R}_1^3$ and x lies in the time-like cone. Then the time-like constant slope surface S can be reparametrized by*

$$x(u,v) = Q_1(u,v)\times Q_2(u,v),$$

where $Q_1(u,v) = \cosh\xi_1(u) - \sinh\xi_1(u)f'(v)$ is a unit time-like split quaternion with the space-like vector part, $Q_2(u,v) = u\cosh\theta f(v)$ is a surface and a pure split quaternion in $\mathbb{R}_1^3$ and f is a unit speed space-like curve on $\mathbb{H}^2$.

Proof. Since $Q_1(u,v) = \cosh\xi_1(u) - \sinh\xi_1(u)f'(v)$ and $Q_2(u,v) = u\cosh\theta f(v)$, we have

$$\begin{aligned} Q_1(u,v)\times Q_2(u,v) &= \left(\cosh\xi_1(u) - \sinh\xi_1(u)f'(v)\right)\times(u\cosh\theta f(v)) \\ &= u\cosh\theta\left(\cosh\xi_1(u) - \sinh\xi_1(u)f'(v)\right)\times f(v) \\ &= u\cosh\theta\left(\cosh\xi_1(u)f(v) - \sinh\xi_1(u)f'(v)\times f(v)\right). \end{aligned} \tag{7}$$

By using the equation (4), we have

$$f'\times f = < f', f > + f'\wedge f.$$

Since f is a unit speed space-like curve on $\mathbb{H}^2$, we get

$$-f'\times f = f\wedge f'. \tag{8}$$

Substituting the equation (8) into the equation (7), we obtain

$$Q_1(u,v)\times Q_2(u,v) = u\cosh\theta\left(\cosh\xi_1(u)f(v) + \sinh\xi_1(u)f(v)\wedge f'(v)\right).$$

Thus, using the equation (1), we conclude that

$$Q_1(u,v)\times Q_2(u,v) = x(u,v).$$

This completes the proof.

From Theorem 3.1, we can see that the time-like constant slope surface S lying in the time-like cone is the split quaternion product of 2-dimensional surfaces $Q_1(u,v)$ on H_1^3 and $Q_2(u,v)$ in $\mathbb{R}_1^3$.

We now consider the relations among rotation matrices R_Q, homothetic motions and time-like constant slope surfaces lying in the time-like cone. We have the following results of Theorem 3.1.

Corollary 3.2. *Let R_Q be the rotation matrix corresponding to the unit time-like split quaternion with the space-like vector part $Q(u,v)$. Then the time-like constant slope surface S can be written as*

$$x(u,v) = R_Q Q_2(u,v).$$

Corollary 3.3. *For the homothetic motion $\tilde{Q}(u,v) = u\cosh\theta Q_1(u,v)$, the time-like constant slope surface S can be reparametrized by $x(u,v) = \tilde{Q}(u,v)\times f(v)$. Therefore, we have*

$$x(u,v) = u\cosh\theta R_Q f(v). \tag{9}$$

Thus, we can give the following example:

Example 3.4. *We consider a unit speed space-like curve on $\mathbb{H}^2$ defined by*

$$f(v) = (\cosh v, \sinh v, 0).$$

If we take $\theta = 2$, the homothetic motion is

$$\tilde{Q}(u,v) = u\cosh 2\left(\cosh\xi_1(u) + (-\sinh\xi_1(u)\sinh v, -\sinh\xi_1(u)\cosh v, 0)\right),$$

