INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 8 (2019), No. 1, 38 - 46

ON SOME RELATIONS FOR A TRIANGLE

YURII N. MALTSEV and ANNA S. MONASTYREVA

Abstract. Let r_a, r_b, r_c be the radii of the tangent circles at the vertices to the circumcircle of a triangle ABC and to the opposite sides. In this article, we prove some relations for the numbers r_a, r_b, r_c.

1. Introduction

Let R and r be the circumradius and the inradius of an arbitrary triangle ABC. Denote the center of the circumcircle of ABC by O. Consider the circle tangent to BC and to the circumcircle of ABC at the vertex A. We denote its center by O_A. Denote by r_a the radius of this circle ($r_a = AO_A$). Analogously, we define r_b and r_c. We will also use the standard notation for lengths and angles of the triangle: $AB = c$, $AC = b$, $BC = a$, $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$, $p = \dfrac{a+b+c}{2}$. Then $\angle BAK = \gamma$, $\angle AKB = \beta - \gamma$.

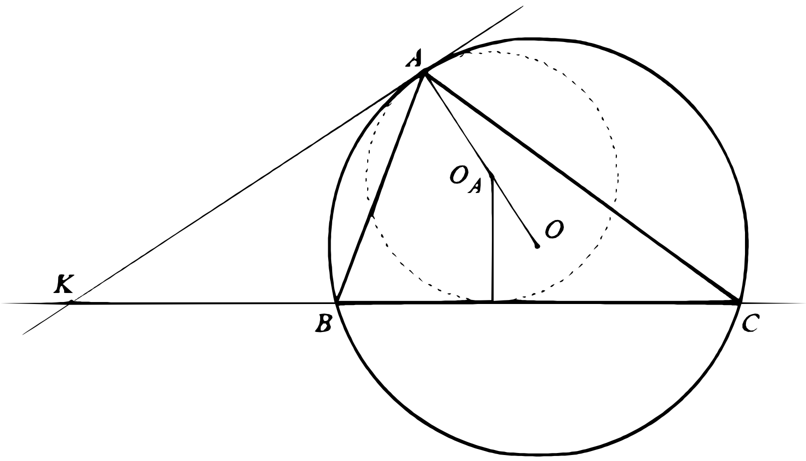

Figure 1

Keywords and phrases: Geometry of triangle, radius of tangent circle.
(2010)Mathematics Subject Classification: 51M04
Received: 29.01.2019. In revised form: 18.03.2019. Accepted: 02.03.2019.

In [1], the following inequalities have been proved

$$\frac{4}{R} \le \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \le \frac{2}{r}. \tag{1}$$

So the authors get a new interpritation for the well-known Euler's inequality $R \ge 2r$. In [3], it was proved the following equation:

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{2}{R} + \frac{1}{r}. \tag{2}$$

The main result of this paper is next theorem.

Theorem 1.1. *The numbers r_a, r_b, r_c are roots of an equation*

$$x^3 - R \cdot \frac{p^4 + p^2(20Rr + 18r^2) + (4R + r)r^3}{(p^2 + r(2R + r))^2} \cdot x^2 + \\ + \frac{(R + 2r)16p^2rR^2}{(p^2 + r(2R + r))^2} \cdot x - \frac{16p^2r^2R^3}{(p^2 + r(2R + r))^2} = 0.$$

2. MAIN RESULT

Before proving the theorem 1.1, let us consider the supplementary lemmas.

Lemma 2.1. [2, 5] *For any triangle ABC the following equalities hold:*

(1) $\sin\alpha + \sin\beta + \sin\gamma = \dfrac{p}{R}$;

(2) $\sin\beta\sin\gamma + \sin\alpha\sin\beta + \sin\alpha\sin\gamma = \dfrac{p^2 + r^2 + 4Rr}{4R^2}$;

(3) $\sin\alpha \cdot \sin\beta \cdot \sin\gamma = \dfrac{pr}{2R^2}$;

(4) $\cos\alpha + \cos\beta + \cos\gamma = \dfrac{r + R}{R}$;

(5) $\cos^3\alpha + \cos^3\beta + \cos^3\gamma = \dfrac{(2R + r)^3 - 3p^2r}{4R^3} - 1$;

(6) $\cos\beta\cos\gamma + \cos\alpha\cos\beta + \cos\alpha\cos\gamma = \dfrac{p^2 + r^2 - 4R^2}{4R^2}$;

(7) $\cos\alpha \cdot \cos\beta \cdot \cos\gamma = \dfrac{p^2 - (2R + r)^2}{4R^2}$.