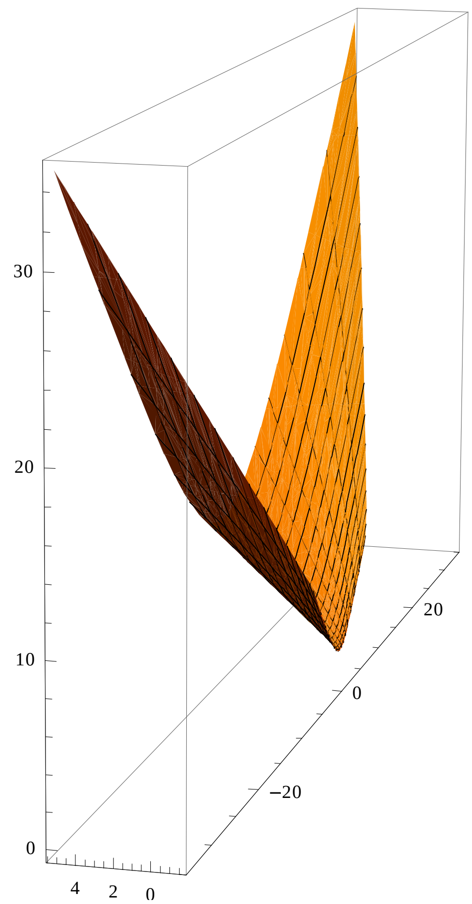
where $\xi_1(u) = \tanh 2\ln u$.
Using the equation (9), we have the following time-like constant slope surface lying in the time-like cone:

$$x(u,v) = u\cosh 2\begin{bmatrix} \cosh^2\frac{\xi_1}{2} + \sinh^2\frac{\xi_1}{2}\cosh 2v & -\sinh^2\frac{\xi_1}{2}\sinh 2v & \sinh\xi_1\cosh v \\ \sinh^2\frac{\xi_1}{2}\sinh 2v & \cosh^2\frac{\xi_1}{2} - \sinh^2\frac{\xi_1}{2}\cosh 2v & \sinh\xi_1\sinh v \\ \sinh\xi_1\cosh v & -\sinh\xi_1\sinh v & \cosh\xi_1 \end{bmatrix}\begin{bmatrix} \cosh v \\ \sinh v \\ 0 \end{bmatrix}.$$

where $\xi_1 = \xi_1(u) = \tanh 2\ln u$. As a result

$$x(u,v) = \begin{bmatrix} u\cosh 2\cosh(\tanh 2\ln u)\cosh v \\ u\cosh 2\cosh(\tanh 2\ln u)\sinh v \\ u\cosh 2\sinh(\tanh 2\ln u) \end{bmatrix}.$$

Thus, the picture of $x(u,v) = u\cosh\theta R_Q f(v)$:



Time-like constant slope surface lying in the time-like cone $x(u,v) = u\cosh\theta R_Q f(v)$,
$f(v) = (\cosh v, \sinh v, 0)$, $\theta = 2$.

We can give the following remarks regarding Theorem 3.1 and Corollary 3.3.

Remark 3.5. *Theorem 3.1 says that the position vectors on the surface* $Q_2(u,v)$ *are rotated by* $Q_1(u,v)$ *through the hyperbolic angle* $\xi_1(u)$ *about the space-like axis* $\operatorname{span}\{f'(v)\}$.

Remark 3.6. *Corollary 3.3 says that the position vector of the space-like curve* $f(v)$ *is rotated by* $\tilde{Q}(u,v)$ *through the hyperbolic angle* $\xi_1(u)$ *about the space-like axis* $\operatorname{span}\{f'(v)\}$ *and extended through the homothetic scale* $u\cosh\theta$.

4 Split Quaternions and Time-like Constant Slope Surfaces Lying in the Space-like Cone

As the previous section, we consider a unit time-like split quaternion with the time-like vector part

$$Q(u,v)=\cos(\xi_2(u)/2)-\sin(\xi_2(u)/2)g'(v),$$

where $\xi_2(u)=\cot\theta\ln u$, θ is a positive constant angle function satisfying $\theta\in(0,\pi/2]$, $g'=(g_1',g_2',g_3')$ and g is a unit speed time-like curve on $\mathbb{S}_1^2$. Thus, using the equation (5), the corresponding rotation matrix can be found as

$$R_Q=\begin{bmatrix} \cos^2\frac{\xi_2}{2}+\sin^2\frac{\xi_2}{2}(g_1'^2+g_2'^2+g_3'^2) & -2\sin^2\frac{\xi_2}{2}g_1'g_2'-\sin\xi_2 g_3' & -2\sin^2\frac{\xi_2}{2}g_1'g_3'+\sin\xi_2 g_2' \\ 2\sin^2\frac{\xi_2}{2}g_1'g_2'-\sin\xi_2 g_3' & \cos^2\frac{\xi_2}{2}+\sin^2\frac{\xi_2}{2}(-g_1'^2-g_2'^2+g_3'^2) & -2\sin^2\frac{\xi_2}{2}g_2'g_3'+\sin\xi_2 g_1' \\ 2\sin^2\frac{\xi_2}{2}g_1'g_3'+\sin\xi_2 g_2' & -2\sin^2\frac{\xi_2}{2}g_2'g_3'-\sin\xi_2 g_1' & \cos^2\frac{\xi_2}{2}+\sin^2\frac{\xi_2}{2}(-g_1'^2+g_2'^2-g_3'^2) \end{bmatrix}. \tag{10}$$

Now we give the relation between unit time-like quaternions with the time-like vector parts and the time-like constant slope surfaces lying in the space-like cone.

Theorem 4.1. *Let $x\ :\ S\to\mathbb{R}_1^3$ be a time-like constant slope surface immersed in Minkowski 3-space $\mathbb{R}_1^3$ and x lies in the space-like cone. Then the time-like constant slope surface S can be reparametrized by*

$$x(u,v)\ =Q_1(u,v)\times Q_2(u,v),$$

where $Q_1(u,v)=\cos\xi_2(u)-\sin\xi_2(u)g'(v)$ is a unit time-like split quaternion with time-like vector part, $Q_2(u,v)=u\sin\theta g(v)$ is a surface and a pure split quaternion in $\mathbb{R}_1^3$ and g is a unit speed time-like curve on $\mathbb{S}_1^2$.

Proof. Since $Q_1(u,v)=\cos\xi_2(u)-\sin\xi_2(u)g'(v)$ and $Q_2(u,v)=u\sin\theta g(v)$, we obtain

$$\begin{aligned} Q_1(u,v)\times Q_2(u,v) &= (\cos\xi_2(u)-\sin\xi_2(u)g'(v))\times(u\sin\theta g(v)) \\ &= u\sin\theta\,(\cos\xi_2(u)-\sin\xi_2(u)g'(v))\times g(v) \\ &= u\sin\theta\,(\cos\xi_2(u)g(v)-\sin\xi_2(u)g'(v)\times g(v))\,. \end{aligned} \tag{11}$$

By using the equation (4), we have

$$g'\times g=<g',g>+g'\wedge g.$$

Since g is a unit speed time-like curve on $\mathbb{S}_1^2$, we have

$$-g'\times g=g\wedge g'. \tag{12}$$

Substituting the equation (12) into the equation (11) gives

$$Q_1(u,v)\times Q_2(u,v) = u\sin\theta\left(\cos\xi_2(u)g(v) + \sin\xi_2(u)g(v)\wedge g'(v)\right).$$

Therefore applying the equation (2), we conclude that

$$Q_1(u,v)\times Q_2(u,v) = x(u,v).$$

Thus the proof is completed.

From Theorem 4.1, we can see that the time-like constant slope surface S lying in the space-like cone is the split quaternion product of 2-dimensional surfaces $Q_1(u,v)$ on $\mathbb{H}_1^3$ and $Q_2(u,v)$ in $\mathbb{R}_1^3$.

Let us consider the relations among rotation matrices R_Q, homothetic motions and time-like constant slope surfaces lying in the space-like cone. We have the following results of Theorem 4.1.