Lemma 2.2. *For any triangle ABC the following equalities hold:*

(1) $\cos(\beta - \gamma) + \cos(\alpha - \gamma) + \cos(\beta - \alpha) = \dfrac{p^2 + r^2 + 2Rr - 2R^2}{2R^2}$;

(2) $\sin(\alpha - \beta) + \sin(\beta - \gamma) + \sin(\gamma - \alpha) = -\dfrac{(a - b)(b - c)(c - a)}{4R^2r}$.

Proof. By Lemma 2.1, we have that

$$\cos(\beta-\gamma)+\cos(\alpha-\gamma)+\cos(\beta-\alpha)=$$
$$=(\cos\beta\cos\gamma+\cos\alpha\cos\beta+\cos\alpha\cos\gamma)+$$
$$+(\sin\beta\sin\gamma+\sin\alpha\sin\beta+\sin\alpha\sin\gamma)=$$
$$=\frac{p^2+r^2-4R^2}{4R^2}+\frac{p^2+r^2+4Rr}{4R^2}=\frac{p^2+r^2+2Rr-2R^2}{2R^2}.$$

Now we prove (2). By law of sines, $\sin\alpha=\dfrac{a}{2R}$, $\sin\beta=\dfrac{b}{2R}$, $\sin\gamma=\dfrac{c}{2R}$. By law of cosine, $\cos\alpha=\dfrac{b^2+c^2-a^2}{2bc}$, $\cos\beta=\dfrac{a^2+c^2-b^2}{2ac}$, $\cos\gamma=\dfrac{a^2+b^2-c^2}{2ab}$. It is known that $abc=4prR$ [2, p. 18]. Therefore,

$$\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)=$$
$$=\sin\alpha(\cos\beta-\cos\gamma)+\sin\beta(\cos\gamma-\cos\alpha)+\sin\gamma(\cos\alpha-\cos\beta)=$$
$$=\frac{a}{2R}\left(\frac{a^2+c^2-b^2}{2ac}-\frac{a^2+b^2-c^2}{2ab}\right)+\frac{b}{2R}\left(\frac{a^2+b^2-c^2}{2ab}-\frac{b^2+c^2-a^2}{2bc}\right)+$$
$$+\frac{c}{2R}\left(\frac{b^2+c^2-a^2}{2bc}-\frac{a^2+c^2-b^2}{2ac}\right)=\frac{1}{4R}\left(\frac{a^2+c^2-b^2}{c}-\frac{b^2+c^2-a^2}{c}+\right.$$
$$\left.+\frac{a^2+b^2-c^2}{a}-\frac{a^2+c^2-b^2}{a}+\frac{b^2+c^2-a^2}{b}-\frac{a^2+b^2-c^2}{b}\right)=$$
$$=\frac{1}{2R}\left(\frac{a^2-b^2}{c}+\frac{b^2-c^2}{a}+\frac{c^2-a^2}{b}\right)=$$
$$=\frac{1}{2Rabc}\left(a^3b-ab^3+b^3c-bc^3+ac^3-ca^3\right)=$$
$$=\frac{-(a+b+c)(a-b)(b-c)(c-a)}{2Rabc}=-\frac{(a-b)(b-c)(c-a)}{4R^2r}.$$

This completes the proof.

By Lemma 2.2(2), the following corollary is true.

Corrolary 2.1. *An triangle ABC is isosceles if and only if*

$$\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)=0.$$

Lemma 2.3. *For any triangle ABC the following equalities hold:*

(1) $\cos(\beta-\gamma)\cos(\alpha-\gamma)+\cos(\beta-\gamma)\cos(\beta-\alpha)+\cos(\alpha-\gamma)\cos(\beta-\alpha)=$
$=\dfrac{p^2(R+6r)-4R^3-16R^2r-11Rr^2-2r^3}{4R^3}$;

(2) $\cos(\alpha-\beta)\cos(\alpha-\gamma)\cos(\beta-\gamma)=$
$=\dfrac{p^4-p^2(6R^2+8Rr-2r^2)+8R^4+24R^3r+22R^2r^2+8Rr^3+r^4}{8R^4}$.

Proof. By Lemma 2.1, we have that

$$\cos(\beta-\gamma)\cos(\alpha-\gamma)+\cos(\beta-\gamma)\cos(\beta-\alpha)+\cos(\alpha-\gamma)\cos(\beta-\alpha)=$$
$$=(\cos\beta\cos\gamma+\sin\beta\sin\gamma)(\cos\alpha\cos\gamma+\sin\alpha\sin\gamma)+$$
$$+(\cos\beta\cos\gamma+\sin\beta\sin\gamma)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+$$

$$+(\cos\alpha\cos\gamma+\sin\alpha\sin\gamma)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=$$

$$=\cos\alpha\cdot\cos\beta\cdot\cos\gamma\cdot(\cos\alpha+\cos\beta+\cos\gamma)+\sin\alpha\cdot\sin\beta\cdot\sin\gamma(\sin\alpha+\sin\beta+\sin\gamma)+$$

$$+\sin\gamma\cos\gamma\sin(\alpha+\beta)+\sin\beta\cos\beta\sin(\alpha+\gamma)+\sin\alpha\cos\alpha\sin(\beta+\gamma)=$$

$$=\frac{p^2(R+r)-(2R+r)^2(R+r)+2p^2r}{4R^3}+$$

$$+(1-\cos^2\alpha)\cos\alpha+(1-\cos^2\beta)\cos\beta+(1-\cos^2\gamma)\cos\gamma=$$

$$=\frac{p^2(R+r)-(2R+r)^2(R+r)+2p^2r}{4R^3}+\frac{R+r}{R}-\left(\frac{(2R+r)^3-3p^2r}{4R^3}-1\right)=$$

$$=\frac{p^2(R+6r)-4R^3-16R^2r-11Rr^2-2r^3}{4R^3}.$$

Now we prove (2). We have that

$$\cos(\alpha-\beta)\cos(\alpha-\gamma)\cos(\beta-\gamma)=$$
$$=(\cos\alpha\cos\beta+\sin\alpha\sin\beta)(\cos\alpha\cos\gamma+\sin\alpha\sin\gamma)(\cos\beta\cos\gamma+\sin\beta\sin\gamma)=$$
$$=(\cos\alpha\cos\beta+\sin\alpha\sin\beta)\big(\cos\alpha\cos\beta\cos^2\gamma+\sin\gamma\cos\gamma\cos\alpha\sin\beta+$$
$$+\sin\gamma\cos\gamma\sin\alpha\cos\beta+\sin^2\gamma\sin\alpha\sin\beta\big)=$$
$$=(\cos\alpha\cdot\cos\beta\cdot\cos\gamma)^2+(\sin\alpha\cdot\sin\beta\cdot\sin\gamma)^2+$$
$$+\frac{\sin2\gamma\cdot\sin2\beta+\sin2\gamma\cdot\sin2\alpha+\sin2\alpha\cdot\sin2\beta}{4}.$$

Since

$$\sin2\gamma\cdot\sin2\beta+\sin2\gamma\cdot\sin2\alpha+\sin2\alpha\cdot\sin2\beta=$$
$$=\frac{1}{4R^4}(p^4+2p^2(r^2-2R^2-4Rr)+16R^3r+20R^2r^2+8Rr^3+r^4)$$

(see [2, p. 36]), by Lemma 2.1, we have that

$$\cos(\alpha-\beta)\cos(\alpha-\gamma)\cos(\beta-\gamma)=\left(\frac{p^2-(2R+r)^2}{4R^2}\right)^2+\left(\frac{pr}{2R^2}\right)^2+$$
$$+\frac{1}{16R^4}(p^4+2p^2(r^2-2R^2-4Rr)+16R^3r+20R^2r^2+8Rr^3+r^4)=$$
$$=\frac{p^4-p^2(6R^2+8Rr-2r^2)+8R^4+24R^3r+22R^2r^2+8Rr^3+r^4}{8R^4}.$$

This completes the proof.