Corollary 4.2. *Let R_Q be the rotation matrix corresponding to the unit time-like split quaternion with time-like vector part $Q(u,v)$. Then the time-like constant slope surface S can be written as*

$$x(u,v) = R_Q Q_2(u,v).$$

Corollary 4.3. *For the homothetic motion $\tilde{Q}(u,v) = u\sin\theta Q_1(u,v)$, the time-like constant slope surface S can be reparametrized by $x(u,v) = \tilde{Q}(u,v)\times g(v)$. Therefore, we have*

$$x(u,v) = u\sin\theta R_Q g(v). \tag{13}$$

Also, we can give an example:

Example 4.4. *We consider a unit speed time-like curve on $\mathbb{S}_1^2$ defined by*

$$g(v) = (\sinh v, \cosh v, 0).$$

If we take $\theta = \pi/3$, the homothetic motion is

$$\tilde{Q}(u,v) = \frac{\sqrt{3}}{2}u\left(\cos(\frac{\ln u}{\sqrt{3}}) + \left(-\sin(\frac{\ln u}{\sqrt{3}})\cosh v, -\sin(\frac{\ln u}{\sqrt{3}})\sinh v, 0\right)\right).$$

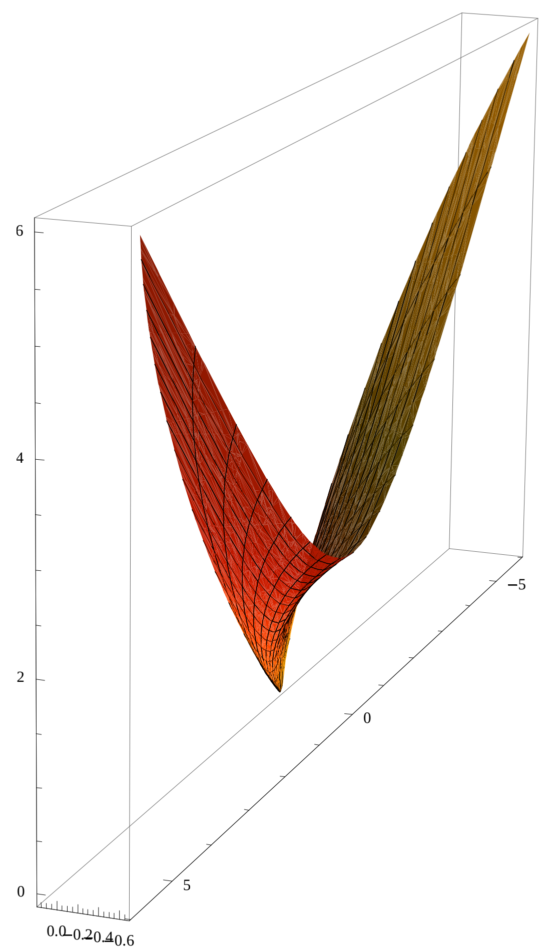
By using the equation (13), we have the following time-like constant slope surface lying in the space-like cone:

$$x(u,v) = \frac{\sqrt{3}}{2}u\begin{bmatrix} \cos^2\frac{\xi_2}{2} + \sin^2\frac{\xi_2}{2}\cosh 2v & -\sin^2\frac{\xi_2}{2}\sinh 2v & \sin\xi_2\sinh v \\ \sin^2\frac{\xi_2}{2}\sinh 2v & \cos^2\frac{\xi_2}{2} - \sin^2\frac{\xi_2}{2}\cosh 2v & \sin\xi_2\cosh v \\ \sin\xi_2\sinh v & -\sin\xi_2\cosh v & \cos\xi_2 \end{bmatrix}\begin{bmatrix} \sinh v \\ \cosh v \\ 0 \end{bmatrix},$$

where $\xi_2 = \xi_2(u) = \frac{\ln u}{\sqrt{3}}$. As a result

$$x(u,v) = \begin{bmatrix} \frac{\sqrt{3}}{2} u \cos(\frac{\ln u}{\sqrt{3}}) \sinh v \\ \frac{\sqrt{3}}{2} u \cos(\frac{\ln u}{\sqrt{3}}) \cosh v \\ -\frac{\sqrt{3}}{2} u \sin(\frac{\ln u}{\sqrt{3}}) \end{bmatrix}.$$

Thus, the picture of $x(u,v) = u \sin\theta R_Q g(v)$:



Time-like constant slope surface lying in the space-like cone $x(u,v) = u \sin\theta R_Q g(v)$, $g(v) = (\sinh v, \cosh v, 0)$, $\theta = \pi/3$.