Theorem 2.1. *For any triangle ABC*

$$r_ar_br_c=\frac{16p^2r^2R^3}{(p^2+r(2R+r))^2}.$$

Proof. In [1], it is proved that $r_a = \dfrac{pr}{a\cdot\cos\dfrac{\beta-\gamma}{2}}$. By Lemmas 2.2–2.3, we get

$$r_ar_br_c = \frac{(pr)^3}{abc\cdot\cos^2\dfrac{\beta-\gamma}{2}\cdot\cos^2\dfrac{\alpha-\gamma}{2}\cdot\cos^2\dfrac{\alpha-\beta}{2}} = \\ = \frac{(pr)^3}{4prR\cdot\dfrac{1+\cos(\beta-\gamma)}{2}\cdot\dfrac{1+\cos(\alpha-\gamma)}{2}\cdot\dfrac{1+\cos(\alpha-\beta)}{2}} = \\ = \frac{2(pr)^2}{R}\cdot\frac{1}{1+A+B+C} = \frac{16p^2r^2R^3}{(p^2+r(2R+r))^2}$$

where

$$A = \cos(\beta-\gamma)+\cos(\alpha-\beta)+\cos(\gamma-\alpha),$$

$$B = \cos(\alpha-\beta)\cos(\alpha-\gamma)+\cos(\alpha-\gamma)\cos(\beta-\gamma)+\cos(\beta-\gamma)\cos(\alpha-\beta),$$

$$C = \cos(\alpha-\beta)\cos(\alpha-\gamma)\cos(\beta-\gamma).$$

This completes the proof.

Theorem 2.2. *For any triangle ABC*

$$r_ar_b + r_ar_c + r_br_c = \frac{(R+2r)\cdot 16p^2R^2r}{(p^2+r(2R+r))^2}.$$

Proof. Using the equality (2) and Theorem 2.1, we get

$$r_ar_b + r_ar_c + r_br_c = \left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\cdot r_ar_br_c = \\ = \left(\frac{2}{R}+\frac{1}{r}\right)\cdot\frac{16p^2r^2R^3}{(p^2+r(2R+r))^2} = \frac{(R+2r)\cdot 16p^2R^2r}{(p^2+r(2R+r))^2}.$$

This completes the proof.

Lemma 2.4. *For any triangle ABC the following equality holds:*

$$ab\cos(\beta-\gamma)\cos(\alpha-\gamma)+ac\cos(\alpha-\beta)\cos(\gamma-\beta)+bc\cos(\beta-\alpha)\cos(\gamma-\alpha) = \\ = \frac{p^4-p^2(4R^2+8Rr-18r^2)+(2R+r)^2(r^2+4Rr)}{4R^2}.$$

Proof. We have that

$$\begin{aligned}
&ab\cos(\beta-\gamma)\cos(\alpha-\gamma)+ac\cos(\alpha-\beta)\cos(\gamma-\beta)+bc\cos(\beta-\alpha)\cos(\gamma-\alpha)=\\
&=ab(\cos\beta\cos\gamma+\sin\beta\sin\gamma)(\cos\alpha\cos\gamma+\sin\alpha\sin\gamma)+\\
&+ac(\cos\alpha\cos\beta+\sin\alpha\sin\beta)(\cos\gamma\cos\beta+\sin\gamma\sin\beta)+\\
&+bc(\cos\alpha\cos\beta+\sin\alpha\sin\beta)(\cos\gamma\cos\alpha+\sin\gamma\sin\alpha)=\\
&ab\left(\cos\alpha\cos\beta\cos^2\gamma+\frac{\sin 2\gamma}{2}\cdot\sin\alpha\cos\beta+\frac{\sin 2\gamma}{2}\cdot\sin\beta\cos\alpha+\sin\alpha\sin\beta\sin^2\gamma\right)+\\
&+ac\left(\cos\alpha\cos\gamma\cos^2\beta+\frac{\sin 2\beta}{2}\cdot\sin\gamma\cos\alpha+\frac{\sin 2\beta}{2}\cdot\sin\alpha\cos\gamma+\sin\alpha\sin\gamma\sin^2\beta\right)+\\
&+bc\left(\cos\beta\cos\gamma\cos^2\alpha+\frac{\sin 2\alpha}{2}\cdot\sin\gamma\cos\beta+\frac{\sin 2\alpha}{2}\cdot\sin\beta\cos\gamma+\sin\beta\sin\gamma\sin^2\alpha\right)=\\
&=\cos\alpha\cos\beta\cos\gamma\cdot(ab\cos\gamma+ac\cos\beta+bc\cos\alpha)+\\
&+\frac{ab\sin 2\gamma\sin(\alpha+\beta)+ac\sin 2\beta\sin(\alpha+\gamma)+bc\sin 2\alpha\sin(\beta+\gamma)}{2}+\\
&+\sin\alpha\sin\beta\sin\gamma\cdot(ab\sin\gamma+ac\sin\beta+bc\sin\alpha)=\\
&=\cos\alpha\cos\beta\cos\gamma\cdot\left(\frac{a^2+b^2-c^2}{2}+\frac{a^2+c^2-b^2}{2}+\frac{b^2+c^2-a^2}{2}\right)+\\
&+\frac{4R^2}{2}\cdot\sin\alpha\sin\beta\sin\gamma\cdot(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)+4R^2\sin\alpha\sin\beta\sin\gamma\cdot 3\sin\alpha\sin\beta\sin\gamma=\\
&=\cos\alpha\cos\beta\cos\gamma\cdot\frac{a^2+b^2+c^2}{2}+2R^2\cdot\sin\alpha\sin\beta\sin\gamma\cdot(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)+\\
&+12R^2(\sin\alpha\sin\beta\sin\gamma)^2.
\end{aligned}$$

By [2, p. 18, 36], $a^2+b^2+c^2=2(p^2-r^2-4Rr)$ and $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=\frac{2pr}{R^2}$. Using Lemma 2.1, we have that

$$\begin{aligned}
&ab\cos(\beta-\gamma)\cos(\alpha-\gamma)+ac\cos(\alpha-\beta)\cos(\gamma-\beta)+bc\cos(\beta-\alpha)\cos(\gamma-\alpha)=\\
&=\frac{p^2-(2R+r)^2}{4R^2}\cdot\frac{2(p^2-r^2-4Rr)}{2}+2R^2\cdot\frac{pr}{2R^2}\cdot\frac{2pr}{R^2}+12R^2\cdot\frac{p^2r^2}{4R^4}=\\
&=\frac{p^2-(2R+r)^2}{4R^2}\cdot(p^2-r^2-4Rr)+\frac{5p^2r^2}{R^2}=\\
&\frac{p^4-p^2(4R^2+8Rr-18r^2)+(2R+r)^2(r^2+4Rr)}{4R^2}.
\end{aligned}$$

This completes the proof.

Lemma 2.5. *For any triangle ABC the following equation holds:*

$$ab\cos(\alpha-\gamma)+ab\cos(\beta-\gamma)+ac\cos(\alpha-\beta)+ac\cos(\gamma-\beta)+bc\cos(\beta-\alpha)+bc\cos(\gamma-\alpha)=$$

$$=\frac{7rp^2}{R}-8Rr-6r^2-\frac{r^3}{R}.$$

Proof. By Lemma 2.1, we get

$$ab\cos(\alpha-\gamma)+ab\cos(\beta-\gamma)+ac\cos(\alpha-\beta)+ac\cos(\gamma-\beta)+bc\cos(\beta-\alpha)+bc\cos(\gamma-\alpha) =$$
$$= ab\cos\alpha\cos\gamma + ab\sin\alpha\sin\gamma + ab\cos\beta\cos\gamma + ab\sin\beta\sin\gamma +$$
$$+ ac\cos\alpha\cos\beta + ac\sin\alpha\sin\beta + ac\cos\gamma\cos\beta + ac\sin\gamma\sin\beta +$$
$$+ bc\cos\beta\cos\alpha + bc\sin\beta\sin\alpha + bc\cos\gamma\cos\alpha + bc\sin\gamma\sin\alpha =$$
$$= 2R\cdot\sin\alpha\sin\beta\sin\gamma\cdot(a+b+c) + 2R\cdot\sin\alpha\sin\beta\sin\gamma\cdot(a+b+c)+$$
$$+a\cos\alpha(b\cos\gamma+c\cos\beta)+b\cos\beta(a\cos\gamma+c\cos\alpha)+c\cos\gamma(a\cos\beta+b\cos\alpha) =$$
$$= 8pR\cdot\sin\alpha\sin\beta\sin\gamma+$$
$$+ 2Ra\cos\alpha\sin(\beta+\gamma) + 2Rb\cos\beta\sin(\alpha+\gamma) + 2Rc\cos\gamma\sin(\alpha+\beta) =$$
$$= 8pR\cdot\frac{pr}{2R^2} + 4R^2\cdot\left(\cos\alpha\sin^2\alpha + \cos\beta\sin^2\beta + \cos\gamma\sin^2\gamma\right) =$$
$$= \frac{4p^2r}{R} + 4R^2\cdot\left((\cos\alpha+\cos\beta+\cos\gamma) - (\cos^3\alpha+\cos^3\beta+\cos^3\gamma)\right) =$$
$$= \frac{4rp^2}{R} + 4R^2\cdot\left(1+\frac{r}{R}+1-\frac{(2R+r)^3-2p^2r}{4R^3}\right) = \frac{7rp^2}{R} - 8Rr - 6r^2 - \frac{r^3}{R}.$$