We can give the following remarks regarding Theorem 4.1 and Corollary 4.3.

Remark 4.5. *Theorem 4.1 says that the position vectors on the surface $Q_2(u,v)$ are rotated by $Q_1(u,v)$ through the spherical angle $\xi_2(u)$ about the time-like axis* $\mathrm{span}\{g'(v)\}$.

Remark 4.6. *Corollary 4.3 says that the position vector of the time-like curve $g(v)$ is rotated by $\tilde{Q}(u,v)$ through the spherical angle $\xi_2(u)$ about the time-like axis* $\mathrm{span}\{g'(v)\}$ *and extended through the homothetic scale $u\sin\theta$.*

Also, we can give the relation between unit time-like split quaternions with the space-like vector parts and the time-like constant slope surfaces lying in the space-like cone. We consider a unit time-like split quaternion with the space-like vector part

$$Q(u,v) = \cosh(\xi_3(u)/2) - \sinh(\xi_3(u)/2)h'(v),$$

where $\xi_3(u) = \coth\theta \ln u$, θ is a positive constant angle function, $h' = (h_1', h_2', h_3')$ and h is a unit speed space-like curve on $\mathbb{S}_1^2$. Thus, using the equation (5), the corresponding rotation matrix can be found as

$$R_Q = \begin{bmatrix} \cosh^2\frac{\xi_3}{2} + \sinh^2\frac{\xi_3}{2}(h_1'^2 + h_2'^2 + h_3'^2) & -2\sinh^2\frac{\xi_3}{2}h_1'h_2' - \sinh\xi_3 h_3' & -2\sinh^2\frac{\xi_3}{2}h_1'h_3' + \sinh\xi_3 h_2' \\ 2\sinh^2\frac{\xi_3}{2}h_1'h_2' - \sinh\xi_3 h_3' & \cosh^2\frac{\xi_3}{2} + \sinh^2\frac{\xi_1}{2}(-h_1'^2 - h_2'^2 + h_3'^2) & -2\sinh^2\frac{\xi_3}{2}h_2'h_3' + \sinh\xi_3 h_1' \\ 2\sinh^2\frac{\xi_3}{2}h_1'h_3' + \sinh\xi_3 h_2' & -2\sinh^2\frac{\xi_3}{2}h_2'h_3' - \sinh\xi_3 h_1' & \cosh^2\frac{\xi_3}{2} + \sinh^2\frac{\xi_3}{2}(-h_1'^2 + h_2'^2 - h_3'^2) \end{bmatrix}. \tag{14}$$

Theorem 4.7. *Let $x : S \to \mathbb{R}_1^3$ be a time-like constant slope surface immersed in Minkowski 3-space $\mathbb{R}_1^3$ and x lies in the space-like cone. Then the time-like constant slope surface S can be reparametrized by*

$$x(u,v) = Q_1(u,v) \times Q_2(u,v),$$

where $Q_1(u,v) = \cosh\xi_3(u) - \sinh\xi_3(u)h'(v)$ is a unit time-like split quaternion with space-like vector part, $Q_2(u,v) = u\sinh\theta h(v)$ is a surface and a pure split quaternion in $\mathbb{R}_1^3$ and h is a unit speed space-like curve on $\mathbb{S}_1^2$.

Proof. Since $Q_1(u,v) = \cosh\xi_3(u) - \sinh\xi_3(u)h'(v)$ and $Q_2(u,v) = u\sinh\theta h(v)$, we obtain

$$\begin{aligned} Q_1(u,v) \times Q_2(u,v) &= (\cosh\xi_3(u) - \sinh\xi_3(u)h'(v)) \times (u\sinh\theta h(v)) \\ &= u\sinh\theta\,(\cosh\xi_3(u) - \sinh\xi_3(u)h'(v)) \times h(v) \\ &= u\sinh\theta\,(\cosh\xi_3(u)h(v) - \sinh\xi_3(u)h'(v) \times h(v)). \end{aligned} \tag{15}$$