This completes the proof.

Theorem 2.3. *For any triangle ABC the following equalities hold:*

(1) $\dfrac{1}{r_ar_b} + \dfrac{1}{r_ar_c} + \dfrac{1}{r_br_c} = \dfrac{p^4 + p^2(20Rr+18r^2) + (4R+r)r^3}{16p^2r^2R^2}$;

(2) $r_a + r_b + r_c = \dfrac{\left(p^4 + p^2(20Rr+18r^2) + (4R+r)r^3\right)\cdot R}{(p^2+r(2R+r))^2}$.

Proof. Using Lemmas 2.4–2.5, we have that

$$\frac{1}{r_ar_b} + \frac{1}{r_ar_c} + \frac{1}{r_br_c} = \frac{1}{(rp)^2}\left(ab\cos^2\frac{\beta-\gamma}{2}\cos^2\frac{\alpha-\gamma}{2}+\right.$$
$$\left.+ac\cos^2\frac{\alpha-\beta}{2}\cos^2\frac{\gamma-\beta}{2} + bc\cos^2\frac{\beta-\alpha}{2}\cos^2\frac{\gamma-\alpha}{2}\right) =$$
$$= \frac{1}{(rp)^2}\left(\frac{ab(1+\cos(\beta-\gamma))(1+\cos(\alpha-\gamma))}{4}+\right.$$
$$\left.+\frac{ac(1+\cos(\alpha-\beta))(1+\cos(\gamma-\beta))}{4}+\frac{bc(1+\cos(\beta-\alpha))(1+\cos(\gamma-\alpha))}{4}\right) =$$
$$= \frac{1}{4p^2r^2}\left(ab+ac+bc+ab\cos(\beta-\gamma)\cos(\alpha-\gamma)+\right.$$
$$+ ac\cos(\alpha-\beta)\cos(\gamma-\beta) + bc\cos(\beta-\alpha)\cos(\gamma-\alpha)+$$
$$+ ab\cos(\alpha-\gamma) + ab\cos(\beta-\gamma) + ac\cos(\gamma-\beta)+$$
$$\left.+ac\cos(\alpha-\beta) + bc\cos(\gamma-\alpha) + bc\cos(\beta-\alpha)\right) =$$

$$= \frac{1}{4p^2r^2}\left(p^2 + r^2 + 4Rr + \frac{p^4 - p^2(4R^2 + 8Rr - 18r^2) + (2R + r)^2(r^2 + 4Rr)}{4R^2} + \right.$$
$$\left. + \frac{7r}{R}p^2 - 8Rr - 6r^2 - \frac{r^3}{R}\right) =$$
$$= \frac{1}{16p^2r^2R^2}\big(4R^2p^2 + 4R^2r^2 + 16R^3r + p^4 - p^2(4R^2 + 8Rr - 18r^2) +$$
$$+ (2R + r)^2(r^2 + 4Rr) + 28p^2Rr - 32R^3r - 24R^2r^2 - 4Rr^3\big) =$$
$$= \frac{1}{16p^2r^2R^2}\big(p^4 + p^2(20Rr + 18r^2) + (2R + r)^2(r^2 + 4Rr) -$$
$$- 16R^3r - 20R^2r^2 - 4Rr^3\big) = \frac{1}{16p^2r^2R^2}(p^4 + p^2(20Rr + 18r^2) + (4R + r)r^3).$$

Now we prove (2). We have that

$$r_a + r_b + r_c = \left(\frac{1}{r_ar_b} + \frac{1}{r_ar_c} + \frac{1}{r_br_c}\right) \cdot r_ar_br_c =$$
$$= \frac{\big(p^4 + p^2(20Rr + 18r^2) + (4R + r)r^3\big)}{16p^2r^2R^2} \cdot \frac{16p^2r^2R^3}{(p^2 + r(2R + r))^2} =$$
$$= \frac{\big(p^4 + p^2(20Rr + 18r^2) + (4R + r)r^3\big) \cdot R}{(p^2 + r(2R + r))^2}.$$

This completes the proof.

Using the formulas of Vieta and Theorems 2.1–2.3, we get the proof of Theorem 1.1.

Let m_a, m_b, m_c be medians of a triangle ABC. It is known that one can to construct a new triangle with sides m_a, m_b, m_c [4, p. 28]. Let h_a, h_b, h_c be the heights and ρ_a, ρ_b, ρ_c be the radii of the escribed circles of the triangle ABC. In [2, p. 56], the authors found necessary and sufficient conditions when (a) $p - a, p - b, p - c$; (b) h_a, h_b, h_c; (c) ρ_a, ρ_b, ρ_c are sides of some triangle. So next question is interesting: when does a triangle with sides r_a, r_b, r_c exist? It is known that the roots of an equation $x^3 + sx^2 + qx + t = 0$ ($s, q, t \in \mathbb{R}$), are sides of some triangle if and only if

(1) $s < 0, q > 0, t < 0$;
(2) $-4s^2t + s^2q^2 + 18sqt - 4q^2 - 27t^2 > 0$;
(3) $s^3 - 4sq + 8t > 0$ (see [2, Theorem 1, p. 53]).

The conditions (1) and (2) of this theorem are true when the roots of the equation $x^3 + sx^2 + qx + t = 0$ are positive real numbers [2, Corollary 1, Lemma 3]. Therefore a triangle with sides r_a, r_b, r_c exists if and only if the condition $s^3 - 4sq + 8t > 0$ is true. So a triangle with sides r_a, r_b, r_c exists if and only if the following inequality holds:

$$(3)\quad -R^3\cdot\frac{(p^4+p^2(20Rr+18r^2)+(4R+r)r^3)^3}{(p^2+r(2R+r))^6}+ \\ +4\cdot\frac{(p^4+p^2(20Rr+18r^2)+(4R+r)r^3)(R+2r)\cdot 16p^2rR^3}{(p^2+r(2R+r))^4}- \\ -\frac{128p^2r^2R^3}{(p^2+r(2R+r))^2}>0.$$

Now we show that the condition (3) is not always true. Fix the number R. Consider an triangle ABC such that $BA = BC$ and R is the circumradius of the triangle ABC. Fix the vertex B. Let the vertex A tends to the vertex C. Then we have that p tends to $2R$ and r tends to zero. Hence the left part of the inequality (3) is negative for some p, R, r. So a triangle with sides r_a, r_b, r_c does not exist for such triangle ABC.

References

[1] Andrica, D., Marinescu, D.S., *New Interpolation Inequalities to Euler's $R \geq 2r$*, Forum Geometricorum, **17 (2017)** 149–156.

[2] Soltan, V., Maidman, S., *Identities and inequalities in a triangle* (Russian), Kishinev, Shtiinca Publishers, 1982.

[3] Isaev, I.M., Maltsev, Yu.N., Monastyreva, A.S., *On Some Geometric Relations of a Triangle*, Journal of Classical Geometry, **4 (2018)**.

[4] Zetel, S.I., *New Geometry in Triangle* (Russian), Moskow, 1962.

[5] Maltsev, Yu.N., Kuzmina (Monastyreva), A.S., *Selected Lectures on the Geometry of Triangle and Quadrangle* (Russian), Barnaul, Altai State Pedagogical University, 2016.

ALTAI STATE PEDAGOGICAL UNIVERSITY
BARNAUL, RUSSIA
E-mail address: `maltsevyn@gmail.com`

ALTAI STATE UNIVERSITY
BARNAUL, RUSSIA
E-mail address: `akuzmina1@yandex.ru`