By using the equation (4), we get

$$h' \times h = < h', h > + h' \wedge h.$$

Since h is a unit speed space-like curve on $\mathbb{S}_1^2$, we have

$$-h' \times h = h \wedge h'. \tag{16}$$

Substituting the equation (16) into the equation (15) gives

$$Q_1(u,v) \times Q_2(u,v) = u \sinh\theta \left(\cosh\xi_3(u) h(v) + \sinh\xi_3(u) h(v) \wedge h'(v)\right).$$

Therefore applying the equation (3) we conclude that

$$Q_1(u,v) \times Q_2(u,v) = x(u,v).$$

This completes the proof.

We have the following results of Theorem 4.7.

Corollary 4.8. *Let R_Q be the rotation matrix corresponding to the unit time-like split quaternion with space-like vector part $Q(u,v)$. Then the time-like constant slope surface S can be written as*

$$x(u,v) = R_Q Q_2(u,v).$$

Corollary 4.9. *For the homothetic motion $\tilde{Q}(u,v) = u \sinh\theta Q_1(u,v)$, the time-like constant slope surface S can be reparametrized by $x(u,v) = \tilde{Q}(u,v) \times h(v)$. Therefore, we have*

$$x(u,v) = u \sinh\theta R_Q h(v). \tag{17}$$

We can give the following example:

Example 4.10. *Let us consider a unit speed space-like curve on $\mathbb{S}_1^2$ defined by*

$$h(v) = (0, \sin v, \cos v).$$

Taking $\theta = 3$, the homothetic motion is equal to

$$\tilde{Q}(u,v) = u \sinh 3 \left(\cosh\xi_3(u) + (0, -\sinh\xi_3(u)\cos v, \sinh\xi_3(u)\sin v)\right).$$

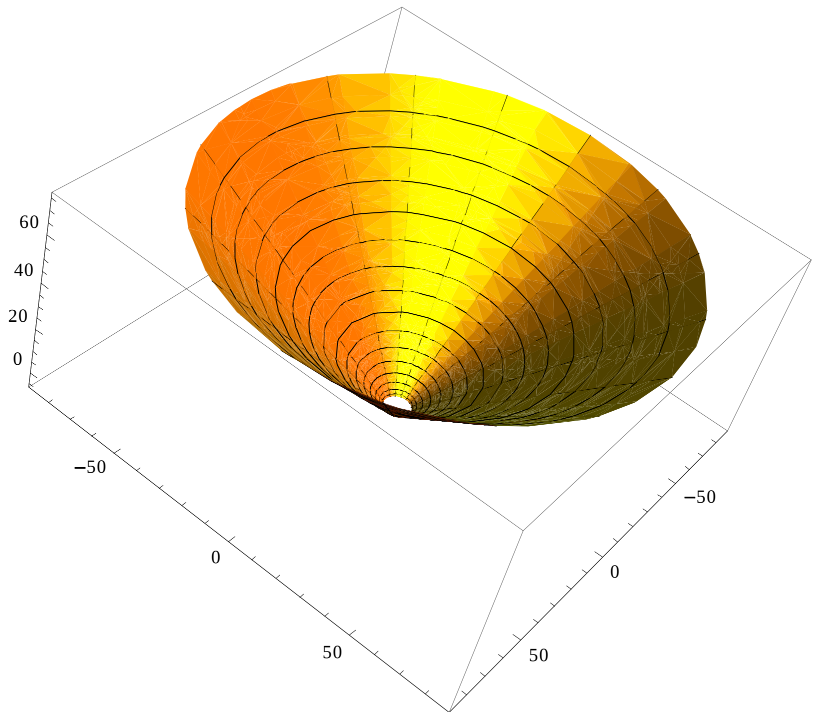
By using the equation (17), we have the following time-like constant slope surface lying in the space-like cone:

$$x(u,v) = u \sinh 3 \begin{bmatrix} \cosh\xi_3 & \sinh\xi_3 \sin v & \sinh\xi_3 \cos v \\ \sinh\xi_3 \sin v & \cosh^2\frac{\xi_3}{2} - \sinh^2\frac{\xi_3}{2}\cos 2v & \sinh^2\frac{\xi_3}{2}\sin 2v \\ \sinh\xi_3 \cos v & \sinh^2\frac{\xi_3}{2}\sin 2v & \cosh^2\frac{\xi_3}{2} + \sinh^2\frac{\xi_3}{2}\cos 2v \end{bmatrix} \begin{bmatrix} 0 \\ \sin v \\ \cos v \end{bmatrix},$$

where $\xi_3 = \xi_3(u) = \coth 3 \ln u$. As a result

$$x(u,v) = \begin{bmatrix} u \sinh 3 \sinh(\coth 3 \ln u) \\ u \sinh 3 \cosh(\coth 3 \ln u) \sin v \\ u \sinh 3 \cosh(\coth 3 \ln u) \cos v \end{bmatrix}.$$

Thus, we can draw the picture of $x(u,v) = u \sinh\theta R_Q h(v)$ as follows:



Time-like constant slope surface lying in the space-like cone $x(u,v) = u\sinh\theta R_Q h(v)$, $h(v) = (0, \sin v, \cos v)$, $\theta = 3$.

We can give the following remarks regarding Theorem 4.7 and Corollary 4.9.

Remark 4.11. *Theorem 4.7 says that the position vectors on the surface $Q_2(u,v)$ are rotated by $Q_1(u,v)$ through the hyperbolic angle $\xi_3(u)$ about the space-like axis* $\operatorname{span}\{h'(v)\}$.

Remark 4.10. *Corollary 4.9 says that the position vector of the space-like curve $h(v)$ is rotated by $\tilde{Q}(u,v)$ through the hyperbolic angle $\xi_3(u)$ about the space-like axis* $\operatorname{span}\{h'(v)\}$ *and extended through the homothetic scale $u\sinh\theta$.*

References

[1] Babaarslan, M. and Yayli, Y., *A new approach to constant slope surfaces with quaternions*, ISRN Geometry, Vol. **2012**, Article ID 126358, 8 pages, 2012. doi:10.5402/2012/126358.

[2] Babaarslan, M. and Yayli, Y., *Split quaternions and space-like constant slope surfaces in Minkowski 3-space*, International Journal of Geometry, **2 (2013)**, 23-33.

[3] Fu, Y. and Yang, D., *On constant slope spacelike surfaces in 3-dimensional Minkowski space*, Journal of Mathematical Analysis and Applications, **385 (2012)**, 208-220.

[4] Fu, Y. and Wang, X., *Classification of time-like constant slope surfaces in 3-dimensional Minkowski space*, Results in Mathematics, **63 (2013)**, 1095-1108.

[5] Inoguchi, J., *Timelike surfaces of constant mean curvature in Minkowski 3-space*, Tokyo Journal of Mathematics, **21 (1998)**, 140-152.

[6] Kula, L. and Yayli, Y., *Split quaternions and rotations in semi Euclidean space* $\mathbb{R}^4_2$, Journal of the Korean Mathematical Society, **44 (2007)**, 1313-1327.

[7] Mukundan, R., *Quaternions: From Classical Mechanics to Computer Graphics, and Beyond*, Proceedings of the 7th Asian Technology Conference in Mathematics 2002.

[8] Munteanu, M. I., *From golden spirals to constant slope surfaces*, Journal of Mathematical Physics, **51 (2010)**, 1-9.

[9] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Pure and Applied Mathematics, 103, Academic Press, New York, 1983.

[10] Ozdemir, M. and Ergin, A. A., *Rotations with unit timelike quaternions in Minkowski 3-space*, Journal of Mathematical Physics, **56 (2006)**, 322-326.

[11] Tosun, M., Kucuk, A. and Gungor, M. A., *The homothetic motions in the Lorentzian 3-space*, Acta Mathematica Scientia, **26B (2006)**, 711-719.

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